
XIII. *Demonstration of the Fundamental Property of the Lever.*

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IT is a singular fact in the history of science, that, after all the attempts of the most eminent modern mathematicians, to obtain a simple and satisfactory demonstration of the fundamental property of the lever, the solution of this problem given by ARCHIMEDES, should still be considered as the most legitimate and elementary. GALILEO, HUYGENS, DE LA HIRE, Sir ISAAC NEWTON, MACLAURIN, LANDEN, and HAMILTON, have directed their attention to this important part of mechanics; but their demonstrations are in general either tedious and abstruse, or founded on assumptions too arbitrary to be recognised as a proper basis for mathematical reasoning. Even the demonstration given by ARCHIMEDES is not free from objections, and is applicable only to the lever, considered as a physical body. GALILEO, though his demonstration is superior in point of simplicity to that of ARCHIMEDES, resorts to the inelegant contrivance, of suspending a solid prism from a mathematical lever, and of dividing the prism into two unequal parts, which act as the power and the weight. The demonstration given by HUYGENS, assumes as an axiom, that a given weight
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removed from the fulcrum, has a greater tendency to turn the lever round its centre of motion, and is, besides, applicable only to a commensurable proportion of the arms. The foundation of Sir ISAAC NEWTON's demonstration is still more inadmissible. He assumes, that if a given power act in any direction upon a lever, and if lines be drawn from the fulcrum to the line of direction, the mechanical effort of the power will be the same when it is applied to the extremity of any of these lines; but it is obvious, that this axiom is as difficult to be proved as the property of the lever itself. M. DE LA HIRE has given a demonstration which is remarkable for its want of elegance. He employs the *reductio ad absurdum*, and thus deduces the proposition from the case where the arms are commensurable. The demonstration given by MACLAURIN has been highly praised; but if it does not involve a *petitio principii*, it has at least the radical defect, of extending only to a commensurable proportion of the arms. The solutions of LANDEN and HAMILTON are peculiarly long and complicated, and resemble more the demonstration of some of the abstrusest points of mechanics, than of one of its simplest and most elementary truths.

IN attempting to give a new demonstration of the fundamental property of the lever, which shall be at the same time simple and legitimate, we shall assume only one principle, which has been universally admitted as axiomatic, namely, *that equal and opposite forces, acting at the extremities of the equal arms of a lever, and at equal angles to these arms, will be in equilibrio*. With the aid of this axiom, the fundamental property of the lever may be established by the three following propositions.

IN PROP. I. the property is deduced in a very simple manner, when the arms of the lever are commensurable.

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IN PROP. II., which is totally independent of the first, the demonstration is general, and extends to any proportion between the arms.

IN PROP. III. the property is established, when the forces act in an oblique direction, and when the lever is either rectilinear, angular, or curvilinear. In the demonstrations which have generally been given of this last proposition, the oblique force has been resolved into two, one of which is directed to the fulcrum, while the other is perpendicular to that direction. It is then assumed, *that the force directed to the fulcrum has no tendency to disturb the equilibrium, even though it acts at the extremity of a bent arm*; and hence it is easy to demonstrate, that the remaining force is proportional to the perpendicular drawn from the fulcrum to the line of direction in which the original force was applied. As the principle thus assumed, however, is totally inadmissible as an intuitive truth, we have attempted to demonstrate the proposition without its assistance.

PROP. I.—*If one arm of a straight lever is any multiple of the other, a force acting at the extremity of the one will be in equilibrio with a force acting at the extremity of the other, when these forces are reciprocally proportioned to the length of the arms to which they are applied.*

LET AB (PLATE XI. fig. 1.) be a lever supported on the two fulcra F, f , so that $Af = fF = FB$. Then, if two equal weights C, D , of 1 pound each, be suspended from the extremities A, B , they will be in equilibrio, since they act at the end of equal arms Af, BF ; and each of the fulcra f, F , will support an equal part of the whole weight, or 1 pound. Let the fulcrum f be now removed, and let a weight E , of 1 pound, act upwards at the point f ; the equilibrium will still continue; but the weight E , of 1 pound, acting upwards at f , is equivalent to a weight G of 1 pound, acting downwards at B . Remove, therefore, the weight E ,

E, and suspend the weight G from B; then, since the equilibrium is still preserved after these two substitutions, we have a weight C, of one pound, acting at the extremity of the arm AF, in equilibrio with the weights D and G, which together make two pounds, acting at the extremity of the arm FB. But FA is to FB as 2 is to 1; therefore an equilibrium takes place, when the weights are reciprocally proportional to the arms, in the particular case when the arms are as 2 to 1. By making Ff successively double, triple, &c. of FB, it may in like manner be shewn, that, in these cases, the proposition holds true.

LEMMA.

If any weight BCc b, (fig. 2. No. 1.), of uniform shape and density, is placed on a lever A ϕ , whose fulcrum is ϕ , it has the same tendency to turn the lever round ϕ , as if it were suspended from a point G, so taken that $bG = Gc$.

If a weight W, of the same magnitude with BC, acts upwards at the point G, it will be in equilibrium with the weight BC, and will therefore destroy the tendency of that weight to turn the lever round ϕ . But the weight W, acting upwards at the point G, has the same power to turn the lever round ϕ , as an equal weight w , acting downwards at G. Consequently the tendency of the weight BC to turn the lever round ϕ , is the same as the tendency of an equal weight w , acting downwards at G.

PROP. II.

If two forces applied to a lever, and acting at right angles to it, have the same tendency to turn the lever round its centre of motion, they are reciprocally proportional to the distances of the points at which they are applied from the centre of motion.

LET A ϕ d, (fig. 2. No. 2.) be a lever whose fulcrum is ϕ , and let it be loaded with a weight BD d b of uniform shape and density.

fity. Then by the lemma, this weight has the same tendency to turn the lever round, as if it were suspended from the point n , so taken that $bn = dn$. Make $\phi c = \phi d$, and let the weight $BD db$ be divided at the points C and F , by the lines Cc , $F\phi$. The weights $CF \phi c$, $DF \phi d$, being in equilibrio, by the axiom, have no tendency to turn the lever round ϕ , consequently the remaining weight $BC cb$, has the same tendency to turn the lever round ϕ as the whole weight $BD db$. Hence if $bm = cm$, the weight $BC cb$ acting at the point m , will have the same tendency to turn the lever round ϕ , as the weight $BD db$ acting at n . Now $BD db : BC cb = bd : bc = nd : mc$; and since $bc = bd - cd$, we have $mc = \frac{1}{2} bd - \frac{1}{2} cd = nd - \frac{1}{2} cd = n\phi$, and $nd = n\phi + \frac{1}{2} cd = mc + \frac{1}{2} cd = m\phi$. Consequently,

$$BD db : BC cb = m\phi : n\phi.$$

LEMMA.

Two equal forces acting at the same point of the arm of a lever, and in directions which form equal angles with a perpendicular drawn through that point of the arm, will have equal tendencies to turn the lever round its centre of motion.

LET AB (fig. 3.) be a lever with equal arms AF , FB . Through the points A , B , draw AD , BE , perpendicular to AB , and AP , $A\rho$, BW , Bw , forming equal angles with the lines AD , BE . Produce PA to M . Then, equal forces acting in the directions AP , Bw , will be in equilibrio. But a force M equal to P , and acting in the direction AM , will counteract the force P , acting in the direction AB , or will have the same tendency to turn the lever round F ; and the force W , acting in the direction BW , will have the same tendency to turn the lever round F as the

the force M : Consequently the force W will have the same tendency to turn the lever round F as the force w ; and this will hold true, whether the arms AF , FB , are straight or curvilinear, provided that they are both of the same form.

PROP. III.—*If a force acts in different directions at the same point in the arm of a lever, its tendency to turn the lever round its centre of motion, will be proportional to the perpendiculars let fall from that centre on the lines of direction in which the force is applied.*

LET AB , (fig. 4.) be the lever, and let the two equal forces BM , Bm , act upon it at the point B , in the direction of the lines BM , Bm . Draw BN , Bn , respectively equal to BM , Bm , and forming the same angles with the line PB perpendicular to AB . To BM , Bm , BN , Bn , produced, draw the perpendiculars AY , Ay , AX , Ax . Now, the side $AX = AY$, and $Ax = Ay$, on account of the equality of the triangles ABX , ABY ; and if Bl , $B\lambda$, be drawn perpendicular to $B\omega$, the triangles ABY , BMl , will be similar, and also the triangles ABy , $Bm\lambda$: Hence we obtain

$$AB : AY = BM : Bl, \text{ and}$$

$$AB : Ay = BM : B\lambda$$

Therefore, *ex æquo*, $AY : Ay = Bl : B\lambda$.

Complete the parallelograms $BM\omega N$, $Bm\omega n$, and Bl , $B\lambda$ will be respectively one-half of the diagonals $B\omega$, $B\omega$.

Now let two equal forces BM , BN , act in these directions upon the lever at B , their joint force will be represented by the diagonal $B\omega$, and consequently one of the forces BM will
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be represented by $Bl = \frac{1}{2} Bo$. In the same manner, if the two equal forces Bm, Bn , act upon the lever at B , their joint force will be represented by $B\omega$, and one of them, Bm , will be represented by $B\lambda = \frac{1}{2} B\omega$. Consequently the power of the two forces BM, Bm , to turn the lever round its centre of motion, is represented by $Bl, B\lambda$, respectively; that is, the force BM is to the force Bm as Bl is to $B\lambda$; that is, as AY is to Ay , the perpendiculars let fall upon the lines of their direction.

