NOTE ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS

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I SHOULD like to have the opportunity of acknowledging, what I learned in October, 1903, from a paper of Professor Bôcher's (*Amer. Jour.*, Vol. xxiv., p. 311), that the matrizant solution of a system of linear differential equations given in *Proc. London Math. Soc.*, Vol. xxxiv., p. 354 and p. 356, footnote, which grew up naturally in my mind in connection with Schur's series for continuous groups (*Proc. London Math. Soc.*, February 14th, 1901, Vol. xxxiv., p. 97, and Vol. xxxiv., p. 348) had been previously given by Peano, *Math. Ann.*, Vol. xxxii., 1888, pp. 455, 456, with the unimportant limitation, in the statement, to coefficients which are real functions of the variable continuous in an interval for which the convergence of the series is to be proved.

Indeed the idea of using series of repeated integrations, for a single difference equation and a single differential equation regarded as a limit of this, is at least as old as the paper of Caqué, *Liouville's Journal*, 2nd Series, t. ix., 1864, p. 194; while in the paper written by Fuchs in 1870, to give a more general aspect to Caqué's method (*Ann. d. Mat.*, ii. Ser., t. iv., p. 36), the convergence of these series for all finite values of the argument other than the singularities is clearly recognized. My ignorance of this paper of Fuchs, at the time of the last note on linear differential equations (*Proc. London Math. Soc.*, Vol. xxxv., p. 333, where, p. 378, I have collected references to papers seeming to be in connection with the method of the paper), is the more inexcusable in that Fuchs's results are expounded in Schlesinger's treatise (Vol. i., pp. 370 and 389), with an application to equations of rank unity. Perhaps, however, the connexion of Fuchs's formula with the matrizant solution is not very obvious; and it may be worth showing that Fuchs's generalized form of Caqué's solution for a single differential equation is a particular case of a general formula given in the note just referred to (*Proc. London Math. Soc.*, Vol. xxxv., p. 339). This is what is proved below.

Let a, β be two matrices of the same number n of rows and columns, of which each element is a function of t, and let σ be a matrix of constants; then

$$\Delta = \Omega(a)\sigma$$

is a matrix whose columns are sets of solutions of the linear system

$$\frac{dx}{dt} = ax.$$

By easily verified formulæ given in *Proc. London Math. Soc.*, Vol. xxxv., 1902, pp. 339, 337, namely,

$$\Omega(a+\beta) = \Omega(a)\,\Omega[\Omega^{-1}(a)\beta\,\Omega(a)], \qquad \sigma^{-1}\Omega(u)\sigma = \Omega(\sigma^{-1}u\sigma),$$

we have $\Omega(a+\beta)\,\sigma = \Omega(a)\,\sigma\,.\,\sigma^{-1}\Omega[\Omega^{-1}(a)\beta\Omega(a)]\sigma$

$$= \Delta\Omega[\sigma^{-1}\Omega^{-1}(a)\,\beta\Omega(a)\sigma] = \Delta\Omega(\Delta^{-1}\beta\Delta);$$

now let $w = \Delta^{-1}\beta\Delta$, or $w\Delta^{-1} = \Delta^{-1}\beta$, and, denoting a row $(h_1 \ldots h_n)$ of constants by h, put

$$(u_1^{(0)} \ldots u_n^{(0)}) = \Delta h,$$

(A) $$(u_1^{(i)} \ldots u_n^{(i)}) = \Delta Q\,[\Delta^{-1}\beta(u_1^{(i-1)} \ldots u_n^{(i-1)})] \quad (i = 1, 2, \ldots, \infty),$$

so that $u_1^{(0)} \ldots u_n^{(0)}$ form a set of solutions of the linear system $dx/dt = ax$, and the successive sets $u_1^{(i)} \ldots u_n^{(i)}$ are determined each from the preceding by a single quadrature, denoted by Q. Then we have

$$\Omega(a+\beta)\sigma h = \Delta\Omega(w)\,h = \Delta h + \Delta Qwh + \Delta Qw\,Qwh + \ldots,$$

where

$$\Delta h = (u_1^{(0)} \ldots u_n^{(0)}),$$

$$\Delta Qwh = \Delta Q(\Delta^{-1}\beta\Delta h) = \Delta Q[\Delta^{-1}\beta(u_1^{(0)} \ldots u_n^{(0)})] = (u_1^{(1)} \ldots u_2^{(1)}),$$

$$\Delta Qw\,Qwh = \Delta Q[w\Delta^{-1}(u_1^{(1)} \ldots u_n^{(1)})] = \Delta Q[\Delta^{-1}\beta(u_1^{(1)} \ldots u_n^{(1)})] = (u_1^{(2)} \ldots u_n^{(2)}),$$

and so on. Thus a set of solutions of the linear system $dx/dt = (a+\beta)\,x$ is given by

$$(x_1 \ldots x_n) = (u_1^{(0)} \ldots u_n^{(0)}) + (u_1^{(1)} \ldots u_n^{(1)}) + (u_1^{(2)} \ldots u_n^{(2)}) + \ldots,$$

that is, by

(B) $$x_j = u_j^{(0)} + u_j^{(1)} + u_j^{(2)} + \ldots.$$

By choosing the matrix σ suitably, with non-vanishing determinant, Δ may be regarded as having for its columns any fundamental set of integrals of the system $dx/dt = ax$, and, by choosing h suitably, the solution (B), with the law of recurrence (A), represents any set of integrals of the compound system $dx/dt = (\alpha+\beta)x$.

This includes, as we next show, the results of Fuchs, *Ann. d. Mat.*, 1870–71, p. 43, given in Schlesinger, Treatise I., p. 373; we limit ourselves to the case of a homogeneous differential equation.

The single linear equation

$$y^{(n)} = (a_n+b_n)y^{(n-1)}+(a_{n+1}+b_{n-1})y^{(n-2)}+\dots+(a+b)y$$

becomes, by $x_1 = y,\ x_2 = y',\ \dots,\ x_n = y^{(n-1)}$,

$$\frac{dx}{dt} = \begin{Bmatrix} 0 & 1 & 0 & . & . \\ 0 & 0 & 1 & 0 & . \\ . & . & . & . & . \\ a+b & a_1+b_1 & . & . & a_n+b_n \end{Bmatrix} x,$$

$$= \left[\begin{Bmatrix} 0 & 1 & 0 & . & . \\ 0 & 0 & 1 & 0 & . \\ . & . & . & . & . \\ a_1 & a_2 & a_3 & . & a_n \end{Bmatrix} + \begin{Bmatrix} 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \\ b_1 & b_2 & b_3 & . & b_n \end{Bmatrix}\right] x = (\alpha+\beta)\,x, \text{ say};$$

let $y_1, \dots, y_n$ be a set of independent integrals of $y^{(n)} = a_n y^{(n-1)}+\dots+a_1 y$; for the matrix Δ we can then put

$$\Delta = \begin{Bmatrix} y_1 & \cdots & y_n \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{Bmatrix},$$

and then, if $D_1, \dots, D_n$ be the determinants of the minors of the elements in the last row of Δ, and $D, = |\,\Delta\,|$, be the determinant of Δ, and the inverse matrix Δ^{-1} be (ϕ_{ij}), so that

$$\phi_{1n} = \frac{D_1}{D},\quad \phi_{2n} = \frac{D_2}{D},\quad \dots,\quad \phi_{nn} = \frac{D_n}{D},$$

we have, for arbitrary $v_1 \dots v_n$,

$$\Delta^{-1}\beta(v_1, v_2, \dots, v_n) = \begin{Bmatrix} \phi_{1n}b_1 & \dots & \phi_{1n}b_n \\ \cdots & \cdots & \cdots \\ \phi_{nn}b_1 & \dots & \phi_{nn}b_n \end{Bmatrix} (v_1 \dots v_n)$$

$$= \left[\frac{D_1}{D}(b_1v_1+\dots+b_nv_n),\ \dots,\ \frac{D_n}{D}(b_1v_1+\dots+b_nv_n)\right];$$

hence the elements of $\Delta Q[\Delta^{-1}\beta(v_1, \ldots, v_n)]$ are respectively

$$y_1 Q\left[\frac{D_1}{D}(b_1 v_1 + \ldots + b_n v_n)\right] + \ldots + y_n Q\left[\frac{D_n}{D}(b_1 v_1 + \ldots + b_n v_n)\right],$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y_1^{(n-1)} Q\left[\frac{D_1}{D}(b_1 v_1 + \ldots + b_n v_n)\right] + \ldots + y_n^{(n-1)} Q\left[\frac{D_n}{D}(b_1 v_1 + \ldots + b_n v_n)\right],$$

of which manifestly each is the differential coefficient of the preceding. Put now in particular $(v_1, \ldots, v_n) = (y, y', \ldots, y^{(n-1)})$; the first element of $\Delta Q[\Delta^{-1}\beta(v, \ldots, v_n)]$ becomes

$$y_1 Q\left[\frac{D_1}{D}(b_1 y + \ldots + b_n y^{(n-1)})\right] + \ldots + y_n Q\left[\frac{D_n}{D}(b_1 y + \ldots + b_n y^{(n-1)})\right],$$

or, if $B[y(t)] = b_1 y + b_2 \dfrac{dy}{dt} + \ldots + b_n \dfrac{d^{n-1}y}{dt^{n-1}}$, it becomes

$$Y_1(t) = \int^t B[y(\xi)] \frac{\begin{vmatrix} y_1(\xi) & \ldots & y_n(\xi) \\ \cdot & \ldots & \cdot \\ y_1^{(n-2)}(\xi) & \ldots & y_n^{(n-2)}(\xi) \\ y_1(t) & \ldots & y_n(t) \end{vmatrix}}{\begin{vmatrix} y_1(\xi) & \ldots & y_n(\xi) \\ \cdot & \ldots & \cdot \\ y_1^{(n-1)}(\xi) & \ldots & y_n^{(n-1)}(\xi) \end{vmatrix}} d\xi = \int^t B[y(\xi)] \frac{H(\xi, t)}{D(\xi)} d\xi, \text{ say.}$$

Put next $(v_1, v_2, \ldots, v_n) = (Y_1, Y_1, \ldots, Y_1^{(n-1)})$; then the first element of $\Delta Q[\Delta^{-1}\beta(v_1, \ldots, v_n)]$ becomes

$$y_1 Q\left[\frac{D_1}{D}(b_1 Y_1 + \ldots + b_n Y_1^{(n-1)})\right] + \ldots + y_n Q\left[\frac{D_n}{D}(b_1 Y_1 + \ldots + b_n Y_1^{(n-1)})\right],$$

or
$$Y_2(t) = \int^t B[Y_1(\xi)] \frac{H(\xi, t)}{D(\xi)} d\xi,$$

and so on; which are Fuchs's formulæ.

Similarly, Caqué's formula, obtained by taking $a_1 = 0, \ldots, a_n = 0$; $b_1 = p_1, \ldots, b_n = p_n$, and an application to equations of rank unity, obtained by taking $a_1 = \text{const.}, \ldots, a_n = \text{const.}$; $b_1 = Q_1(1/t), \ldots, b_n = Q_n(1/t)$, may be deduced; but, as they are deduced in Schlesinger's treatise (Vol. I., pp. 377 and 389) from Fuchs's formula, they need not be given here.

Another interesting case is that where a is a matrix of constants with linear invariant factors. Taking this in its canonical form, the matrix $\Delta = \Omega(a)$ has only elements of the form $e^{\theta t}$, occurring in its diagonal.