On Forms of Numbers determined by Continued Fractions.

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1. In my paper "On the Decomposition of Certain Numbers into Sums of Two Square Integers by Continued Fractions,"* I made use of considerations which are evidently susceptible of wider application. A somewhat detailed treatment of a particular case is not unlikely to be useful in connection with a theory which, like that of binary quadratic forms, presents highly condensed and technically expressed results. I shall now give some further illustrations of the subject in a more general aspect.

The theory of Legendre relative to primes of the form $4m+1$ was greatly extended by Goepel in his inaugural dissertation "De æquationibus secundi gradus indeterminatis" (1835).†

The principal aim of the essay was to show that the decomposition of a prime of the form $8m+3$ or its double into a square and the double of a square, or of a prime of the form $8m+7$ or its double into the difference of a square and the double of a square, is obtainable by continued fractions in a manner strictly analogous to that in which the decomposition of a prime of the form $4m+1$ is effected.

Goepel showed, in fact, that such a decomposition can be effected of all numbers whose square roots, developed as continued fractions with unit numerators, present 2 as the denominator of a complete quotient. And, as Prof. H. J. S. Smith has remarked, the method employed is yet more general.

2. In the Report on the Theory of Numbers (Part V., "British Association Reports," 33rd Meeting), Professor Smith has given the more general results to which Goepel's conclusions point. It would be difficult to abridge the reasoning without confusion, and I only mention therefore some of the principal results which have a bearing on what I have to say. The section referred to is entitled "Application of Continued Fractions to the Theory of Quadratic Forms." (pp. 783–6.)

The Author sets out from the following theorem:—

"If (a, b, c), (a', b', c') are two primitive forms of the determinants D, D', whose joint invariant $aa'-2bb'+cc'$ is zero, and if m and m' are the greatest common divisors of a, $2b$, c; a', $2b'$, c'; m^2D' and m'^2D are respectively capable of primitive representation by the duplicates of (a, b, c) and (a', b', c')."

* Proceedings, Vol. IX., p. 187, *et seq.*

† Republished in Crelle's Journal, Vol. XLV., pp. 1—13.

If, then, $\frac{p_n}{q_n}$ is a convergent to $\sqrt{A}$, so that $p_n^2 - Aq_n^2 = (-)^n D_n$, and $(q_n, -p_n, q_nA)$ is properly primitive and of the principal class, or improperly primitive and of the principal class of its order, we may suppose the latter form to be changed into $(\epsilon_s, -\delta_s, \epsilon_{s-1})$ by a substitution $\begin{pmatrix} p_s, & p_{s-1} \\ q_s, & q_{s-1} \end{pmatrix}$, and that we have

$$(-)^{n+1}D_n = \epsilon_s\epsilon_{s-1} - \delta_s^2, \quad \epsilon_{s-1}D_s - 2\delta_s J_s - \epsilon_s D_{s-1} = 0, \quad A = J_s^2 + D_s D_{s-1},$$

and ϵ_s or $\epsilon_{s-1} = 1$ if the form is properly primitive, ϵ_s or $\epsilon_{s-1} = 2$ if it is improperly primitive.

If n is odd, so that $(-)^{n+1}D_n = \Delta$, the following theorem is arrived at:—

"If $\frac{p_n}{q_n}$ is an inferior convergent to $\sqrt{A}$, and $\Delta = q_n^2 A - p_n^2$; when $(q_n, -p_n, q_nA)$ is of the principal class of forms of determinant $-\Delta$, A is of the form $X^2 + \Delta Y^2$, and Y is the denominator of a complete quotient in the development of $\sqrt{A}$; when $(q_n, -p_n, q_nA)$ is of the principal class of improperly primitive forms of determinant $-\Delta$, $2A$ is of the form $2X^2 + 2XY + \frac{\Delta+1}{2}Y^2$, and Y is the denominator of a complete quotient in the development of $\sqrt{A}$."

Similar results are noticed when $(q_n, -p_n, q_nA)$ is ambiguous and properly primitive, but not of the principal class; or ambiguous and improperly primitive, but not of the principal class of improperly primitive forms.*

In the case of n even, corresponding expressions are also given, and, in conclusion, some account is furnished of the speciality of the middle denominator of a period, particularly when it is 1 or 2.

3. The middle denominator in the periods of the development of $\sqrt{A}$ as a continued fraction has naturally some characteristics deserving of attention.

We may consider the set of denominators, commencing with the first denominator unity, and ending with the denominator immediately preceding the next denominator whose value is unity, and whose place is odd, as constituting the first period. Thus considered, the period has a middle term. When this term is odd, it is a factor of A; and when it is even, its half is a factor of A.

If, therefore, (t, u) is the least solution of $x^2 - Ay^2 = 1$, we shall have (except as hereinafter mentioned), for some values t', u' less than

* In this connection should be mentioned Goepel's theorem $\Delta > -\delta_s\delta_{s+1}$, δ_{s+1} being negative (p. 784, *loc. cit.*). The usefulness of this is easy to see, especially when Δ is small.

t, u, either $$Mt'^2 - Nu'^2 = \pm 1, \quad MN = A \;\ldots\ldots\ldots\ldots\ldots\ldots\; (a),$$
or $$Mt'^2 - Nu'^2 = \pm 2, \quad MN = A \;\ldots\ldots\ldots\ldots\ldots\ldots\; (b).$$
If A is the double of an odd number, so that $N = 2K$, both forms of equation hold good, and we have, from (b) for instance,

$$2Mt'^2 - Ku'^2 = \pm 1.$$

This is the case also whenever $A = MN$ is even, and $(M, 0, -N)$ is properly primitive and gives (b), or when (b) is of the form

$$2^m Pt'^2 - 2Qu'^2 = \pm 2,$$

and P, Q are odd, and t' is even.

It is necessary, on account of our original assumption, to except the use of $M = 1$, $t'^2 - Nu'^2 = 1$, $N = A$.

If $M = 1$ and $t'^2 - Au'^2 = -1$, the period of denominators possesses a double symmetry, and consists of two subordinate periods without middle terms.

4. After showing that the middle denominator, or the half of it, is a factor of A, and thus establishing that any number not a square can be separated into two factors M, N, so that

$$Mx^2 - Ny^2 = \pm 1 \quad \text{or} \quad Mx^2 - Ny^2 = \pm 2,$$

Legendre remarks as follows:—*

"Il faut d'ailleurs observer—1°, que pour un même nombre $A = MN$, il n'y aura jamais q'une manière de satisfaire à l'une de ces équations; car il n'y a q'un quotient moyen qui résulte du développement de $\sqrt{A}$ en fraction continue."

The above reasoning seems to require at least some expansion. A denominator may be a factor of A, and yet not a middle denominator corresponding to a middle quotient; viz., it may be the factor of A conjugate to the middle denominator, *e.g.*, in the case of $\sqrt{45}$. Or, again, the middle denominator may occur again in the period, and therefore not as a middle term, as in the case of $\sqrt{153}$. The conclusion that M or N, or the double of M or N, must be a middle denominator, is therefore not obvious, and requires further consideration.

5. I shall now show that, if the equation (a) is resoluble, and M is $< N$, *i.e.*, $< \sqrt{A}$, M is the middle denominator of the period arising from the development of $\sqrt{A}$ as a continued fraction.

In fact, the form $(M, 0, -N)$ or else $(N, 0, -M)$ is of the principal class, and the continued fraction representing $\frac{\sqrt{A}}{M}$ has the same denominator periods as that representing $\sqrt{A}$, beginning however with the

* Théorie des Nombres, Tome I., p. 69.

denominator M, which may be regarded as a middle term of denominator periods, and is consequently the middle term of the denominator period of $\sqrt{A}$. The middle term, then, if it is odd, or if it is even, and A is the double of an odd number; or if A is even, and $(M, 0, -N)$ in (b) is properly primitive, or if (b) is of the form

$$2^m P t'^2 - 2Qu'^2 = \pm 2,$$

and P, Q are odd and t' is even, is $< \sqrt{A}$.

6. When (b) holds, but not (a), the middle denominator belonging to $\sqrt{A}$ is $2M$, M being $< N$. For, if we develope $\frac{\sqrt{A}}{M}$ as a continued fraction, and $\frac{p_{s-2}}{q_{s-2}}, \frac{p_{s-1}}{q_{s-1}}$ are consecutive convergents, with $\frac{\sqrt{A}+J_s}{D_s}$ for the next complete quotient, we have

$$\frac{\sqrt{A}}{M} = \frac{p_{s-1}(\sqrt{A}+J_s)+p_{s-2}D_s}{q_{s-1}(\sqrt{A}+J_s)+q_{s-2}D_s},$$

and thence

$$q_{s-1}J_s + q_{s-2}D_s = Mp_{s-1},$$
$$p_{s-1}J_s + p_{s-2}D_s = Nq_{s-1}.$$

When $D_s = 2$,

$$J_s + 2\frac{q_{s-2}}{q_{s-1}} = M\frac{p_{s-1}}{q_{s-1}}.$$

Let a be the greatest integer in $\sqrt{A}$. Since

$$(Mp_{s-1})^2 - Aq_{s-1}^2 = \pm 2M, \quad M\frac{p_{s-1}}{q_{s-1}} \text{ is } \overline{\overline{>}}\, a \,\overline{\overline{<}}\, a+1.$$

For, if we take the upper sign, $\frac{Mp_{s-1}}{q_{s-1}}$ is evidently $> a \,\overline{\overline{<}}\, a+1$. If the lower sign be taken, we have

$$(Mp_{s-1})^2 = Aq_{s-1}^2 - 2M \,\overline{\overline{>}}\, Aq_{s-1}^2 - 2Mp_{s-1},$$

which is positive for $A > 1$, since $2Mp_{s-1}$ is $< 2\sqrt{Aq_{s-1}^2}$.

Therefore

$$(Mp_{s-1}+1)^2 > Aq_{s-1}^2,$$
$$Mp_{s-1} < \sqrt{Aq_{s-1}^2},$$

so that

$$Mp_{s-1} \,\overline{\overline{>}}\, aq_{s-1} < (a+1)\, q_{s-1}.$$

Also $\frac{2q_{s-2}}{q_{s-1}}$ is < 2, and consequently, if A and a are of the same species, $J_s = a$, and if they are of different species $J_s = a-1$, because $A - J_s^2$ is even.

In developing $\frac{\sqrt{A}}{M}$, we shall arrive, then, either at a complete quotient $\frac{\sqrt{A}+a}{2}$, or a complete quotient $\frac{\sqrt{A}+a-1}{2}$, and the corresponding partial quotients will be a or $a-1$. There is, therefore, only one form of equation (b), and $2M$ is the middle denominator belonging to $\sqrt{A}$ and may be $>\sqrt{A}$; *e.g.*, in the case of $\sqrt{91}$, and generally when $2M$ is $>N$ and $N>M$.

If M and N are both even, so that (b) is of the form

$$2^m Px^2 - 2Qy^2 = \pm 2,$$

where P and Q are odd and x^2 is also odd, $m>1$, the middle denominator will be the least of the numbers $2^{m-1}P$ and 2^2Q. If x is even, and $Q<2^{m+1}P$, Q will be the middle denominator, except that, if $Q=1$, it may hold an odd place.

The numbers 1 and 2 can only appear as middle denominators in the development of $\frac{\sqrt{A}}{M}$. In fact, if μ_s is the partial quotient corresponding to J_s, D_s, we have $\mu_s D_s = J_s + J_{s-1}$. Hence, if $D_s = 2$, $\mu_s = a$ or $a-1$, and $J_s = J_{s-1}$; D_s is therefore a middle term.

The case of 1 as a denominator is similar.

7. If the number A is an odd power of an odd prime, or the double of an odd or even power of such a prime, its middle denominator is 1 or 2.* In particular, if A is an odd power of a prime of the form $4m+1$, or the double of an odd power of a prime of the form $8m+5$, the middle denominator is unity.

The last conclusion follows from the consideration that, if P is a prime of the form $8m+5$, we must have either

$$x^2-2Py^2=-1 \quad \text{or} \quad x^2-2Py^2=\pm 2 \;\ldots\ldots\ldots\ldots\; (1, 2),$$

or

$$2x^2-Py^2=\pm 1 \quad \text{or} \quad 2x^2-Py^2=\pm 2 \;\ldots\ldots\ldots\ldots\; (3, 4).$$

But (2) and (3) are equivalent, since, in (2), x is even. Also (4) gives y even, so that $x^2-2Py'^2=\pm 1$, and the upper sign is excluded because (x, y) may be taken less than the least solution of $x_1^2-2Py_1^2=1$. Hence we have the alternatives $x^2-2Py^2=-1$ or $2x^2-Py^2=\pm 1$. The latter form is found to be impossible by exclusion of multiples of 8, and the former equation must hold.

If A is an odd power of a prime of the form $4m+3$, or the double of such a number, the middle denominator is 2. This is also the case when A is the double of an even power of such a prime.

In the cases of the double of an odd or even power of a prime of the form $8m+1$, and of the double of an even power of a prime of the form $8m+5$, the middle denominator may be 1 or 2.

8. If $Mx^2-Ny^2=\pm 1$ is resoluble, we may write

$$(\sqrt{M}.x-\sqrt{N}.y)^{2m+1} = \sqrt{M}.X-\sqrt{N}.Y,$$

$$MX^2-NY^2=\pm 1.$$

* This follows immediately from Legendre's theorem mentioned in § 4.

Hence, if $2m+1$ contains any odd factor of M or N, X or Y will contain that factor. It follows that, if we augment the index of any power of an odd prime contained in M or N as a factor by an even number, the equation remains resoluble.

In like manner, if the equation

$$Mx^2-Ny^2=\pm 2$$

is resoluble, we may write

$$\left(\sqrt{\frac{M}{2}}\cdot x-\sqrt{\frac{N}{2}}\cdot y\right)^{2m+1}=\sqrt{\frac{M}{2}}\cdot X-\sqrt{\frac{N}{2}}\,Y,$$

$$MX^2-NY^2=\pm 2.$$

And, therefore, if we augment the index of a power of an odd prime contained as a factor in M or N by an even number, the equation remains resoluble.

Hence, if M or $2M$ is the middle denominator in $\sqrt{A}$, and we augment the index of any odd prime factor of N by an even number, the middle denominator remains the same. But if we do the same thing with regard to factors of M, so that the augmented number becomes $>N$, the middle denominator will be N or $2N$. And generally, if we so augment the factors of both M and N, the middle denominator will be the least of the numbers so augmented, or the double of it. Thus, if P, P' are primes of the form $4m+3$, the development of $\sqrt{P^{2p+1}.P'^{2p'+1}}$ has the least of the numbers P^{2p+1}, $P'^{2p'+1}$ for the middle denominator. Again, $51=7^2+2$ gives $3^{2a+1}.17^{2b+1}=$ square and double of a square determined by continued fractions.

9. The identity

$$(au\alpha+b\,(u\beta+v\alpha)+cv\beta)^2-(b^2-ac)\,(u\beta-v\alpha)^2$$
$$=(au^2+2buv+cv^2)\,(a\alpha^2+2b\alpha\beta+c\beta^2)$$

gives
$$X^2-\frac{au^2+2buv+cv^2}{a\alpha^2+2b\alpha\beta+c\beta^2}\,Y^2=-\Delta,$$

or
$$X^2-AY^2=-\Delta,$$

when we write X for $au\alpha+b\,(u\beta+v\alpha)+cv\beta$, Y for $a\alpha^2+2b\alpha\beta+c\beta^2$ Δ for $ac-b^2$, and assume

$$u\beta-v\alpha=\pm 1.$$

We may use, instead of the form (a, b, c), any equivalent form. Perhaps the most interesting case is when (a, b, c) belongs to the principal class. The equation may then be written

$$X^2-\frac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2}\,Y^2=-\Delta,$$

$$X=u\alpha+\Delta v\beta,\quad Y=\alpha^2+\Delta\beta^2,\quad u\beta-v\alpha=\pm 1,$$

and the resulting decomposition will be binomial.

In fact, the form $(q_n, -p_n, q_n A)$ is here identical with

$$[a^2+\Delta\beta^2, \quad -(ua+\Delta v\beta), \quad u^2+\Delta v^2],$$

and is transformed into $(1, 0, \Delta)$ by $\begin{pmatrix} \pm v, & u \\ \pm\beta, & a \end{pmatrix}$.* It belongs, then, to the principal class. We shall, therefore, arrive at a representation of A of the form $P^2+\Delta Q^2$.

For simplicity's sake, and on account of the special interest of the case, I take Δ positive.

10. If $p^2+\Delta q^2$ be a factor of Y, so that

$$A = \frac{u^2+\Delta v^2}{(a^2+\Delta\beta^2)(p^2+\Delta q^2)} = \frac{u^2+\Delta v^2}{(ap+\Delta\beta q)^2+\Delta(aq-\beta p)^2}$$

with the condition

$$u(aq-\beta p)-v(ap+\Delta\beta q) = \pm 1,$$

then $(p^2+\Delta q^2)^2 A = \dfrac{(u^2+\Delta v^2)(p^2+\Delta q^2)}{a^2+\Delta\beta^2} = \dfrac{(up+\Delta vq)^2+\Delta(uq-vp)^2}{a^2+\Delta\beta^2}$,

and the condition above written is

$$(up+\Delta vq)\beta-(uq-vp)a = \mp 1,$$

so that $(p^2+\Delta q^2)^2 A$ is such a number that a similar decomposition of it will be obtained by developing its square root as a continued fraction.

We may therefore multiply A by the square of any factor of Y having the form $p^2+\Delta q^2$, Y being any integer value giving

$$X^2-AY^2 = -\Delta.$$

11. We have a condition in order that $\dfrac{u^2+\Delta v^2}{a^2+\Delta\beta^2}$ may be an integer.

Let $u = \epsilon+az$, $v = \eta+\beta z$, where ϵ, η may or may not be $< a, \beta$ respectively.

Then $$A = z^2+\frac{2z(\epsilon a+\Delta\eta\beta)+\epsilon^2+\Delta\eta^2}{a^2+\Delta\beta^2},$$

and we must solve in integers

$$2z(\epsilon a+\Delta\eta\beta)-\zeta(a^2+\Delta\beta^2) = -(\epsilon^2+\Delta\eta^2);$$

but $$(\epsilon a+\Delta\eta\beta)^2-(\epsilon^2+\Delta\eta^2)(a^2+\Delta\beta^2) = -\Delta,$$

and consequently

$$2\Delta z = (\epsilon a+\Delta\eta\beta)(\epsilon^2+\Delta\eta^2)+m(a^2+\Delta\beta^2),$$
$$\Delta\zeta = (\epsilon^2+\Delta\eta^2)^2+m(\epsilon a+\Delta\eta\beta).$$

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* From Problem III., Article 2, of the Additions to Euler's Algebra, it appears that if $Ap^2-2Bpq+Cq^2$ is a minimum, $\frac{p}{q}$ is convergent to $\frac{B}{A}$. Hence $\frac{v}{\beta}$ is a convergent to $\frac{p_n}{q_n}$ in this case.

In order that these may be divisible by Δ, we must have further

$$m\alpha+\epsilon^3 \equiv 0 \pmod{\Delta}.$$

The condition $u\beta-v\alpha=\pm 1$ implies that $\alpha^2+\Delta\beta^2$ is odd, for if A is integer and $\alpha^2+\Delta\beta^2$ is even, $u^2+\Delta v^2$ must also be even, and $u\beta-v\alpha=\pm 1$ cannot hold.

If Δ is odd, m is even; if Δ is even and ϵ is odd, m is odd; if Δ is even and ϵ is even, m is even.

We now have the means of forming as many numbers as we please to illustrate the foregoing results. It will be as well, however, to give the values of P and Q belonging to the representation which will be given by the continued fraction.

The equation $$P^2+\Delta Q^2 = \frac{u^2+\Delta v^2}{\alpha^2+\Delta\beta^2}$$

gives $$(P+i\sqrt{\Delta}.\,Q)\,(\alpha-i\sqrt{A}.\,\beta) = u+i\sqrt{\Delta}.\,v,$$

and thence $$P\alpha+\Delta\beta Q = u, \quad Q\alpha-P\beta = v,$$

or $$P = \frac{\alpha u-\Delta\beta v}{\alpha^2+\Delta\beta^2}, \qquad Q = \frac{\beta u+\alpha v}{\alpha^2+\Delta\beta^2}$$
$$= \mp(uv-A\alpha\beta), \quad = \mp(v^2-A\beta^2).$$

If we substitute $\epsilon+\alpha z$ for u, and $\eta+\beta z$ for v, and make the expression homogeneous by means of $(\epsilon\beta-\eta\alpha)^2=1$, we get

$$2\Delta P = m\,(\alpha^2-\Delta\beta^2)+3\epsilon\eta\Delta\,(\alpha\eta-\epsilon\beta)+\epsilon^3\alpha-\Delta^2\eta^3\beta,$$
$$\Delta Q = m\alpha\beta+\epsilon^3\beta+\Delta\eta^3\alpha.$$

As an example, we may take one of the simplest cases. Let $\epsilon=0$, $\eta=\alpha=1$, $\beta=2\beta_1$, $m=2p\Delta$; then

$$A = \{p\,(1-4\Delta\beta_1^2)-\Delta\beta_1\}^2+\Delta\,\{4p\beta_1+1\}^2.$$

If $p=3$, $\beta_1=2$, $\Delta=3$, we have

$$A = 147^2+3\,.\,25^2 = 23484.$$

Developing $\sqrt{A}$, we have accordingly

Partial Quotient.	J.	Denominator.
153,	153,	75,
4,	147,	25,
12,	153,	3,
&c.	&c.	

Again, let $\epsilon=5$, $\alpha=6$, $\eta=\beta=1$, $\Delta=5$, $m=10p$;

then $$A = (41p+105)^2+70p+180.$$

The development is

Partial Quotient.	*J*.	Denominator.
$41p+105$,	$41p+105$,	$70p+180$,
1,	$29p+\ 75$,	$12p+\ 31$,
5,	$31p+\ 80$,	$60p+155$,
1,	$29p+\ 75$,	$14p+\ 36$,
5,	$41p+105$,	5,
	&c.	&c.,

and $$A = (31p+80)^2+5\,(12p+31)^2.$$

13. Let us suppose that Δ is the middle denominator belonging to $\sqrt{\Delta\,.\,A}$, we then have $\Delta X^2 - \dfrac{u^2+\Delta v^2}{a^2+\Delta\beta^2}\,Y^2 = -1$,

with the conditions $ua - \Delta v\beta = \pm 1$, $X = u\beta + va$, $Y = a^2+\Delta\beta^2$.

Proceeding as before, we have, in order that $A = \dfrac{u^2+\Delta v^2}{a^2+\Delta\beta^2}$ may be an integer,

$$2z = (\eta a+\epsilon\beta)\,(\epsilon^2+\Delta\eta^2)+m\,(a^2+\Delta\beta^2),$$
$$\zeta = (\epsilon^2+\Delta\eta^2)^2+m\Delta\,(\eta a+\epsilon\beta),$$
$$A = \Delta z^2+\zeta,\quad u = \epsilon+\Delta\beta z,\quad v = \eta+az.$$

If Δ is odd, m must be even. For, $a^2+\Delta\beta^2$ being odd, a or β is odd and the other even. If a is odd and β even, ϵ is odd, and η odd or even. If η is odd, $\epsilon^2+\Delta\eta^2$ is even; and if η is even, $\eta a+\epsilon\beta$ is even. The reasoning is similar if a is even and β odd.

If Δ is even, a will be odd and ϵ odd, and m will be even or odd according as $\eta a+\beta\epsilon$ is even or odd, or according as η, β are or are not of the same species.

From $\Delta A = P^2+\Delta Q^2$, we have

$$(P+i\sqrt{\Delta}Q)\,(a-i\sqrt{\Delta}\beta) = \Delta v+i\sqrt{\Delta}\,u,$$
$$P = \Delta\,\frac{va-u\beta}{a^2+\Delta\beta^2},\qquad Q = \frac{ua+\Delta v\beta}{a^2+\Delta\beta^2},$$
$$= \pm\Delta\,(uv-Aa\beta),\qquad = \pm(u^2-\Delta A\beta^2);$$

and, writing $\epsilon+\Delta\beta z$ for u, $\eta+az$ for v, and making the expression homogeneous by $(\epsilon a-\Delta\eta\beta)^2 = 1$, we get

$$2P = \Delta\,\{m\,(a^2-\Delta\beta^2)+3\epsilon\eta\,(\epsilon a-\Delta\eta\beta)-\epsilon^3\beta+\Delta\eta^3 a\},$$
$$Q = \Delta m a\beta+\epsilon^3 a+\Delta^2\eta^3\beta.$$

14. We have still to consider the case when the middle denominator is even and equation (*b*) holds.

Let $\Delta = 2\Pi$, and the equation to be treated is

$$\Pi\,(au+2\beta v)^2 - \frac{2v^2+\Pi u^2}{a^2+2\Pi\beta^2}\,(a^2+2\Pi\beta^2)^2 = -2,$$

or
$$\Pi X^2 - \frac{2v^2+\Pi u^2}{\alpha^2+2\Pi\beta^2} Y^2 = -2,$$
with
$$\alpha v - \Pi\beta u = \pm 1.$$

In the same manner as before, we get

$$4z = (2\epsilon\beta+\eta\alpha)(2\epsilon^2+\Pi\eta^2)+m(\alpha^2+2\Pi\beta^2),$$
$$2\zeta = (2\epsilon^2+\Pi\eta^2)^2+m\Pi(2\epsilon\beta+\eta\alpha),$$
$$A = \Pi z^2+\zeta, \quad \epsilon\alpha-\Pi\beta\eta = \pm 1.$$

And if $\Pi A = P^2+2\Pi Q^2$ is the decomposition in question, we have also, in the same manner as before,

$$4P = \Pi\{m(\alpha^2-2\Pi\beta^2)+6\epsilon\eta(\alpha\epsilon-\Pi\eta\beta)+\Pi\eta^3\alpha-4\epsilon^3\beta\}$$
$$2Q = \Pi m\alpha\beta+2\epsilon^3\alpha+\Pi^2\eta^3\beta,$$

subject to the minor conditions necessary that A, P, and Q may be integers.

As examples of the last two cases, (1) let $\alpha=4$, $\epsilon=\eta=\beta=1$, $\Delta=5$, $m=2n$, so that

$$z = 21n+15, \quad \zeta = 50n+36, \quad 5A = (105n+75)^2+250n+180.$$

The development of $\sqrt{5A}$ is

Partial Quotient.	J.	Denominator.
$105n+76$,	$105n+76$,	$40n+29$,
5,	$95n+69$,	$50n+36$,
4,	$105n+75$,	5,
$42n+30$,	$105n+75$,	$50n+36$,
&c.	&c.,	

$$5A = (55n+40)^2+5(40n+29)^2,$$
$$95n+69-(40n+29) = 55n+40;$$

(2) let $\Pi=3$, $\epsilon=2$, $\alpha=\beta=\eta=1$, $m=3$, giving

$$z = 19, \quad \zeta = 83, \quad A = 19^2+83, \quad 3A = 3498.$$

The development of $\sqrt{3A}$ is

Partial Quotient.	J.	Denominator.
59,	59,	17,
6,	43,	97,
1,	54,	6,
18,	54,	97,
&c.	&c.,	

and
$$3A = 42^2+6\,.\,17^2.$$

15. Since, if
$$A = \frac{u^2+\Delta v^2}{(\alpha^2+\Delta\beta^2)(p^2+\Delta q^2)} = \frac{u^2+\Delta v^2}{(\alpha p-\Delta\beta q)^2+\Delta(\alpha q+\beta p)^2}$$

complies with the condition

$$u\,(ap-\Delta\beta q)-\Delta v\,(aq+\beta p)=\pm 1,$$

then $$A'=\frac{(u^2+\Delta v^2)\,(p^2+\Delta q^2)}{a^2+\Delta\beta^2}=\frac{(up-\Delta vq)^2+\Delta\,(uq+vp)^2}{a^2+\Delta\beta^2}$$

complies with the condition

$$(up-\Delta qv)\,a-\Delta\,(uq+vp)\,\beta=\pm 1\,;$$

it follows that, if $p^2+\Delta q^2$ is a factor of Y in some solution of $\Delta X^2-AY^2=\pm 1$, we may multiply A by an even power of that factor, and a similar decomposition can be effected of the augmented number. In the case of § 13, it appears, by § 8, that any odd factor of A may be constituted a factor of Y.

Hence, when Δ is the middle term, we may augment the index of any odd factor of A of the form $p^2+\Delta q^2$ by an even number. For example, $\sqrt{39}$ has 3 for the middle denominator, and satisfies $3x^2-13y^2=-1$. The continued fraction gives the decomposition $6^2+3\,.\,1^2$. The factor 13 is $3\,.\,2^2+1$ of the same form. Hence the continued fraction for $\sqrt{3\,.\,13^3}$ or $\sqrt{6591}$ gives a similar decomposition, namely, $54^2+3\,.\,35^2$.

16. Like considerations apply in the case of § 14; for if

$$A=\frac{2v^2+\Pi u^2}{(a^2+2\Pi\beta^2)(p^2+2\Pi q^2)}=\frac{2v^2+\Pi u^2}{(ap-2\Pi q\beta)^2+2\Pi\,(aq+\beta p)^2},$$

with $$v\,(ap-2\Pi q\beta)-\Pi\,(aq+\beta p)\,u=\pm 1\,;$$

then $$A'=\frac{(2v^2+\Pi u^2)(p^2+2\Pi q^2)}{a^2+2\Pi\beta^2}=\frac{\Pi\,(up+2vq)^2+2\,(vp-\Pi uq)^2}{a^2+2\Pi\beta^2},$$

with $$(vp-\Pi uq)\,a-\Pi\,(up+2vq)\,\beta=\pm 1.$$

Hence, if we augment the index of any odd factor of A having the form $p^2+2\Pi q^2$ by an even number, the number so changed, A', multiplied by Π, can be decomposed as before by taking the continued fraction for $\sqrt{\Pi A'}$.

17. When, as in the preceding articles, we take Δ positive, the decomposition into the form $P^2+\Delta Q^2$ can be made in a finite number of ways, only one of which is given by the process indicated. Similar considerations, however, to those by which other decompositions of the form P^2+Q^2 were arrived at, are applicable to the more general form ("Proceedings," Vol. ix., p. 187 *et seq.*). Having already occupied as much space as the subject probably deserves, I abstain from details which in great part would be a repetition of those given for the more limited case. It will be observed that the equation (§ 9) from which we set out by no means requires that the coefficient of Y^2 should be an integer in the first instance.

Following out this remark, we can see that another similar decomposition will be obtained by dividing the square root of A by a factor of the same form and $< \sqrt{A}$.

An example will be sufficient to explain the process.

Suppose we take $A = 22116 = 2^2 . 3 . 19 . 97$;

then $\frac{\sqrt{A}}{1}$ gives $21^2+3 . 85^2$, $\frac{\sqrt{A}}{4}$ gives $117^2+3 . 53^2$, $\frac{\sqrt{A}}{19}$ gives $93^2+3 . 67^2$, $\frac{\sqrt{A}}{97}$ gives $147^2+3 . 13^2$.

If we take factors which are multiples of 3, the same decompositions are obtained. So also, if we take $4 . 19$, the conjugate factor of 97 in $\frac{A}{3}$, we get again the decomposition given by $\frac{\sqrt{A}}{97}$.

18. We might now take n even, so that $(-)^n D = \Delta$, with much the same result; the decomposition would then be of the form $P^2 - \Delta Q^2$. The number of such decompositions is infinite when one is possible. I omit the discussion of this case, and confine myself to a very brief account of the forms corresponding to the second part of the second theorem of § 2.

The form (a, b, c) may be supposed to be of the principal class of improperly primitive forms. We then have

$$X^2 - \frac{2u^2+2uv+\frac{\Delta+1}{2}v^2}{2a^2+2a\beta+\frac{\Delta+1}{2}\beta^2} . Y^2 = -\Delta,$$

with $$u\beta - va = \pm 1.$$

And, writing $\frac{U}{V}$ for the coefficient of Y^2, we find that the form $(V, -X, U)$ is of the principal class of improperly primitive forms, Δ being of the form $4m-1$.

To make the coefficient of Y^2 integral, put $u = \epsilon + az$, $v = \eta + \beta z$; then

$$2\Delta z = \left(2\epsilon a + \epsilon\beta + \eta a + \frac{\Delta+1}{2}\eta\beta\right)\left(2\epsilon^2 + 2\epsilon\eta + \frac{\Delta+1}{2}\eta^2\right) + m\left(2a^2 + 2a\beta + \frac{\Delta+1}{2}\beta^2\right),$$

$$\Delta\zeta = \left(2\epsilon^2 + 2\epsilon\eta + \frac{\Delta+1}{2}\eta^2\right)^2 + m\left(2\epsilon a + \epsilon\beta + \eta a + \frac{\Delta+1}{2}\eta\beta\right),$$

$$A = z^2 + \zeta.$$

The condition follows

$$(2\epsilon + \eta)^2 + 2m(2a+\beta) \equiv 0 \pmod{4\Delta}.$$

If Δ is the middle denominator belonging to $\sqrt{A}$, the equation is

$$\Delta X^2 - \frac{2u^2+2uv+\frac{\Delta+1}{2}v^2}{2\alpha^2-2\alpha\beta+\frac{\Delta+1}{2}\beta^2} Y^2 = -1,$$

$$X = u\beta+v\alpha,\quad Y = 2\alpha^2-2\alpha\beta+\frac{\Delta+1}{2}\beta^2,\quad 2u\alpha+v\alpha-u\beta-\frac{\Delta+1}{2}v\beta = \pm 1,$$

the form

$$\left\{\left(2\alpha^2-2\alpha\beta+\frac{\Delta+1}{2}\beta^2,\quad -\Delta(u\beta+v\alpha),\quad \Delta\left(2u^2+2uv+\frac{\Delta+1}{2}v^2\right)\right\}\right.,$$

reduces to $\left(2,\ 1,\ \frac{\Delta+1}{2}\right)$ by the substitution

$$\begin{pmatrix} 2u+v, & u+\frac{\Delta+1}{2}v \\ \beta, & \alpha \end{pmatrix} = \pm 1,$$

and $2z = (\epsilon\beta+\eta\alpha)\left(2\epsilon^2+2\epsilon\eta+\frac{\Delta+1}{2}\eta^2\right)+m\left(2\alpha^2-2\alpha\beta+\frac{\Delta+1}{2}\beta^2\right),$

$$\zeta = \left(2\epsilon^2+2\epsilon\eta+\frac{\Delta+1}{2}\eta^2\right)^2+m\Delta(\epsilon\beta+\eta\alpha),$$

$$A = \Delta z^2+\zeta,\quad 2\epsilon\alpha-\epsilon\beta+\eta\alpha-\frac{\Delta+1}{2}\eta\beta = \pm 1.$$

Then, if $\eta = 0$, $\alpha = \beta = \epsilon = 1$, $\Delta = 7$, $z = 2m+1$, $\zeta = 7m+4$,

we have $$7A = 7^2(2m+1)^2+7^2m+28;$$

and the development of $\sqrt{7A}$ gives

Partial Quotient.	*J.*	Denominator.
$14m+8$,	$14m+8$,	$21m+13$,
1,	$7m+5$,	$7m+4$,
3,	$14m+7$,	7,
$4m+2$,	$14m+7$,	$7m+4$,

and $$7A = (7m+5)^2+(7m+5)(7m+4)+2(7m+4)^2.$$

It will be observed that, if we take $m = -1$, the decomposition is still affected, but the middle denominator is 4, not 7. In fact, in this case the coefficient of Y^2 is < 7, and the middle denominator is changed to 4, the smaller factor. This was to be expected by § 5, but otherwise we may take m negative. When the middle term is even, we can obtain corresponding expressions.

January 9th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Dr. J. Hopkinson, F.R.S., was admitted into the Society.

Letters from the Auditor, Prof. W. G. Adams, F.R.S., and from the Chief Accountant of the Bank of England, were read; and as the Treasurer's accounts were found to be correct, it was agreed that his Report should be adopted.

Prof. Cayley read a paper "On a Theorem in Elliptic Functions." Prof. Smith spoke upon a correction in Sohncke's Tables, and also upon a Modular Equation. Prof. Greenhill communicated a paper "On Coefficients of Induction and Capacity of two Electrified Spheres." Mr. Tucker read an abstract of a paper, by Prof. Lloyd Tanner, "On certain systems of Partial Differential Equations of the First Order with several dependent variables."

The following presents were received:—

"Memoire sur l'approximation des fonctions de tres-grands nombres, et sur une classe étendue de développements en série," par M. G. Darboux (from "Journal de Mathématiques pures et appliquées," Tome iv., 1878, pp. 1–56; Deuxième Partie, Tome iv., 1878, pp. 377–416).

"Educational Times," January, 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome ii., Août, Sept., 1878 ; Paris.

"Bulletin des Sciences Mathématiques et Astronomiques : Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré," par M. G. Darboux.

"Proceedings of the Royal Society," Vol. xxviii., No. 190.

"Annali di Matematica," Serie 11ª, Tome ix., Fasc. 2, Nov. 1878; Milano.

"American Journal of Mathematics Pure and Applied," No. iv., Vol. i.; Baltimore.

"Atti della R. Accademia dei Lincei anno cclxxvi.," 1878–9, Serie terza. "Transunti," Vol. iii., Fasc i., Dic. 1878; Vol. iii., Fasc. 2°, Gennajo 1879; Roma, 1879.

Crelle's "Journal," 86 Band, drittes Heft; Berlin, 1878.

"Monatsbericht," Sept., Oct., 1879; Berlin, 1879.

"Tidsskrift for Mathematik," Fjerde Række, Anden Aargang, Første, Andet, Tredie, Fjerde, Femte, Sjette Hefte, udgifet af H. B. Zeuthen.

"Ueber die Erniedrigung der Modular-Gleichungen," von F. Klein, in München; and

"Ueber die Transformation siebenter Ordnung der elliptischen Funktionen," von demselben ("Math. Annalen," Bd. xiv.)

"Beiblätter zu den Annalen der Physik und Chemie," edited by G. and E. Wiedemann, Band iii., Stück i.; Leipzig, 1879.

"Analisi Matematica—sulle equazioni modulari," nota del S. C. F. Klein (read before R. Istituto Lombardo, 2 Gennajo, 1879).

A Theorem in Elliptic Functions. By Prof. Cayley.

[*Read January 8th*, 1879.]

The theorem is as follows: If $u+v+r+s=0$, then

$$\begin{aligned} &-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s \\ &+ \quad \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \\ &-\frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s \\ =&-\frac{k'^2}{k^2}. \end{aligned}$$

It is easy to see that, if a linear relation exists between the three products, then it must be this relation: for the relation must be satisfied on writing therein $v=-u$, $s=-r$, and the only linear relation connecting $\operatorname{sn}^2 u \operatorname{sn}^2 r$, $\operatorname{cn}^2 u \operatorname{cn}^2 r$, $\operatorname{dn}^2 u \operatorname{dn}^2 r$ is the relation in question

$$\begin{aligned} &-k'^2 \operatorname{sn}^2 u \operatorname{sn}^2 r \\ &+ \quad \operatorname{cn}^2 u \operatorname{cn}^2 r \\ &-\frac{1}{k^2} \operatorname{dn}^2 u \operatorname{dn}^2 r \\ =&-\frac{k'^2}{k^2}. \end{aligned}$$

A demonstration of the theorem was recently communicated to me by Mr. Glaisher, and this led me to the somewhat more general theorem

$$\begin{aligned} &-k'^2 \operatorname{sn}(\alpha+\beta) \operatorname{sn}(\alpha-\beta) \operatorname{sn}(\gamma+\delta) \operatorname{sn}(\gamma-\delta) \\ &+ \quad \operatorname{cn}(\alpha+\beta) \operatorname{cn}(\alpha-\beta) \operatorname{cn}(\gamma+\delta) \operatorname{cn}(\gamma-\delta) \\ &-\frac{1}{k^2} \operatorname{dn}(\alpha+\beta) \operatorname{dn}(\alpha-\beta) \operatorname{dn}(\gamma+\delta) \operatorname{dn}(\gamma-\delta) \\ =&-\frac{k'^2}{k^2}-\frac{2k'^2(\operatorname{sn}^2\alpha-\operatorname{sn}^2\gamma)(\operatorname{sn}^2\beta-\operatorname{sn}^2\delta)}{1-k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta \, . \, 1-k^2\operatorname{sn}^2\gamma\operatorname{sn}^2\delta}. \end{aligned}$$

In fact, writing herein $\alpha+\gamma=0$, that is $\gamma=-\alpha$, the right-hand side becomes $=0$; and the arcs on the left-hand side are $\alpha+\beta$, $\alpha-\beta$,