Note on the Functions $Z(u)$, $\Theta(u)$, $\Pi(u, a)$.

By J. W. L. Glaisher, M.A., F.R.S.

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§ 1. The present note relates to the three fundamental formulæ

$$\text{(i.)}\ \mathrm{Z}(u)+\mathrm{Z}(v)-\mathrm{Z}(u+v) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v),$$

$$\text{(ii.)}\ \frac{\Theta(u+a)\,\Theta(u-a)\,\Theta^2(0)}{\Theta^2(u)\,\Theta^2(a)} = 1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a,$$

$$\text{(iii.)}\ \Pi(u, a) = a\mathrm{Z}(u)+\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}.$$

The first of these formulæ is in effect Legendre's addition-equation for the second elliptic integral. The second and third were given in the *Fundamenta Nova*, the second being deduced from the third in § 53, and the third being established in §§ 51, 52, by means of q-series. In § 53, Jacobi deduces the first formula from the third, but he had previously given, in § 49, an independent proof of the first formula by Elliptic Functions, the notation employed being however Legendrian, in order no doubt that the result might be obtained in Legendre's form.

The object of this note is to show how extremely simply the three formulæ may be established independently of each other by elementary Elliptic Functions, and to point out the close connexion existing between (i.) and (ii.).

§ 2. The definitions of the functions $\mathrm{Z}(u)$, $\Theta(u)$, $\Pi(u, a)$ are taken to be

$$\mathrm{Z}(u) = \int_0^u \operatorname{dn}^2 u\,du - \frac{E}{K}\,u,$$

$$\Theta(u) = e^{\int_0^u \mathrm{Z}(u)\,du},$$

$$\Pi(u, a) = \int^u \frac{k^2 \operatorname{sn}^2 u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a\,du}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a};$$

and I take as starting point the fundamental formula

$$\begin{aligned} & k^2 \operatorname{sn}^2(u-a) - k^2 \operatorname{sn}^2(u+a) \\ &= k^2 \operatorname{cn}^2(u+a) - k^2 \operatorname{cn}^2(u-a) \\ &= \quad \operatorname{dn}^2(u+a) - \operatorname{dn}^2(u-a) \\ &= -\frac{4k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{(1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a)^2} \quad \text{(A)}, \end{aligned}$$

which is derivable at sight from the ordinary addition formulæ, giving the sn, cn or dn of $u \pm a$ in terms of the sn's, cn's, and dn's of u and a.

§ 3. Integrating (A) with respect to a, we find

$$Z(u+a) + Z(u-a) = C - \frac{1}{\operatorname{sn}^2 u} \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \quad \text{(A}_1\text{)},$$

where C is the constant of integration, and is therefore independent of a.

1°. Putting $a = u$ in (A_1), we have

$$Z(2u) = C - \frac{1}{\operatorname{sn}^2 u} \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^4 u},$$

whence, by subtraction,

$$\begin{aligned} & Z(u+a) + Z(u-a) - Z(2u) \\ &= \frac{1}{\operatorname{sn}^2 u} \left\{ \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^4 u} - \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \right\} \\ &= \operatorname{sn} 2u \frac{k^2 (\operatorname{sn}^2 u - \operatorname{sn}^2 a)}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \\ &= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u \quad (\alpha). \end{aligned}$$

2°. Putting $a = 0$ in (A_1), we have

$$2Z(u) = C - \frac{1}{\operatorname{sn}^2 u} 2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u,$$

whence, by subtraction,

$$Z(u+a) + Z(u-a) - 2Z(u) = -\frac{2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \quad (\beta).$$

The equation (α) is equivalent to the addition-equation (i.), for, on replacing $u+a$ and $u-a$ by u and v, it becomes

$$Z(u)+Z(v)-Z(u+v) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v).$$

By integrating the equation (β) with respect to u between the limits u and 0, we find

$$\int_0^u Z(u+a)\,du+\int_0^u Z(u-a)\,du-2\int_0^u Z(u)\,du = \log(1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a).$$

Now
$$\int_0^u Z(u+a)\,du = \log\frac{\Theta(u+a)}{\Theta(a)},$$

$$\int_0^u Z(u-a)\,du = \log\frac{\Theta(u-a)}{\Theta(a)}.$$

The equation just obtained may therefore be written

$$\log\frac{\Theta(u+a)}{\Theta(a)}+\log\frac{\Theta(u-a)}{\Theta(a)}-2\log\frac{\Theta(u)}{\Theta(0)} = \log(1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a),$$

that is,
$$\frac{\Theta(u+a)\,\Theta(u-a)\,\Theta^2(0)}{\Theta^2(u)\,\Theta^2(a)} = 1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a \ldots\ldots\ldots\ldots(\beta_1),$$

which is the formula (ii.).

By integrating (β) with respect to a, instead of with respect to u, we find

$$\int_0^a Z(u+a)\,da+\int_0^a Z(u-a)\,da-2a\,Z(u)$$

$$= -2\int_0^a \frac{k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a\,da}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a},$$

that is,
$$\log\frac{\Theta(a+u)}{\Theta(a-u)}-2aZ(u) = -2\Pi(a, u),$$

or, on transposing u and a,

$$\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}+aZ(u) = \Pi(u, a)\ldots\ldots\ldots\ldots\ldots\ldots(\beta_2),$$

which is the formula (iii.).

Thus, starting with the identical equation (A), and integrating it with respect to a, we obtain the two forms (α) and (β) of the addition-equation; of these (α) is the same as (i.), and (β) gives rise to (ii.) by integration with respect to u, and to (iii.) by integration with respect to a. The three formulæ (i.), (ii.), (iii.) have therefore been derived independently from the elementary identity (A).

§ 4. In the preceding investigation, the results (α) and (β) were obtained by integrating (A) with respect to a, and determining the constant of integration by putting $a = u$ to obtain (α), and by putting $a = 0$ to obtain (β). We may however, if we please, derive (α) from (β), instead of deducing it independently from (A) by a separate determination of the constant; for, putting $u = a$ in (β), we

have
$$Z(2u) - 2Z(u) = -\frac{2k^2 \operatorname{sn}^3 u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^4 u},$$

whence, by subtraction,

$$Z(u+a) + Z(u-a) - Z(2u) = \frac{2k^2 \operatorname{sn}^3 u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^4 u} - \frac{2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a}$$
$$= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u.$$

If, therefore, starting with (A), we integrate it with respect to a, between the limits a and 0, thus obtaining (β), we may deduce therefrom all three formulæ (i.), (ii.), (iii.); viz., (i.) by putting $u = a$ and subtracting, (ii.) by integrating (β) with respect to u, and (iii.) by integrating (β) with respect to a.

§ 5. To deduce (ii.) from (i.), we may proceed as follows:—

Substituting $u+a$ and $u-a$ for u and v, (i.) becomes

$$Z(u+a) + Z(u-a) - Z(2u) = k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u;$$

whence, putting $a = 0$,

$$2Z(u) - Z(2u) = k^2 \operatorname{sn}^2 u \operatorname{sn} 2u,$$

and therefore, by subtraction,

$$Z(u+a) + Z(u-a) - 2Z(u) = k^2 \operatorname{sn} 2u \left\{\operatorname{sn}(u+a) \operatorname{sn}(u-a) - \operatorname{sn}^2 u\right\}$$
$$= -\frac{2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a},$$

from which (ii.) follows at once by integration with respect to a, as in § 3.

But, by the following process we may derive (ii.) even more directly from (i.) by integrating (i.) while it still remains in the form of an addition-equation. Writing (i.) in the form

$$Z(u+a) + Z(u-a) - Z(2u) = k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u,$$

we notice that the right-hand member of this equation

$$= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn}\{(u+a)+(u-a)\}$$

$$= k^2 \frac{\operatorname{sn}^2(u+a)\operatorname{sn}(u-a)\operatorname{cn}(u-a)\operatorname{dn}(u-a) + \operatorname{sn}^2(u-a)\operatorname{sn}(u+a)\operatorname{cn}(u+a)\operatorname{dn}(u+a)}{1-k^2\operatorname{sn}^2(u+a)\operatorname{sn}^2(u-a)}$$

$$= -\tfrac{1}{2}\frac{d}{du}\log\{1-k^2\operatorname{sn}^2(u+a)\operatorname{sn}^2(u-a)\}.$$

Integrating therefore the equation, as it stands, with respect to u between the limits u and 0, we find

$$\frac{\Theta^2(u+a)\,\Theta^2(u-a)\,\Theta(0)}{\Theta^4(a)\,\Theta(2u)} = \frac{1-k^2\operatorname{sn}^4 a}{1-k^2\operatorname{sn}^2(u+a)\operatorname{sn}^2(u-a)}.$$

Putting $a=0$, this equation becomes

$$\frac{\Theta^4(u)\,\Theta(0)}{\Theta^4(0)\,\Theta(2u)} = \frac{1}{1-k^2\operatorname{sn}^4 u},$$

whence, by division,

$$\frac{\Theta^2(u+a)\,\Theta^2(u-a)\,\Theta^4(0)}{\Theta^4(u)\,\Theta^4(a)} = \frac{(1-k^2\operatorname{sn}^4 u)(1-k^2\operatorname{sn}^4 a)}{1-k^2\operatorname{sn}^2(u+a)\operatorname{sn}^2(u-a)}$$

$$= (1-k^2\operatorname{sn}^2 u\operatorname{sn}^2 a)^2,$$

which is equivalent to the formula (ii.).

§ 6. So far as I know, the close connexion between the "addition-equation" (ii.) for the function Θ, and the addition-equation for the second elliptic integral, has not been specially remarked. In my lectures on Elliptic Functions, I have been in the habit of following Jacobi in proving (ii.) by means of the third elliptic integral. While working at formulæ connected with the Zeta Function, I recently noticed that (ii.) was derivable immediately from (i.) by integration, so that, in order to prove (ii.), it was unnecessary either to have recourse to the third elliptic integral or to use q-series for the Θ's; and this led me to remark how simply (i.), (ii.), and (iii.) may all be deduced from the identity (A), and how closely they are related. It will be observed that the method of proving (i.) in § 3 is practically the same as the method given by Jacobi, in Legendrian notation, in § 49 of the *Fundamenta Nova*.

§ 7. The quantity $k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v)$, which forms the right-hand member of the addition-equation

$$Z(u)+Z(v)-Z(u+v) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v),$$

does not, when so expressed, appear to possess any interesting or remarkable property. If, however, we transform the arguments by replacing u and v by $u+a$ and $u+b$ (so that the letter u occurs in each argument), we see that it is an expression which possesses the remarkable property of being an exact differential coefficient with respect to u;

$$\text{viz., it} \quad = k^2 \operatorname{sn}(u+a) \operatorname{sn}(u+b) \operatorname{sn}(2u+a+b)$$

$$= \frac{k^2 s_1 s_2 (s_1 c_2 d_2 + s_2 c_1 d_1)}{1 - k^2 s_1^2 s_2^2}$$

$$= -\tfrac{1}{2} \frac{d}{du} \log(1 - k^2 s_1^2 s_2^2),$$

where s_1, c_1, d_1, s_2, c_2, d_2 denote the sn, cn, dn of $u+a$ and $u-a$ respectively.

§ 8. The arguments $u+a$ and $u+b$ are not more general than $u+a$ and $u-a$, but it may be worth while perhaps to notice the result obtained by integrating the addition-equation in the form

$$Z(u+a) + Z(u+b) - Z(2u+a+b)$$
$$= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u+b) \operatorname{sn}(2u+a+b)$$

with respect to u.

The limits being u and 0, we thus find

$$\frac{\Theta^2(u+a)\ \Theta^2(u+b)\ \Theta(a+b)}{\Theta^2(a)\ \Theta^2(b)\ \Theta(2u+a+b)} = \frac{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 b}{1 - k^2 \operatorname{sn}^2(u+a) \operatorname{sn}^2(u+b)},$$

from which (ii.) follows at once by putting $u = -a$ or $u = -b$.

Replacing $u+a$ and $u+b$ by x and y, and finally writing $-a$ instead of u, this equation becomes

$$\frac{\Theta^2(x)\ \Theta^2(y)\ \Theta(x+y+2a)}{\Theta^2(x+a)\ \Theta^2(y+a)\ \Theta(x+y)} = \frac{1 - k^2 \operatorname{sn}^2(x+a) \operatorname{sn}^2(y+a)}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y},$$

which, making a slight further change of notation, may be written also in the form

$$\frac{\Theta^2(x+a)\ \Theta^2(y+a)\ \Theta(x+y-2a)}{\Theta^2(x-a)\ \Theta^2(y-a)\ \Theta(x+y+2a)} = \frac{1 - k^2 \operatorname{sn}^2(x-a) \operatorname{sn}^2(y-a)}{1 - k^2 \operatorname{sn}^2(x+a) \operatorname{sn}^2(y+a)}.$$

These two formulæ are therefore deducible by direct integration from the addition-equation for the second elliptic integral, subject only to changes in the letters by which the arguments are expressed.