where the area of integration includes the whole of the region through which z_1 varies.*

Finally it may be noticed that the preceding methods are applicable in two dimensions, when we replace x and y by any conjugate functions α, β of xy. By this transformation the scope of the analysis may be considerably increased, but to enlarge upon this would take us too far from the principal subject of the paper.

The Solution of Partial Differential Equations of the Second Order, with any number of variables, when there is a general first integral. By H. W. Lloyd Tanner, M.A.

[*Read 13th January*, 1876.]

In the first part of this paper it is proved that any equation of the second order which has a first integral

$$F\{u_1, \ldots\ldots u_n\} = 0,$$

consists of $\frac{1}{2} \cdot \frac{\lfloor 2n}{\{\lfloor n\}^2} + 2^{n-1}$ terms. One factor of each of these is the determinant $\left(s_{ij} \equiv \frac{d^2 z}{dx_i dx_j}\right)$

$$\begin{vmatrix} s_{11}, & s_{12} & \ldots\ldots & s_{1n} \\ s_{12}, & s_{22} & \ldots\ldots & \\ \ldots & \ldots & \ldots & \ldots \\ s_{1n} & \ldots & \ldots & s_{nn} \end{vmatrix}$$

or a minor of this determinant. The other factor is a function of the derivatives of $u_1 \ldots u_n$, whose form is specified for each term.

In the second part we form $\frac{\lfloor 2n}{\lfloor n+1 \; \lfloor n-1}$ linear equations of the first order which serve to determine $u_1 \ldots u_n$. Of these equations n, and only n, are independent, and their coefficients are expressed directly or indirectly in terms of the coefficients of the given equation of the second order. There is always a second set of equations corresponding to another first integral; but, except possibly in one case, there are not more than two first integrals. The general theory is then applied to the case of two variables, and the results agree with those given by Boole; but with the advantage that here there are no irrelevant equations to be removed, and there is no need to consider special cases of some of the coefficients vanishing.

In the third part we consider the theory of the second integration.

---

* Math. Tripos, 1876, Jan. 21, $1\frac{1}{2}$ to 4, Question x.

If there be only one first integral (as distinguished from two identical first integrals) we can only obtain as many particular solutions as we please.

If two first integrals occur, it is shown that the arguments of the one furnish the equations required for the integration of the other. But in this case it is suggested that the equation should be solved by the method employed by Imschenetsky in the case of two variables.

If the two first integrals be identical, the complete primitive is obtained by equating to constants the different arguments of F; hence finding the values of $p_1 \dots p_n$, and integrating the expression

$$dz - p_1 dx_1 \dots - p_n dx_n,$$

which is then an exact differential.

The dependent variable being z, the n independent variables $x_1 \dots x_n$, we shall write p_i for $\frac{dz}{dx_i}$, and s_{ij} for $\frac{d^2 z}{dx_i dx_j}$.

By $\frac{du}{dx_i}$ we shall indicate the result of differentiating u with respect to x_i explicitly involved in u; by $\left(\frac{du}{dx_i}\right)$ we indicate that the differentiation is with respect to x_i as it occurs explicitly, and implicitly in z; in $\frac{d \,.\, u}{dx_i}$ (a notation suggested by Imschenetsky*) the differentiation is with respect to x_i, however it occurs. Thus, if u be a function of $\dots p_j \dots x_i \dots z$, we should write

$$\frac{d \,.\, u}{dx_i} \equiv \dots + \frac{du}{dp_j} s_{ij} + \dots + \left(\frac{du}{dx_i}\right)$$
$$\equiv \dots\dots\dots\dots\dots\dots + p_i \frac{du}{dz} + \frac{du}{dx_i}.$$

I. *On the Genesis of the Equation.*

(1.) Let $u_1 \dots u_n$ represent n mutually independent functions of $x_1 \dots x_n, z, p_1 \dots p_n$. Then, representing by F an arbitrary function, the general form of a first integral may be written

$$F(u_1, \dots u_n) = 0 \dots\dots\dots\dots\dots\dots (1).$$

Differentiate this with respect to $x_1 \dots x_n$, and from the n equations thus formed eliminate the arbitrary functions $\frac{dF}{du_1} \dots \frac{dF}{du_n} \dots$, which are equivalent to $n-1$ only since the equations are homogeneous. The resulting equation will be the general form of the equation of the second order which admits of a first integral (1). This form is

* And De Morgan, *Diff. and Int. Calc.*, p. 90.

$$\begin{vmatrix} \frac{d.u_1}{dx_1}, & \frac{d.u_1}{dx_2}, & \cdots & \frac{d.u_1}{dx_n} \\ \frac{d.u_2}{dx_1}, & \cdots\cdots & & \\ \cdots\cdots & & & \\ \frac{d.u_n}{dx_1} & \cdots\cdots & & \frac{d.u_n}{dx_n} \end{vmatrix} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $\frac{d.u_k}{dx_l} = \frac{du_k}{dp_1}.s_{1l} + \frac{du_k}{dp_2}.s_{2l} + \ldots\ldots + \frac{du_k}{dp_n}.s_{nl} + \left(\frac{du_k}{dx_l}\right).$

(2.) In order to reduce this determinant to a more convenient form, we make use of the following generalization of the theorem implied in the rule for the multiplication of determinants. If

$$c_{kl} = \sum_{i=1}^{i=m} a_{ki}.b_{il},$$

the determinant $(c_{11} \ldots c_{nn})$ vanishes when m is less than n; but if m be not less than n its value is

$$\Sigma (a_{1i} \ldots a_{nj})(b_{i1} \ldots b_{jn}) \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where the Σ extends to all the different products that can be formed by giving the n quantities $i \ldots j$, n different values between 1 and m inclusive. Hence the number of terms is $\frac{\lfloor m}{\lfloor n \lfloor m-n}$, which, when $m = n$ reduces to 1.

The determinant on the left of (2) is a particular case of this. For, i, j being any numbers not greater than n, it is only necessary to put

$$a_{ij} = \frac{du_i}{dp_j}, \quad b_{ij} = s_{ij},$$

and $\quad a_{i,n+1}\, b_{n+1,j} + \ldots + a_{i,n+j}\, b_{n+j,j} + \ldots + a_{i,m}\, b_{m,j} = \left(\frac{du_i}{dx_j}\right),$

for all values of i, j to make the two determinants identical. Now the last condition is satisfied identically if we assume

$$a_{i,n+j} = \left(\frac{du_i}{dx_j}\right),$$

$$b_{n+i,j} = 0, \text{ unless } j = i \text{ when } b_{n+j,j} = 1.$$

The value of m must be $2n$. It cannot be less, for we require the equation $a_{i,2n} = \left(\frac{du_i}{dx_n}\right)$. It cannot be more, for all the b's of the form $b_{2n+i,j}$ vanish identically in virtue of the last condition, and the fact that j is less than n. Hence in the expansion of (2) there will be $\frac{\lfloor 2n}{\{\lfloor n\}^2}$ terms.

(3.) Let us now seek to determine the constitution of these terms. From the theorem quoted it is clear that each term will be the product of two determinants. The first or a-determinant will involve only the derivatives of $u_1 \dots u_n$. We shall write these determinants in the form $\frac{d\,.\,(u_1 \;\dots\; u_j \;\dots\; u_n)}{d\,.\,(p_1 \dots (x_\iota) \dots p_n)}$, where the x is put in brackets to indicate that the symbols $\left(\frac{du}{dx_\iota}\right)$ are used, and it is placed under the u_j to show that the differentials $\frac{du}{dp_j}$ do not occur, $\left(\frac{du}{dx_\iota}\right)$ taking their place. We shall occasionally find it convenient to write the determinants in the form $\begin{vmatrix} j \dots \\ \iota \dots \end{vmatrix}$ the upper line showing the p's which do not occur, the lower line the x's by which they are respectively replaced, the original determinant being $\frac{d\,(u_1 \dots u_\iota \dots u_n)}{d\,(p_1 \dots p_\iota \dots p_n)}$.

(4.) The second factor of each term is a determinant involving only the quantities ... s_{ij} ..., unity and zero.

To determine the general form we may write down such a term as the following:—

$$\frac{d\,(u_1 \;\dots\; u_j \;\dots\; u_n)}{d\,(p_1 \dots (x_\iota) \dots p_n)} \times \begin{vmatrix} s_{11}, & \dots & s_{1\iota} \dots s_{1n} \\ \dots & \dots & \dots \\ s_{j-1,1} \dots & s_{j-1,} & \dots \\ 0 \;\dots & 1 & \dots\; 0 \\ \dots & \dots & \dots \\ s_{n,1} \dots & s_{n\iota} & \dots s_{nn} \end{vmatrix}$$

the second factor being written down by a reference to (3) and the values subsequently assigned to the b's. In this second factor remove the ι^{th} column to the front and then the j^{th} row to the to It is, therefore,

$$(-)^{\iota+j} \begin{vmatrix} 1, & 0, & 0, & \dots 0 \\ s_{1\iota}, & s_{11}, & s_{12}, & \dots s_{1n} \\ s_{2\iota}, & s_{21}, & s_{22} & \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & s_{nn} \end{vmatrix}$$

which is the minor of $(s_{11} \dots s_{nn})$ formed by erasing the ι^{th} column and j^{th} row, and affected with the sign $(-)^{\iota+j}$.

Again, suppose the first factor is $\begin{vmatrix} j & k \\ \iota & \kappa \end{vmatrix}$ or

$$\frac{d\,(u_1 \dots \;u_j \;\,..\;\, u_k \;..\;\; u_n)}{d\,(p_1 \dots (x_\iota) \dots (x_\kappa) \dots p_n)},$$

where we assume $k > j$, $\kappa > \iota$. In the second the factor the $\left.\begin{matrix} j \\ k \end{matrix}\right\}^{\text{th}}$ row vanishes, with the exception of the quantity common to it and the

$\left.\begin{matrix}\iota\\ \kappa\end{matrix}\right\}^{\text{th}}$ column, which is unity. By bringing these rows and columns to the edges, we find that the second factor is the minor of $(s_{11} \dots s_{nn})$ formed by erasing the rows j, k, and the columns ι, κ, and affected with the sign $(-)^{\iota+\kappa+j+k}$. The assumption as to the order of j, k, ι, κ does not affect this result; for interchanging ι κ (say) is simply equivalent to interchanging two columns in each of the determinants; viz., it changes the sign only of both factors, so that the product is unaltered.

This process is clearly general. Hence the s-factors in the expansion of (2) are the minors that can be obtained from $(s_{11} \dots s_{nn})$ including this determinant itself and unity as extremes. The coefficient of the minor formed by erasing the rows j, k, l ..., and the columns ι, κ, λ ... is the Jacobian

$$(-)^{\iota+\kappa+\lambda\dots+i+j+k-\dots} \left| \begin{matrix} j, k, l, \dots \\ \iota, \kappa, \lambda, \dots \end{matrix} \right|,$$

or

$$(-)^{\dots\dots} \frac{d\,(\dots u_j \dots u_k \dots u_l \dots)}{d\,(\dots(x_\iota) \dots (x_\kappa) \dots (x_\lambda) \dots)},$$

where j, k, l ... and ι, κ, λ ... are in ascending order of magnitude.

It is easy to verify that this gives the same number of terms as we obtained before. For there are $\left\{\frac{\lfloor n}{\lfloor n-r \lfloor r}\right\}^2$ minors of the r^{th} order, since these may be formed by retaining any r rows and any r columns of the n; and

$$1^2+n^2+\left\{\frac{n(n-1)}{1\,.\,2}\right\}^2+\dots\dots+n^2+1^2=\frac{\lfloor 2n}{\{\lfloor n\}^2},$$

which agrees with the result given at the conclusion of Art. 2.

(5.) Let us now take account of the fact that $s_{ij}=s_{ji}$, or that the determinant $(s_{11} \dots s_{nn})$ is symmetrical. The symmetrical minors of the r^{th} order are of the form $(s_{ii}, \dots s_{kk} \dots s_{ll})$, and there are $\frac{\lfloor n}{\lfloor r \lfloor n-r}$ of them, since any combination r together of the n suffixes may be taken. Altogether, then, there are $1+n+\frac{n\,(n-1)}{1\,.\,2}+\dots+n+1$, or 2^n symmetrical minors. The coefficients of these are of the form

$$\frac{d\,(\dots u_j \dots u_k \dots)}{d\,(\dots (x_j) \dots (x_k) \dots)}.$$

The sign is $(-)^{2\,(j+k+\dots)}$ or positive.

All the other terms in the expansion of (2) involve unsymmetrical minors. These occur in pairs: for the minor formed by erasing the rows j, k, ... and the columns ι, κ, ... is identical with that formed by erasing the rows ι, κ, ... and the columns j, k, ...

Taking each pair together, we have $\frac{1}{2}\cdot\left\{\frac{\lfloor 2n}{\{\lfloor n\}^2}-2^n\right\}$ terms involving unsymmetrical minors of $(s_{11}\dots s_{nn})$, and the coefficient of each will be of the form

$$(-)^{j+k+\dots+\iota+\kappa}\left\{\frac{d\,(\dots u_j\,\dots\,u_k\,\dots)}{d\,(\dots(x_\iota)\dots(x_\kappa)\dots)}+\frac{d\,(\dots u_\iota\,\dots\,u_\kappa\,\dots)}{d\,(\dots(x_j)\dots(x_k)\dots)}\right\}.$$

(6.) These results enable us to write down the form of the equation of the second order which includes all that admit of a first integral of the form (1). For instance, in the case of three independent variables, this is

$$\mathrm{U}\begin{vmatrix} s_{11}, & s_{12}, & s_{13} \\ s_{12}, & s_{22}, & s_{23} \\ s_{13}, & s_{23}, & s_{33}\end{vmatrix}+\mathrm{U}_1\begin{vmatrix} s_{22}, & s_{23} \\ s_{23}, & s_{33}\end{vmatrix}+\mathrm{U}_2\begin{vmatrix} s_{11}, & s_{13} \\ s_{13}, & s_{33}\end{vmatrix}+\mathrm{U}_3\begin{vmatrix} s_{11}, & s_{12} \\ s_{12}, & s_{22}\end{vmatrix}$$

$$-2\mathrm{U}_{\substack{2\\3}}\begin{vmatrix} s_{11}, & s_{12} \\ s_{13}, & s_{23}\end{vmatrix}+2\mathrm{U}_{\substack{1\\3}}\begin{vmatrix} s_{12}, & s_{22} \\ s_{13}, & s_{23}\end{vmatrix}-2\mathrm{U}_{\substack{1\\2}}\begin{vmatrix} s_{12}, & s_{23} \\ s_{13}, & s_{33}\end{vmatrix}$$

$$+\mathrm{U}_{23}.s_{11}+\mathrm{U}_{13}.s_{22}+\mathrm{U}_{12}.s_{33}-2\mathrm{U}_{\substack{12\\13}}.s_{23}+2\mathrm{U}_{\substack{12\\23}}.s_{31}-2\mathrm{U}_{\substack{13\\23}}.s_{12}+\mathrm{U}_{123}=0\dots(4),$$

where, if μ be any indeterminate multiplier,

$$\mathrm{U}=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\cdot p_2\cdot p_3)},\quad \mathrm{U}_{123}=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,(x_2)\,.\,(x_3))},$$

$$\mathrm{U}_1=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\cdot p_2\cdot p_3)},\quad 2\mathrm{U}_{\substack{2\\3}}=\mu\left\{\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\cdot(x_3)\cdot p_3)}+\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\cdot p_2\cdot(x_2))}\right\},$$

$$\mathrm{U}_{23}=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\,.\,(x_2)\,.\,(x_3))},$$

$$\mathrm{U}_{\substack{12\\13}}=\mu\left\{\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,(x_3)\,.\,p_3)}+\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,p_2\,.\,(x_2))}\right\},$$

&c. = &c.

That the given equation of the second order should be of the general form indicated by the equation (4), regarding therein the coefficients U as arbitrary, is necessary, but not sufficient to ensure the existence of a general first integral. In the case of n independent variables we should have $\frac{1}{2}\cdot\frac{\lfloor 2n}{\{\lfloor n\}^2}+2^{n-1}$ coefficients; but these are expressible in terms of $n+1$ independent quantities $u_1\dots u_n$, μ. Thus the coefficients must satisfy

$$\frac{1}{2}\cdot\frac{\lfloor 2n}{\{\lfloor n\}^2}+2^{n-1}-n-1 \text{ conditions.}$$

II. *Determination of the First Integral.*

(7.) We now attack the problem converse to that discussed in the first part of this paper; viz., starting with a known equation of the second order, we endeavour to find a first integral

$$F\{u_1 \ldots\ldots u_n\} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

when one exists. This will be accomplished by discovering the n quantities $u_1 \ldots u_n$.

Now the equation

$$\left|\begin{array}{c|ccc} \left(\frac{du}{dx_j}\right), & \frac{du}{dp_1} & \cdots\cdots & \frac{du}{dp_n} \\ \left(\frac{du_1}{dx_j}\right), & \frac{du_1}{dp_1} & \cdots\cdots & \frac{du_1}{dp_n} \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots \\ \left(\frac{du_n}{dx_j}\right), & \frac{du_n}{dp_1} & \cdots\cdots & \frac{du_n}{dp_n} \end{array}\right| = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

is evidently satisfied when $u = u_1, \ldots$ or u_n; for then two rows of the determinant become identical.

In this equation the differentiations are performed with respect to $n+1$ only of the $2n$ quantities $x_1 \ldots x_n, p_1 \ldots p_n$; and since we may form a similar equation, using *any* $n+1$ of these quantities, there are evidently $\frac{\lfloor 2n}{\lfloor n+1 \,\lfloor n-1}$ equations of the same type as (5).

(8.) The equation (5) may be written in a more convenient form. It is evidently homogeneous of the first degree in the derivatives of u. The coefficient of $\left(\frac{du}{dx_j}\right)$ is the Jacobian $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$. The coefficient of $\frac{du}{dp_r}$, as may be seen by interchanging the columns containing $\left(\frac{du}{dx_j}\right)$, $\frac{du}{dp_r}$, is the same Jacobian with $\left(\frac{du}{dx_j}\right)$, $\left(\frac{du_2}{dx_j}\right) \ldots$ in the place of $\frac{du_1}{dp_r}, \frac{du_2}{dp_r}, \ldots$, and affected with a negative sign. We shall represent, as before, the result of this substitution by the symbol $\left|\begin{smallmatrix} r \\ j \end{smallmatrix}\right|$, the Jacobian $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$ being accordingly represented by the symbol $|\quad|$. The equation (5) may then be written in the form

$$|\quad| + \left(\frac{du}{dx_j}\right) - \Sigma_r \left|\begin{matrix} r \\ j \end{matrix}\right| \frac{du}{dp_r} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Concerning the $\frac{\lfloor 2n}{\lfloor n+1 \,\lfloor n-1}$ equations of the form (5) or (6), we shall now prove:—that n of them are independent; that not more than n are independent; and that the coefficients can be expressed in terms of the coefficients of the given equation of the second order.

(9.) *There are n independent equations.* From (5) we can obtain n equations by putting $j=1, \ldots n$, in succession. Now, if $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$ does not vanish, these equations will be algebraically independent, for each will contain a quantity $\left(\frac{du}{dx_1}\right)$, $\left(\frac{du}{dx_2}\right)$ ..., which occurs in none of other equations of the system. But the marked minor in (5) may be replaced by any of the Jacobians of $u_1 \ldots u_n$, and if the one selected does not vanish, we get n independent equations by using for the first column differentials with respect to the n quantities that do not occur in the Jacobian. Such a system is always obtainable, since, if all the Jacobians of $u_1 \ldots u_n$ vanish, the same is true of the coefficients of the equation of the second order by the first part of this paper.

(10.) *Only n of the equations are independent.* For we have a matrix, $|a_{i,j}|$, $n+1$ deep by $m+n$ broad (here $m=n$). The m determinants obtained by taking in succession each of the first m columns with the last n columns all vanish: *i.e.*, $A_1, A_2 \ldots A_{n+1}$ being the $n+1$ determinants (of order n) of the last n columns, we have for all values of j,

$$A_1 a_{1,j} + A_2 a_{2,j} + \ldots + A_{n+1} a_{n+1,j} = 0.$$

But $A_1, A_2 \ldots A_{n+1}$ do not all vanish (for the marked minor is to be a Jacobian which does not vanish): therefore all the determinants of the matrix vanish. Hence, if the n equations of Art. 9 be satisfied, the other equations of the system become identities.

(11.) If we express that the $\frac{\underline{|2n}}{\underline{|n+1}\ \underline{|n-1}}$ equations for u are algebraically equivalent to n only, we obtain a remarkable series of theorems which enable us to express the coefficients of these equations in terms of the coefficients of the given equation of the second order, and which give implicitly some of the necessary relations between the coefficients of the given equation.

For instance, supposing that $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$ or $\left|\ \cdot\ \right|$ does not vanish, let us eliminate $\frac{du}{dp_\kappa}$ from the two equations

$$\left|\ \ \right| \cdot \left(\frac{du}{dx_j}\right) - \Sigma_r \left|\begin{matrix} r \\ j \end{matrix}\right| \frac{du}{dp_r} = 0,$$

$$\left|\ \ \right| \cdot \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix} r \\ k \end{matrix}\right| \frac{du}{dp_r} = 0.$$

The result is

$$\left|\ \ \right| \times \left|\begin{matrix} \kappa \\ k \end{matrix}\right| \times \left(\frac{du}{dx_j}\right) - \left|\ \ \right| \times \left|\begin{matrix} \kappa \\ j \end{matrix}\right| \times \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix} \left|\begin{matrix} r \\ j \end{matrix}\right|, & \left|\begin{matrix} r \\ k \end{matrix}\right| \\ \left|\begin{matrix} \kappa \\ j \end{matrix}\right|, & \left|\begin{matrix} \kappa \\ k \end{matrix}\right| \end{matrix}\right| \frac{du}{dp_r} = 0.$$

Now this must be identical with the equation formed from the determinant whose first row is

$$\left(\frac{du}{dx_j}\right),\ \frac{du}{dp_1}\ \cdots\cdots\ \frac{du}{dp_{\kappa-1}},\ \left(\frac{du}{dx_k}\right),\ \cdots\cdots,$$

or

$$\left|\begin{matrix}\kappa\\k\end{matrix}\right| \times \left(\frac{du}{dx_j}\right) - \left|\begin{matrix}\kappa\\j\end{matrix}\right| \times \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix}r\ \kappa\\j\ k\end{matrix}\right| \cdot \frac{du}{dp_r} = 0\,;$$

for otherwise more than n equations of the system would be independent.

Comparing coefficients, we get, taking $r = \iota$,

$$\left|\begin{matrix}\left|\begin{matrix}\iota\\j\end{matrix}\right|, & \left|\begin{matrix}\iota\\k\end{matrix}\right|\\ \left|\begin{matrix}\kappa\\j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\k\end{matrix}\right|\end{matrix}\right| = \left|\ \ \right| \times \left|\begin{matrix}\iota\ \kappa\\j\ k\end{matrix}\right| \quad\cdots\cdots\cdots\cdots\cdots\cdots (7),$$

where in analogy with the previous notation we write $\left|\begin{matrix}\iota\ \kappa\\j\ k\end{matrix}\right|$ for the result of substituting $\left(\frac{du_1}{dx_j}\right)$ ... and $\left(\frac{du_1}{dx_k}\right)$ ... for $\frac{du_1}{dp_\iota}$ ... and $\frac{du_1}{dp_\kappa}$ ... respectively in $|\ |$.

Again, from the three equations

$$\left|\ \right| \left(\frac{du}{dx_j}\right) - \Sigma_r \left|\begin{matrix}r\\j\end{matrix}\right| \frac{du}{dp_r} = 0,$$

$$\left|\ \right| \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix}r\\k\end{matrix}\right| \frac{du}{dp_r} = 0,$$

$$\left|\ \right| \left(\frac{du}{dx_l}\right) - \Sigma_r \left|\begin{matrix}r\\l\end{matrix}\right| \frac{du}{dp_r} = 0,$$

eliminating $\frac{du}{dp_\kappa}$, $\frac{du}{dp_\lambda}$, we obtain the equation

$$\left|\ \right| \times \left\{ \left(\frac{du}{dx_j}\right) \left|\begin{matrix}\left|\begin{matrix}\kappa\\k\end{matrix}\right|, & \left|\begin{matrix}\kappa\\l\end{matrix}\right|\\ \left|\begin{matrix}\lambda\\k\end{matrix}\right|, & \left|\begin{matrix}\lambda\\l\end{matrix}\right|\end{matrix}\right| - \left(\frac{du}{dx_k}\right) \left|\begin{matrix}\left|\begin{matrix}\kappa\\j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\l\end{matrix}\right|\\ \left|\begin{matrix}\lambda\\j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\l\end{matrix}\right|\end{matrix}\right| + \left(\frac{du}{dx_l}\right) \left|\begin{matrix}\left|\begin{matrix}\kappa\\j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\k\end{matrix}\right|\\ \left|\begin{matrix}\lambda\\j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\k\end{matrix}\right|\end{matrix}\right| \right\}$$

$$- \Sigma_r \left|\begin{matrix}\left|\begin{matrix}r\\j\end{matrix}\right|, & \left|\begin{matrix}r\\k\end{matrix}\right|, & \left|\begin{matrix}r\\l\end{matrix}\right|\\ \left|\begin{matrix}\kappa\\j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\k\end{matrix}\right|, & \left|\begin{matrix}\kappa\\l\end{matrix}\right|\\ \left|\begin{matrix}\lambda\\j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\k\end{matrix}\right|, & \left|\begin{matrix}\lambda\\l\end{matrix}\right|\end{matrix}\right| \frac{du}{dp_r} = 0.$$

The first three terms of this equation reduce by (7) to

$$\left\{\left|\ \right|\right\}^2 \times \left\{ \left(\frac{du}{dx_j}\right) \left|\begin{matrix}\kappa\ \lambda\\k\ l\end{matrix}\right| - \left(\frac{du}{dx_k}\right) \left|\begin{matrix}\kappa\ \lambda\\j\ l\end{matrix}\right| - \left(\frac{du}{dx_l}\right) \left|\begin{matrix}\kappa\ \lambda\\k\ j\end{matrix}\right| \right\}.$$

Compare this with the equation derived from the determinant whose top row is

$$\left(\frac{du}{dx_j}\right),\ \frac{du}{dp_1},\ \dots\ \frac{du}{dp_{\kappa-1}},\ \left(\frac{du}{dx_k}\right),\ \dots\ \frac{du}{dp_{\lambda-1}},\ \left(\frac{du}{dx_l}\right),\ \dots\ \frac{du}{dp_n};$$

viz., with the equation

$$\left|\begin{matrix}\kappa\,\lambda\\ k\,l\end{matrix}\right|\left(\frac{du}{dx_j}\right)-\left|\begin{matrix}\kappa\,\lambda\\ j\,l\end{matrix}\right|\left(\frac{du}{dx_k}\right)-\left|\begin{matrix}\kappa\,\lambda\\ k\,j\end{matrix}\right|\left(\frac{du}{dx_l}\right)-\Sigma_r\left|\begin{matrix}r\,\kappa\,\lambda\\ j\,k\,l\end{matrix}\right|\frac{du}{dp_r}=0;$$

and we obtain the relation $(r=\iota)$

$$\left|\begin{matrix}\left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ k\end{matrix}\right|, & \left|\begin{matrix}\iota\\ l\end{matrix}\right|\\ \left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ l\end{matrix}\right|\\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ k\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|\end{matrix}\right| = \Big\{\big|\,\big|\Big\}^2\times\left|\begin{matrix}\iota\,\kappa\,\lambda\\ j\,k\,l\end{matrix}\right| \quad\dots\dots (8).$$

By an inductive proof precisely similar to the above, we infer generally that

$$\left|\begin{matrix}\left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ k\end{matrix}\right|, & \dots\\ \left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right|, & \dots\\ \vdots & \vdots & \end{matrix}\right| = \Big\{\big|\,\big|\Big\}^{t-1}\times\left|\begin{matrix}\iota\,\kappa\,\dots\\ j\,k\,\dots\end{matrix}\right| \quad\dots\dots (9),$$

the determinant on the left being of the t^{th} order.

More generally, starting with the determinant $\left|\begin{matrix}a\,\dots\,\beta\\ a\,\dots\,b\end{matrix}\right|$ in the place of the marked minor in (5), we get a series of theorems of the form

$$\left|\begin{matrix}\left|\begin{matrix}a\,\dots\,\beta\iota\\ a\,\dots\,bj\end{matrix}\right|, & \left|\begin{matrix}a\,\dots\,\beta\iota\\ a\,\dots\,bk\end{matrix}\right|, & \dots\\ \left|\begin{matrix}a\,\dots\,\beta\kappa\\ a\,\dots\,bj\end{matrix}\right|, & \left|\begin{matrix}a\,\dots\,\beta_\kappa\\ a\,\dots\,bj\end{matrix}\right|, & \dots\\ \vdots & \vdots & \end{matrix}\right| = \left\{\left|\begin{matrix}a\,\dots\,\beta\\ a\,\dots\,b\end{matrix}\right|\right\}^{t-1}\times\left|\begin{matrix}a\,\dots\,\beta\iota\kappa\,\dots\\ a\,\dots\,bjk\,\dots\end{matrix}\right| \quad\dots\dots (10),$$

t being the number of quantities $\iota, \kappa, \dots$.

The proof we have given of these theorems ceases to be rigorous when $\big|\,\big|$ or $\left|\begin{matrix}a\,\dots\,\beta\\ a\,\dots\,b\end{matrix}\right|$ respectively vanish. But the truth of these theorems evidently depends upon the manner in which the derivatives of $u_1 \dots u_n$ are involved, and not upon their magnitudes, and it is therefore probable that they remain true even in this case.

It is obvious that we can obtain fresh series of theorems either by expanding (9) or (10) in products of conjugate minors and replacing these by their values found from the same equations, or we may start from a fresh set of equations and combine them in various ways. Thus, for example, multiplying $n+1$ equations by $n+1$ indeterminate multipliers and eliminating these multipliers between the $2n$ equations formed by equating to zero the coefficient of each term in the sum, we shall obtain n identical relations.

(12.) Supposing, as before, that $|\,|$ does not vanish, let us, in (7), put $\iota=j$, $\kappa=k$, and write the determinant at length. Thus we get

$$\begin{vmatrix} j \\ k \end{vmatrix} \cdot \begin{vmatrix} k \\ j \end{vmatrix} = \begin{vmatrix} j \\ j \end{vmatrix} \cdot \begin{vmatrix} k \\ k \end{vmatrix} - |\,| \cdot \begin{vmatrix} jk \\ jk \end{vmatrix} \quad \ldots\ldots\ldots\ldots(11).$$

Now the determinants on the right are known, for they are the coefficients (divided by μ) of symmetrical minors of $(s_{11} \ldots s_{nn})$ in the given equation of the second order. Also the sum of $\begin{vmatrix} j \\ k \end{vmatrix}$, $\begin{vmatrix} k \\ j \end{vmatrix}$ is known, since it is the coefficient of a certain unsymmetrical minor of $(s_{11} \ldots s_{nn})$ divided by μ. Hence $\mu\begin{vmatrix} j \\ k \end{vmatrix}$, $\mu\begin{vmatrix} k \\ j \end{vmatrix}$ are two roots of a certain quadratic equation whose coefficients are given in terms of the coefficients of the equation of the second order to be solved.

For example, in the equation with three independent variables given in (4) we find that $\mu\begin{vmatrix} 2 \\ 3 \end{vmatrix}$, $\mu\begin{vmatrix} 3 \\ 2 \end{vmatrix}$ are the two roots of the quadratic

$$\xi^2 - 2U_{\substack{2\\3}} \, . \, \xi + U_2U_3 - UU_{23} = 0.$$

Again, supposing that $\begin{vmatrix} a \ldots \beta \\ a \ldots b \end{vmatrix}$ does not vanish, we obtain from (10), on putting $t=2$, $\iota=j$, $\kappa=k$, the equation

$$\begin{vmatrix} a \ldots \beta j \\ a \ldots bk \end{vmatrix} \begin{vmatrix} a \ldots \beta k \\ a \ldots bj \end{vmatrix} = \begin{vmatrix} a \ldots \beta j \\ a \ldots bj \end{vmatrix} \times \begin{vmatrix} a \ldots \beta k \\ a \ldots bk \end{vmatrix} - \begin{vmatrix} a \ldots \beta \\ a \ldots b \end{vmatrix} \times \begin{vmatrix} a \ldots \beta jk \\ a \ldots bjk \end{vmatrix};$$

whence, if $a=a$, ... $b=\beta$, we can find the two factors on the left.

Thus, in the equation (4), the quantities $\mu\begin{vmatrix} 1\,2 \\ 1\,3 \end{vmatrix}$, $\mu\begin{vmatrix} 1\,3 \\ 1\,2 \end{vmatrix}$ are the roots of the quadratic equation

$$\xi^2 - 2U_{\substack{12\\13}} \, . \, \xi + U_{12} \, . \, U_{13} - U_1 \, . \, U_{123} = 0.$$

The occurrence of an indeterminate factor μ in these determinations is immaterial; for the equations for u are homogeneous in μ after substitution, so that this factor divides out.

(13.) The determination of the values of, say, $\begin{vmatrix} j \\ k \end{vmatrix}$, $\begin{vmatrix} k \\ j \end{vmatrix}$, as far as it is indicated above, leaves us free to assign to each of these functions one of a pair of values; and it is now necessary to examine if this ambiguity is limited in any way. In the first place, it is evident that when we have assigned values to all the determinants $\begin{vmatrix} j \\ k \end{vmatrix}$, &c., or $\begin{vmatrix} a \ldots bj \\ a \ldots bk \end{vmatrix}$,

&c., that the others are determinate, supposing that $\left|\ \right|$ or $\left|\begin{matrix} a \dots b \\ a \dots b \end{matrix}\right|$ respectively do not vanish. In fact, a single value of each is given by (9) or (10). We shall now show that in general, when $\left|\begin{matrix} j \\ k \end{matrix}\right| \cdot \left|\begin{matrix} k \\ j \end{matrix}\right|$ have values assigned to them, all the Jacobians $\left|\begin{matrix} a \\ a \end{matrix}\right|$ will admit of only single values.

In (8) put $\iota = j$, $\kappa = k$, $\lambda = l$. Expand the determinant and transpose to the right side all the terms such as $\left|\begin{matrix} j \\ j \end{matrix}\right| \left|\begin{matrix} k \\ k \end{matrix}\right| \left|\begin{matrix} l \\ l \end{matrix}\right|$, $\left|\begin{matrix} j \\ j \end{matrix}\right| \left|\begin{matrix} k \\ l \end{matrix}\right| \left|\begin{matrix} l \\ k \end{matrix}\right|$, whose values are free from ambiguity. Thus we find that

$$\left|\begin{matrix} j \\ k \end{matrix}\right| \left|\begin{matrix} k \\ l \end{matrix}\right| \left|\begin{matrix} l \\ j \end{matrix}\right| + \left|\begin{matrix} j \\ l \end{matrix}\right| \left|\begin{matrix} k \\ j \end{matrix}\right| \left|\begin{matrix} l \\ k \end{matrix}\right|$$

has a determinate value for each value of l.

Now these equations, and others which we obtain by putting $t = 4, 5$, &c. in (9), are not altered if we change all the quantities $\left|\begin{matrix} j \\ l \end{matrix}\right|$, &c. into the quantities $\left|\begin{matrix} l \\ j \end{matrix}\right|$; but they generally cease to be satisfied when some only of these quantities are thus interchanged. Thus from the equations (6) we can always form a second set of equations, the solutions of which will give a second first integral; and in general there are only two such sets of equations.

I have attempted to form equations such as to admit of more than one pair of first integrals, by making for all values of l

$$\left|\begin{matrix} k \\ l \end{matrix}\right| \left|\begin{matrix} l \\ j \end{matrix}\right| = \left|\begin{matrix} j \\ l \end{matrix}\right| \left|\begin{matrix} k \\ j \end{matrix}\right|.$$

In all cases I have found, however, that, though two pairs of integrals were obtained, yet in virtue of these very conditions they were identical; but I am not prepared to prove that such is always the case.

(14.) The theorems (9), (10), and others allied to them, serve also to indicate the necessary relations existing between the coefficients of the given equation of the second order. In fact, if we express these coefficients in terms of Jacobians of $u_1 \dots u_n$, as in the first part, we shall have a system of equations between which we can eliminate u_1, &c., leaving a series of necessary relations between the original coefficients. But besides these relations, which do not involve the derivatives of the coefficients, there are also others which are necessary to insure that the n equations (6) should have n common integrals. These conditions are very complex in form and do not invite a direct investigation.

It may, however, be noted that one of the conditions that the n

equations should have $n+1$ common integrals is that $\left|\begin{matrix} k \\ j \end{matrix}\right| = \left|\begin{matrix} j \\ k \end{matrix}\right|$ for all values of j, k. This implies that the quantity

$$dz - p_1 dx_1 \ldots - p_n dx_n,$$

which is the differential of the $n+1^{\text{th}}$ integral, is in this case an exact differential; a result which will be verified hereafter.

But the expression of these conditions is fortunately not necessary for our purpose. To find a first integral we take any set of n equations (5) which are independent. If these have not n common integrals there is no first integral. If they have, substitute them for $u_1 \ldots u_n$ in (1), and find if the equation is satisfied by this integral. It is generally convenient to render our trial integral as free from ambiguity as possible by means of the theorems (9) or (10) as explained above; but if this be not done a comparison with the original equation at once serves to remove all inadmissible ambiguity.

(15.) It may be well to illustrate the results obtained by an application of them to the general equation with two variables,

$$U(rt-s^2)+Rr-2Ss+Tt+V = 0.$$

The arguments of the first integral of this equation satisfy the four equations

$$U\left(\frac{du}{dx}\right) - T\frac{du}{dp} - m_1\frac{du}{dq} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$U\left(\frac{du}{dy}\right) - m_2\frac{du}{dp} - R\frac{du}{dq} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

$$R\left(\frac{du}{dx}\right) - m_1\left(\frac{du}{dy}\right) + V\frac{du}{dp} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma),$$

$$m_2\left(\frac{du}{dx}\right) - T\left(\frac{du}{dy}\right) - V\frac{du}{dq} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\delta),$$

where $m_1+m_2 = 2S$.

These equations reduce to two, provided

$$m_1 m_2 = RT-UV,$$

so that m_1, m_2 are the roots of

$$\xi^2-2S\xi+RT-UV = 0.$$

These results agree exactly with those given by Boole (Supp. Vol., chaps. xxviii., xxix.), making allowance for different notation. It may be interesting to notice that in chap. xxviii. Boole makes use of (α), (β) except when $U=0$, when he employs (α) or (β) with (γ), and when $V=0$, when either (γ) or (δ) is used with (β). In chap. xxix. equations (γ), (δ) are used; but the case of $V=0$, when they become identical, is not specially mentioned.

III. *Determination of the Second Integral.*

(16.) In order to integrate the non-linear differential equation of the first order $F\{u_1 \dots u_n\} = 0$ ....(1),

we must find $n-1$ functions of p_1, &c. such that, v representing any one of these quantities,

$$\sum_{i=1}^{i=n}\left\{\left(\frac{dF}{dx_i}\right)\frac{dv}{dp_i}-\frac{dF}{dp_i}\left(\frac{dv}{dx_i}\right)\right\}=0 \quad \dots\dots\dots\dots (12).$$

Now, in general, the values of v will depend upon the particular form of F, and in this case the general method for obtaining a second integral is to assign to F a particular form and then integrate. We thus obtain as many particular solutions of the given equation of the second order as we please.

But it may happen that the equation (12) is satisfied independently of the form of F. In order that this may be so, we require that the coefficients of the arbitrary functions $\frac{dF}{du_1}$, $\frac{dF}{du_2}$, &c. should simultaneously vanish. This gives for v the n equations obtained by putting k successively equal to 1, 2 ... n in the equation

$$\sum_{i=1}^{i=n}\left\{\left(\frac{du_k}{dx_i}\right)\frac{dv}{dp_i}-\left(\frac{dv}{dx_i}\right)\frac{du_k}{dp_i}\right\}=0.$$

Now, if from these n equations we eliminate $n-1$ of the quantities $\frac{dv}{dp_i}$, $\left(\frac{dv}{dx_i}\right)$, say all the $\left(\frac{dv}{dx}\right)$'s except $\left(\frac{dv}{dx_j}\right)$, we get an equation such as

$$\big|\,\big| \times \left(\frac{dv}{dx_j}\right) - \Sigma \left|\begin{matrix} j \\ r \end{matrix}\right| \frac{dv}{dp_r} = 0.$$

But this equation is simply (6) with $\left|\begin{matrix} j \\ r \end{matrix}\right|$ substituted for $\left|\begin{matrix} r \\ j \end{matrix}\right|$, and we therefore conclude that, (1) being one of the first integrals of the given equation, the equations for integrating (1) are simply those which express that the arguments of the other first integral are constant; and in the particular case in which the two first integrals become identical, that the arguments $u_1 \dots u_n$ equated to constants will yield values of $p_1 \dots p_n$ which render

$$dz - p_1\,dx_1 \dots - p_n\,dx_n$$

an exact differential, so that a complete primitive can be found.

(17.) As an example of the simpler case we take the equation

$$x_1 x_2 x_3 \begin{vmatrix} s_{11}, & s_{12}, & s_{13} \\ s_{12}, & s_{22}, & s_{23} \\ s_{13}, & s_{23}, & s_{33} \end{vmatrix} - p_1 x_2 x_3 \begin{vmatrix} s_{22}, & s_{23} \\ s_{23}, & s_{33} \end{vmatrix} - x_1 p_2 x_3 \begin{vmatrix} s_{11}, & s_{13} \\ s_{13}, & s_{33} \end{vmatrix}$$

$$- x_1 x_2 p_3 \begin{vmatrix} s_{11}, & s_{12} \\ s_{12}, & s_{22} \end{vmatrix}$$

$$+ x_1 p_2 p_3 s_{11} + p_1 x_2 p_3 s_{22} + p_1 p_2 x_3 s_{33} - p_1 p_2 p_3 = 0.$$

Since no unsymmetrical minors occur in this equation the sum of each pair of Jacobians such as $\begin{vmatrix} 1 \\ 2 \end{vmatrix}$, $\begin{vmatrix} 2 \\ 1 \end{vmatrix}$ vanishes. Also their product vanishes, that of $\begin{vmatrix} 1 \\ 2 \end{vmatrix} \times \begin{vmatrix} 2 \\ 1 \end{vmatrix}$, for example, being

$$p_1 x_2 x_3 \,.\, x_1 p_2 x_3 - x_1 x_2 x_3 \,.\, p_1 p_2 x_3.$$

Hence all such Jacobians vanish.

Now, forming the equations (6), we get

$$x_1 \left(\frac{du}{dx_1}\right) + p_1 \frac{du}{dp_1} = 0,$$

$$x_2 \left(\frac{du}{dx_2}\right) + p_2 \frac{du}{dp_2} = 0,$$

$$x_3 \left(\frac{du}{dx_3}\right) + p_3 \frac{du}{dp_3} = 0.$$

These give for a first integral

$$F\left\{\frac{p_1}{x_1}, \frac{p_2}{x_2}, \frac{p_3}{x_3}\right\} = 0,$$

which satisfies the given equation. The other first integral is identical with this.

To find the complete primitive, put

$$\frac{p_1}{x_1} = 2a_1, \quad \frac{p_2}{x_2} = 2a_2, \quad \frac{p_3}{x_3} = 2a_3;$$

therefore $\qquad dz = 2a_1 x_1 dx_1 + 2a_2 x_2 dx_2 + 2a_3 x_3 dx_3,$

$$z = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4;$$

and the general integral is expressed by the equations

$$z = a_1 x_1^2 + a_2 x_2^2 + \phi(a_1, a_2)\, x_3^2 + \psi(a_1, a_2),$$

$$0 = x_1^2 + \frac{d\phi}{da_1} x_3^2 + \frac{d\psi}{da_1},$$

$$0 = x_2^2 + \frac{d\phi}{da_2} x_3^2 + \frac{d\psi}{da_2},$$

where a_1, a_2 are functions of x_1, x_2, x_3 defined by the last two equations when the forms of ϕ, ψ are specified.

(18.) When the first integrals are not identical, a general integral *may* be found in the same way as Boole has treated this case for two variables. But the results are very complex, and I believe that the best way to treat this case is to use a generalisation of Imschenetsky's method of variation of constants; viz., taking any particular solution involving $n+1$ constants, we find the most general relation between these constants, consistent with the given equation, is expressed by an equation of the second order, in which n constants are independent variables, and which is generally of an easily soluble form.

February 10*th*, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Messrs. Arthur Cockshott, M.A., late Fellow of Trinity College, Cambridge, Mathematical Master at Eton College, and Richard Thomas Wright, M.A., Fellow and Tutor of Christ's College, Cambridge, were proposed for election.

The following communications were made to the Society:—"Loci connected with the Rectangular Hyperbola, being inverses with respect to its centre and vertices:" Prof. Wolstenholme.—"On Determinants of Alternate Numbers:" W. Spottiswoode, F.R.S.—"On the Transformation of Gauss's Hypergeometric Series into a Continued Fraction:" Mr. T. Muir, M.A.—"On the Partition of Geometrical Curves:" the President.—"On the Sum of the Products of r different terms of a Series:" Mr. J. Hammond, B.A.—"On Pendular Motion:" Prof. Clifford, F.R.S.

A portion of a fly-sheet, by the Comte Léopold Hugo, Member of the Paris Mathematical Society, entitled "The Pan-imaginary Theory" (in which he speaks of space of $\frac{l}{m}$ dimensions), was read by the President. Messrs. Cayley, Clifford, Cotterill, Roberts, and the President took part in the discussions on the papers.

The following presents were made to the Society's Library:—

"The Journal of Education" (Jan., Feb., March, 1876), by Mr. Tucker, the Mathematical Editor.

"Proceedings of the Royal Society," Vol. xxiv., No. 166.

"Bulletin des Sciences," tome neuvième, Nov., Dec., 1875; and "Table des Matières et Noms d'Auteurs," tome viii., 1° sémestre, 1875.

"Mémoires de la Société des Sciences Physiques et Naturelles de

Bordeaux," tome i. (2e série) 2e cahier, 1876; and "Extrait des Procès-verbaux des Séances," Bordeaux.

"Annual Report of the Board of Regents of the Smithsonian Institution," for the year 1874, Washington, 1875.

"Jornal de Sciencias, Mathematicas, Physicas, e Naturaes, poblicado sob os auspicios da Academia Real das Sciencias de Lisboa," tomo i., Nov. 1866—Dez. 1867, Lisboa, 1868; Tomo ii., Agosto 1868—Dez. 1869, Lisboa, 1870; Tomo iii., Junho 1870—Dez. 1871, Lisboa, 1871: Presented by W. S. W. Vaux, Esq.

On Theorems relating to the Circular Cubics which are the Inverses of a Lemniscate with respect to its Vertices. By Professor Wolstenholme, M.A.

[*Read February 10th,* 1876.]

1. Let S, H (Fig. 1) be two fixed points; R a variable point moving so that SR.HR : $SH^2 = 1 : 4n$; and let the circle SRH meet the lemniscate which is the locus of R again in R′ (on the same side of SH as R). Then let P, P′, Q, Q′ be points on this circle such that the distance of any one of them from R is a mean proportional between its distances from S, H; Q being always taken on the arc HR, Q′ on the arc RS, P, P′ on the arc of the circle opposite R, P being the one nearer to H in passing from H to S along this arc. This paper proposes to investigate the loci of these points and to prove various relations between their loci and the locus of R.

To determine the position of these points for a given position of R, refer the system to areal coordinates measured on the triangle RHS (these answering for the sake of familiar notation to A, B, C, so that for the present HS$=a$, SR$=b$, RH$=c$, where $a^2 = 4nbc$. Then, at Q, x, y will be positive and z negative, and we shall have, from well known properties of the circle,

$$x : y : z = \frac{a}{RQ} : \frac{b}{HQ} : \frac{-c}{SQ}$$

or, since $RQ^2 = SQ.HQ$, $\quad \frac{x^2}{a^2} = \frac{-yz}{bc}$;

but at any point of the circle

$$a^2yz + b^2zx + c^2xy = 0,$$

whence, at Q,

$$-bcx^2 + b^2zx + c^2xy = 0, \quad \text{or} \quad x = \frac{b}{c}z + \frac{c}{b}y.$$