lastly, the singular lines generate a scroll, which is common to every congruence that represents a line.

12. The special cases where the thirteen conditions are such that the surface representing an arbitrary point in either space is of the *first class*, are particularly interesting; since they furnish us with examples of the rational transformation of the two spaces, and consequently, also, of the point to point representation of surfaces on planes.

13. A fuller exposition of the method pursued, and a more complete statement of the results obtained thereby, are reserved for a future communication.

A New View of the Porism of the In- and Circum-scribed Triangle.

By J. WOLSTENHOLME, M.A.

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Starting with the system of equations

$$\frac{\tan\dfrac{\beta+\gamma}{2}}{\tan\alpha} = \frac{\tan\dfrac{\gamma+\alpha}{2}}{\tan\beta} = \frac{\tan\dfrac{\alpha+\beta}{2}}{\tan\gamma} = p \quad \text{............ (1)}$$

(which is equivalent to only a two-fold relation between α, β, γ), I propose to investigate the different forms of equations equivalent to these; and afterwards to give a geometrical interpretation, which gives a complete account of the porism of the in- and circum-scribed triangle, to a pair of coaxal conics.

The angles α, β, γ are throughout supposed unequal and less than 2π.

From (1) we get at once

$$\tan\frac{\beta-\gamma}{2} = \tan\left(\frac{\alpha+\beta}{2} - \frac{\alpha+\gamma}{2}\right) = \frac{p(\tan\gamma - \tan\beta)}{1+p^2\tan\beta\tan\gamma}$$

$$= \frac{p\sin(\gamma-\beta)}{\cos\beta\cos\gamma + p^2\sin\beta\sin\gamma},$$

or $$\cos\beta\cos\gamma + p^2\sin\beta\sin\gamma = -p\{1+\cos(\beta-\gamma)\},$$

or $$\cos\beta\cos\gamma + p\sin\beta\sin\gamma + \frac{p}{1+p} = 0 \quad \text{............... (2)},$$

and the two like equations.

From (2) we get, since β, γ are the two roots of the equation,

$$\cos\alpha\cos\theta + p\sin\alpha\sin\theta + \frac{p}{1+p} = 0,$$

$$\frac{\cos\dfrac{\beta+\gamma}{2}}{\cos\alpha} = \frac{\sin\dfrac{\beta+\gamma}{2}}{p\sin\alpha} = \frac{\cos\dfrac{\beta-\gamma}{2}}{-\dfrac{p}{1+p}};$$

whence the two systems

$$\frac{\cos\frac{\beta+\gamma}{2}}{\cos\alpha\cos\frac{\beta-\gamma}{2}}=\frac{\cos\frac{\gamma+\alpha}{2}}{\cos\beta\cos\frac{\gamma-\alpha}{2}}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\gamma\cos\frac{\alpha-\beta}{2}}=-\frac{1+p}{p}\ \ldots\ldots (3),$$

$$\frac{\sin\frac{\beta+\gamma}{2}}{\sin\alpha\cos\frac{\beta-\gamma}{2}}=\frac{\sin\frac{\gamma+\alpha}{2}}{\sin\beta\cos\frac{\gamma-\alpha}{2}}=\frac{\sin\frac{\alpha+\beta}{2}}{\sin\gamma\cos\frac{\alpha-\beta}{2}}=-(1+p)\ \ldots (4).$$

Next consider the system

$$\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$=\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=q,$$

which give the equation in β, γ,

$$\left(\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right)\left(\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\right)$$

$$=\left(q-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)\left(q-\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\right),$$

or $$4\sin^2\frac{\beta+\gamma}{2}=\left\{(q-1)\cos\frac{\beta-\gamma}{2}+(q+1)\cos\frac{\beta+\gamma}{2}\right\}$$

$$\times\left\{(q-1)\cos\frac{\beta-\gamma}{2}-(q+1)\cos\frac{\beta+\gamma}{2}\right\},$$

or $$4\{1-\cos(\beta+\gamma)\}=(q-1)^2\{1+\cos(\beta-\gamma)\}$$

$$-(q+1)^2\{1+\cos(\beta+\gamma)\},$$

or $$2(q-1)\cos\beta\cos\gamma+(1-q^2)\sin\beta\sin\gamma+2(1+q)=0,$$

which is equivalent to (2) if

$$p=-\frac{1+q}{2},\ \text{ or }\ q=-(2p+1).$$

Hence, the system

$$\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$=\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-(2p+1)\ \ldots (5).$$

One of the system (1) gives us

$$\tan\frac{\gamma+\alpha}{2}\tan\gamma=\tan\frac{\alpha+\beta}{2}\tan\beta;$$

whence
$$\frac{\cos\frac{\gamma-\alpha}{2}}{\cos\frac{3\gamma+\alpha}{2}} = \frac{\cos\frac{\beta-\alpha}{2}}{\cos\frac{3\beta+\alpha}{2}},$$

or
$$\cos\frac{3\beta+\gamma}{2} + \cos\left(\frac{3\beta-\gamma}{2}+\alpha\right) = \cos\frac{3\gamma+\beta}{2} + \cos\left(\frac{3\gamma-\beta}{2}+\alpha\right),$$

or
$$2\sin\frac{\beta-\gamma}{2}\sin(\beta+\gamma) = \sin(\gamma-\beta)\sin\left(\frac{\beta+\gamma}{2}+\alpha\right),$$

or
$$\sin(\beta+\gamma)+2\cos\frac{\beta-\gamma}{2}\sin\left(\frac{\beta+\gamma}{2}+\alpha\right) = 0,$$

or
$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha\quad\beta) = 0.$$

Also, if
$$x = \tan\frac{\alpha}{2},\quad y = \tan\frac{\beta}{2},\quad z = \tan\frac{\gamma}{2},$$

$$\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = \frac{(1+yz)(1+zx)(1+xy)}{(1+x^2)(1+y^2)(1+z^2)};$$

and since
$$yz+zx+xy = \frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy} = q,$$

$$\text{the numerator} = (1+q)(1+x^2y^2z^2),$$

and the denominator
$$= 1+(q^2x^2y^2z^2-2q)+(q^2-2qx^2y^2z^2)+x^2y^2z^2 \equiv (1-q)^2(1+x^2y^2z^2),$$

or
$$\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = \frac{1+q}{(1-q)^2} = -\frac{p}{2(1+p)^2}.$$

Hence the system
$$\left.\begin{aligned} \sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta) &= 0 \\ 2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} &= \frac{-p}{(1+p)^2} \end{aligned}\right\} \quad \ldots\ldots (6).$$

Since $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta) = 0$, we have
$$\sin(\alpha+\beta+\gamma)(\cos\alpha+\cos\beta+\cos\gamma)$$
$$-\cos(\alpha+\beta+\gamma)(\sin\alpha+\sin\beta+\sin\gamma) = 0,$$

and we may write
$$\cos\alpha+\cos\beta+\cos\gamma = m\cos(\alpha+\beta+\gamma),$$
$$\sin\alpha+\sin\beta+\sin\gamma = m\sin(\alpha+\beta+\gamma);$$

and multiplying by $\cos\alpha$, $\sin\alpha$, and adding, we get
$$1+\cos(\alpha-\beta)+\cos(\alpha-\gamma) = m\cos(\beta+\gamma);$$

whence
$$1+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)$$
$$= \cos(\beta-\gamma)+m\cos(\beta+\gamma)$$
$$= 4\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = -\frac{2p}{(1+p)^2},$$

which will coincide with (2) if

$$\frac{1-m}{1+m}=p, \quad \text{or} \quad m=\frac{1-p}{1+p}.$$

Hence

$$\left.\begin{aligned} \cos\alpha+\cos\beta+\cos\gamma &= \frac{1-p}{1+p}\cos(\alpha+\beta+\gamma) \\ \sin\alpha+\sin\beta+\sin\gamma &= \frac{1-p}{1+p}\sin(\alpha+\beta+\gamma) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7).$$

This leads also to $\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)$

$$= \cos(\alpha+\beta+\gamma)(\cos\alpha+\cos\beta+\cos\gamma)$$
$$+\sin(\alpha+\beta+\gamma)(\sin\alpha+\sin\beta+\sin\gamma) = \frac{1-p}{1+p};$$

and therefore, by (2), since

$$\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta+p(\sin\beta\sin\gamma+\ldots) = -\frac{3p}{1+p},$$

we have

$$\left.\begin{aligned} \cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta &= -\frac{p(p+2)}{(1+p)^2} \\ \sin\beta\sin\gamma+\sin\gamma\sin\alpha+\sin\alpha\sin\beta &= -\frac{2p+1}{(1+p)^2} \end{aligned}\right\} \quad \ldots\ldots (8).$$

Again, from (2), multiplying by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and adding, we have

$$3\cos\alpha\cos\beta\cos\gamma+p\{\cos\alpha\cos\beta\cos\gamma-\cos(\alpha+\beta+\gamma)\}$$

$$= -\frac{p}{1+p}(\cos\alpha+\cos\beta+\cos\gamma),$$

or

$$(3+p)\cos\alpha\cos\beta\cos\gamma = \left(p-\frac{p(1-p)}{(1+p)^2}\right)\cos(\alpha+\beta+\gamma)$$

$$= \frac{p^2(3+p)}{(1+p)^2}\cos(\alpha+\beta+\gamma),$$

or

$$\left.\begin{aligned} \cos\alpha\cos\beta\cos\gamma &= \frac{p^2}{(1+p)^2}\cos(\alpha+\beta+\gamma) \\ \text{and similarly} \quad \sin\alpha\sin\beta\sin\gamma &= -\frac{1}{(1+p)^2}\sin(\alpha+\beta+\gamma) \end{aligned}\right\} \quad \ldots\ldots (9).$$

I may observe that, since the original equations (1) are unaltered, if we write $\frac{\pi}{2}-\alpha$, $\frac{\pi}{2}-\beta$, $\frac{\pi}{2}-\gamma$ for α, β, γ, and $\frac{1}{p}$ for p, we may deduce several of the above equations by these substitutions.

Next consider the equation in θ,

$$p\frac{\cos\phi}{\cos\theta}+\frac{\sin\phi}{\sin\theta} = -(1+p),$$

a biquadratic, of which one root is $\pi+\phi$, and for the other three we

have
$$p\cos\frac{\phi+\theta}{2}\sin\theta+\sin\frac{\phi+\theta}{2}\cos\theta=0,$$

or $$p\left(1-z\tan\frac{\phi}{2}\right)2z+\left(z+\tan\frac{\phi}{2}\right)(1-z^2)=0,$$
writing $z=\tan\frac{\theta}{2}$,

or
$$z^3+z^2(2p+1)\tan\frac{\phi}{2}-(2p+1)z-\tan\frac{\phi}{2}=0;$$

whence we see, by (5), that the roots may be taken to be α, β, γ; and if α, β, γ, δ be the four roots of the original equation in θ, it is readily proved that $\alpha+\beta+\gamma+\delta=\pi$ (or an odd multiple of π), so that $\alpha+\beta+\gamma+\phi=0$ (or $2r\pi$); whence we see also that

$$\left.\begin{aligned}\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}&=-(2p+1)\tan\frac{\phi}{2}\\&=(2p+1)\tan\frac{\alpha+\beta+\gamma}{2}\\ \text{and}\quad\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}&=-(2p+1)\cot\frac{\phi}{2}\\&=(2p+1)\cot\frac{\alpha+\beta+\gamma}{2}\end{aligned}\right\}\quad\ldots\ldots(10).$$

A similar relation between the tangents will follow at once from (7), since

$$\left.\begin{aligned}\tan\alpha+\tan\beta+\tan\gamma&=\frac{\sin\alpha\cos\beta\cos\gamma+\ldots}{\cos\alpha\cos\beta\cos\gamma}\\&=\frac{\sin(\alpha+\beta+\gamma)+\sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}\\&=\tan(\alpha+\beta+\gamma)\frac{1-\dfrac{1}{(1+p)^2}}{\dfrac{p^2}{(1+p)^2}}=\frac{p+2}{p}\tan(\alpha+\beta+\gamma)\\ \text{and}\quad\cot\alpha+\cot\beta+\cot\gamma&=(2p+1)\cot(\alpha+\beta+\gamma)\end{aligned}\right\}\quad\ldots\ldots(11).$$

These, however, would be better deduced from the equation in $\tan\theta$, deduced from $\tan\frac{\phi+\theta}{2}=-p\tan\theta$, which leads to ($z\equiv\tan\theta$, $c\equiv\tan\phi$),

$$p^2z^3+p(p+2)cz^2-(2p+1)z-c=0,$$

giving the two preceding, and also

$$\left.\begin{aligned}\tan\beta\tan\gamma+\tan\gamma\tan\alpha+\tan\alpha\tan\beta&=-\frac{2p+1}{p^2}\\ \cot\beta\cot\gamma+\cot\gamma\cot\alpha+\cot\alpha\cot\beta&=-p(p+2)\end{aligned}\right\}\quad\ldots\ldots(12).$$

It is probable that relations of the same form would be found for the tangents and cotangents of 2α, 2β, 2γ, &c., and the number of different forms in which the system may be written seems to be unlimited.

Now suppose along the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point $a\cos z$, $b\sin z$, we measure inwards a length PQ equal to k times the corresponding conjugate semi-diameter CD; the coordinates of Q will be $(a-kb)\cos z$, $(b-ka)\sin z$, and the equation determining the normals which can be drawn from this point will be

$$a(a-kb)\frac{\cos z}{\cos\theta}-b(b-ka)\frac{\sin z}{\sin\theta}=a^2-b^2,$$

and the roots of this equation, other than z, will be α, β, γ, if $p=-\frac{a(a-kb)}{b(b-ka)}$, the z of this equation differing by π from the ϕ of the former, so that $\alpha+\beta+\gamma=\pi-z$. The straight lines joining the points α, β, γ will all touch the ellipse whose semi-axes are

$$a'=\frac{a^2}{a^2-b^2}(a-kb),\quad b'=\frac{b^2}{b^2-a^2}(b-ka),$$

the coordinates of a point of contact being $a'\cos(\pi+\alpha)$, $b'\sin(\pi+\alpha)$; and the tangents at β, γ will intersect in the point $\mathrm{A}\cos(\pi+\alpha)$, $\mathrm{B}\sin(\pi+\alpha)$, where $\mathrm{A}=\frac{a^2}{a'}$, $\mathrm{B}=\frac{b^2}{b'}$. I have thought it better to take such values for a', b' that the equation connecting them may be $\frac{a'}{a}+\frac{b'}{b}=1$. This may give negative values for the lengths of the axes, but the coordinates of the angular points of the several triangles will be in all cases correct. Thus the tangents at β, γ intersect in the

point $$\frac{a\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}},\quad\frac{b\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}},$$

which, by (3), (4), is the point

$$-\frac{(1+p)}{p}\cos\alpha,\quad-(1+p)\sin\alpha,$$

or $$\frac{a^2-b^2}{(a-kb)}\cos(\pi+\alpha),\quad\frac{b^2-a^2}{b-ka}\sin(\pi+\alpha),$$

which is the point stated above. All cases of the in- and circumscribed triangles to two coaxal conics are particular cases of this. The following theorems are only geometrical interpretations of equations already investigated.

If ABC be a triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so that the normals at A, B, C meet in a point Q, and from Q the fourth normal QP be drawn, QP will bear to the semi-diameter CD conjugate to CP, the ratio $k : 1$, k being found from either of the equations

$$X = -(a-kb)\cos(\alpha+\beta+\gamma),\quad Y = (b-ka)\sin(\alpha+\beta+\gamma),$$

where (X, Y) are the coordinates of Q, and α, β, γ the excentric angles of ABC.

The sides of the triangle ABC will touch the ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$ in points whose excentric angles are $\pi+\alpha$, $\pi+\beta$, $\pi+\gamma$, and

$$a' = \frac{a^2}{a^2-b^2}(a-kb),\quad b' = \frac{b^2}{b^2-a^2}(b-ka).$$

The tangents at A, B, C will form a triangle whose angular points lie on the ellipse $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$, and whose excentric angles are $\pi+\alpha$, $\pi+\beta$, $\pi+\gamma$; A being $=\frac{a^2}{a'}$, $B=\frac{b^2}{b'}$.

An infinite number of such triangles ABC can be inscribed to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and circumscribed to the ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$, the excentric angles of the points A, B, C satisfying all the relations investigated, any two of which involve all the others. Hence the ratio k remains the same for all these triangles; and if A′, B′, C′ be the points of contact, the ratio of the areas of the triangles A′B′C′, ABC is always the same, being $a'b' : ab$ or $\frac{ab(a-kb)(b-ka)}{(a^2-b^2)^2} : 1$.

The four points related to each triangle—(1) the centroid, (2) the centre of perpendiculars, (3) the centre of the circumscribed circle, (4) the point of concourse of the normals—all lie on fixed ellipses co-axal with the original, and the excentric angle is always the excess of the sum of the excentric angles of A, B, C above π; while the semi-axes are

$$(1)\quad \frac{a(a^2+b^2-2kab)}{3(a^2-b^2)},\quad \frac{b(a^2+b^2-2kab)}{3(a^2-b^2)};$$

$$(2)\quad \frac{(a^2+b^2)(a^2+b^2-2kab)+(a^2-b^2)^2}{2a(a^2-b^2)},$$

$$\frac{(a^2+b^2)(a^2+b^2-2kab)+(a^2-b^2)^2}{2b(a^2-b^2)};$$

$$(3)\quad \frac{b(b-ka)}{2a},\quad \frac{a(kb-a)}{2b};\qquad (4)\quad a-kb,\ ka-b.$$

Of course also the corresponding points for the triangle A′B′C′, and for the triangle formed by the tangents at A, B, C, will trace out corresponding ellipses which may be found by writing a', b', k' for a, b, k, where

$$a' = \frac{a^2}{a^2-b^2}(a-kb), \quad b' = \frac{b^2}{b^2-a^2}(b-ka), \quad \frac{a'(a'-k'b')}{b'(b'-k'a')} = \frac{a(a-kb)}{b(b-ka)},$$

whence
$$k+k' = \frac{a^2+b^2}{ab};$$

or, by writing A, B, K for a, b, k, where $A = \frac{a^2}{a'}$, $B = \frac{b^2}{b'}$,

$$\frac{A(A-KB)}{B(B-KA)} = \frac{a(a-kb)}{b(b-ka)},$$

whence
$$Kk = 1 - \frac{ab(1-k^2)^2}{(a-kb)(b-ka)}.$$

If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be given, the envelope of the coaxal ellipses to which the triangles are circumscribed is $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$; and that of the ellipses, on which lie the corners of the triangles formed by the tangents at A, B, C, is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$. The envelope of the locus of the point of concourse of the normals at ABC is of course the evolute of the given ellipse, since at a point on the evolute two values of k will become equal.

If the tangents at ABC form a triangle $A_1B_1C_1$, and the tangents at $A_1B_1C_1$ to their locus a triangle $A_2B_2C_2$, and so on, we shall have for the locus of $A_nB_nC_n$ an ellipse whose axes are $a\lambda^n$, $b\mu^n$, where

$$\lambda = \frac{A}{a} = \frac{a^2-b^2}{a(a-kb)}, \quad \mu = \frac{b^2-a^2}{b(b-ka)}, \quad \text{so that} \quad \frac{1}{\lambda}+\frac{1}{\mu} = 1,$$

and if k_n be the value of the quantity corresponding to k in this ellipse

$$\frac{a\lambda^n(a\lambda^n - k_n b\mu^n)}{b\mu^n(b\mu^n - k_n a\lambda^n)} = \frac{a(a-kb)}{b(b-ka)} = -\frac{\mu}{\lambda},$$

or
$$k_n = \frac{a^2\lambda^{2n+1} + b^2\mu^{2n+1}}{(\lambda\mu)^n(\lambda+\mu)ab}.$$

Sufficient equations have been found to determine the loci and envelopes of any inscribed and circumscribed triangles, and of the points connected with them. Perhaps the centre of the Nine Points' Circle might have been added, but as this is the centre of the line joining the points (1) and (2), its locus will manifestly be of the same kind.

As it may interest some readers to see equivalent algebraical equations similarly treated, I append, at Prof. Cayley's suggestion, the greater number of the results above obtained, now deduced algebraically from a single system of three equations, equivalent, of course, to two only. If we write x, y, z for $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$ throughout, we shall get a series of algebraical systems of equations, each equivalent to two relations only. The original system (1) becomes

$$\left.\begin{aligned}(1-x^2)(y+z) &= 2p(x-xyz)\\(1-y^2)(z+x) &= 2p(y-xyz)\\(1-z^2)(x+y) &= 2p(z-xyz)\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (\alpha);$$

and multiplying these by $y-z$, $z-x$, $x-y$, and adding, we have an identity, proving that the system is equivalent to a two-fold relation only. Subtracting, we get from any pair

$$yz+zx+xy = -(2p+1) \equiv -q;$$

and multiplying the second by z, and the third by y, and subtracting,

we get $$\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy} = -(2p+1) \quad\ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

These two, giving the relations between x, y, z in the simplest form, are equivalent to (5).

If we eliminate x between these, we shall get a relation which must be equivalent to the first of (2), and we thus have the system

$$\left.\begin{aligned}(y+z)^2 &= (q+yz)(1+qyz)\\(z+x)^2 &= (q+zx)(1+qzx)\\(x+y)^2 &= (q+xy)(1+qxy)\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

From the last two of these (showing that y, z are the two roots of a certain quadratic), we have

$$\frac{1}{1-qx^2} = \frac{y+z}{x(q^2-1)} = \frac{yz}{x^2-q},$$

and therefore $$= \frac{1+yz}{(1+x^2)(1-q)} = \frac{1-yz}{(1-x^2)(1+q)},$$

or $$\frac{(1-yz)(1+x^2)}{(1+yz)(1-x^2)} = \frac{1+q}{1-q} = -\frac{p+1}{p},$$

and the system $$\frac{(1-yz)(1+x^2)}{(1+yz)(1-x^2)} = \frac{(1-zx)(1+y^2)}{(1+zx)(1-y^2)}$$

$$= \frac{(1-xy)(1+z^2)}{(1+xy)(1-z^2)} = -\frac{1+p}{p} \quad\ldots\ldots\ldots\ldots\ldots\ldots (\delta),$$

equivalent to (3).

Combining this with (α), we get

$$\frac{(y+z)(1+x^2)}{2x(1+yz)} = \frac{(z+x)(1+y^2)}{2y(1+zx)} = \frac{(x+y)(1+z^2)}{2z(1+xy)} = -(1+p)\ldots(\epsilon),$$

equivalent to (4).

Of course any of these systems of three equations are equivalent to two only; and, any two being assumed, the third is readily deducible from them.

The system (6) is equivalent to

$$\left.\begin{aligned}(y+z)(1-yz)(1+x^2)+(z+x)(1-zx)(1+y^2)\\ +(x+y)(1-xy)(1+z^2)=0\\ (1+yz)(1+zx)(1+xy)=-\frac{p}{2(1+p)^2}(1+x^2)(1+y^2)(1+z^2)\end{aligned}\right\}\ \dots(\zeta),$$

of which the first is identically

$$x+y+z=xyz(yz+zx+xy),$$

and the second was proved algebraically before.

If we denote $x+y+z$, $yz+zx+xy$, and xyz by a, b, c, the given relations are equivalent to $b=\frac{a}{c}=-(2p+1)$. Expressing the fraction

$$\frac{(1-x^2)(1+y^2)(1+z^2)+(1-y^2)(1+z^2)(1+x^2)+(1-z^2)(1+x^2)(1+y^2)}{(1-x^2)(1-y^2)(1-z^2)-4yz(1-x^2)-4zx(1-y^2)-4xy(1-z^2)}$$

in terms of a, b, c, and then substituting for a its equivalent bc, this becomes

$$\frac{(3-2b-b^2)(1-c^2)}{(1-b)^2(1-c^2)}\left(=\frac{3+b}{1-b}=\frac{1-p}{1+p}\right).$$

Similarly

$$\frac{2x(1+y^2)(1+z^2)+2y(1+z^2)(1+x^2)+2z(1+x^2)(1+y^2)}{-8xyz+2x(1-y^2)(1-z^2)+2y(1-z^2)(1-x^2)+2z(1-x^2)(1-y^2)}$$

$$=\frac{2c(b^2+2b-3)}{-2c(b-1)^2}=\frac{1-p}{1+p},$$

which are the equivalents of

$$\frac{\cos\alpha+\cos\beta+\cos\gamma}{\cos(\alpha+\beta+\gamma)}=\frac{\sin\alpha+\sin\beta+\sin\gamma}{\sin(\alpha+\beta+\gamma)}=\frac{1-p}{1+p}.$$

In the same manner,

$$\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta$$

$$=\frac{(1+x^2)(1-y^2)(1-z^2)+\text{two like terms}}{(1+x^2)(1+y^2)(1+z^2)}$$

$$=\frac{3-(x^2+y^2+z^2)-(y^2z^2+z^2x^2+x^2y^2)+3x^2y^2z^2}{1+(\dots)+(\dots)+x^2y^2z^2}.$$

Now
$$x^2+y^2+z^2=a^2-2b=b^2c^2-2b,$$
$$y^2z^2+z^2x^2+x^2y^2=b^2-2ac=b^2-2bc^2,$$

so that the fraction

$$=\frac{(1+c^2)(3+2b-b^2)}{(1+c^2)(1-2b+b^2)}=\frac{(1+b)(3-b)}{(1-b)^2}=-\frac{p(p+2)}{(1+p)^2};$$

and $$\sin\beta\sin\gamma+\ldots = \frac{4yz(1+x^2)+\ldots}{(1+x^2)(1+y^2)(1+z^2)} = \frac{4(b+ac)}{(1+c^2)(1-b)^2}$$
$$= \frac{4b}{(1-b)^2} = -\frac{(2p+1)}{(1+p)^2},$$

$$\frac{\cos\alpha\cos\beta\cos\gamma}{\cos(\alpha+\beta+\gamma)} = \frac{(1-x^2)(1-y^2)(1-z^2)}{(1-x^2)(1-y^2)(1-z^2)-4yz(1-x^2)-\ldots}$$
$$= \frac{1-b^2c^2+2b+b^2-2bc^2-c^2}{(1-b)^2(1-c^2)} = \left(\frac{1+b}{1-b}\right)^2 = \left(\frac{p}{1+p}\right)^2,$$

and $$\frac{\sin\alpha\sin\beta\sin\gamma}{\sin(\alpha+\beta+\gamma)} = \frac{8xyz}{-8xyz+2x(1-y^2)(1-z^2)+\ldots}$$
$$= \frac{8c}{-8c+2a-2(ab-3c)+2bc} = \frac{8c}{-2c+4bc-2b^2c}$$
$$= \frac{-4}{(1+b)^2} = -\frac{1}{(1+p)^2}.$$

Many more such equivalent systems might be obtained, but I think the above are the most remarkable. The great simplification of some of the fractions, by the use of the single relation

$$x+y+z = xyz(yz+zx+xy),$$

is certainly deserving of attention.

December 10*th*, 1874.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Special Meeting.

The Chairman having stated the purposes for which the Meeting had been made a "Special" one, in accordance with the Resolution carried at the November Meeting, called upon Mr. Merrifield to move the two following Resolutions:—

"That in future there shall be an entrance-fee of one guinea; and

"That the life-composition shall be raised from £10, its present amount, to fifteen guineas."

It was resolved that the Resolutions should be taken separately. After some discussion, in which Mr. Harley, Dr. Hirst, and others took part, the first Resolution was carried by a large majority.

A protracted discussion followed upon the second Resolution; and at last an amendment, proposed by Mr. Harley, and seconded by Mr.

J. W. L. Glaisher, was carried. This amendment recommended that the life-composition be changed from £10 to ten guineas (on account of the annual subscription being now one guinea).

On the motion of Mr. Harley, seconded by Dr. Hirst, Rule 36* was ordered to be abolished.

The proposal to substitute "session" for "year," in Rule 19, was, at the suggestion of Mr. Stirling, ordered to be deferred, that the bearing of the proposed alteration upon some of the other rules might be considered.

The Meeting then became a "General" one, and Mr. W. D. Niven was admitted into the Society. Messrs. Hart and Nanson were elected Members of the Society, and the following gentlemen were nominated for Membership:—John Wesley Russell, B.A., Fellow of Merton College, Oxford; Charles M. Leudesdorf, B.A., Fellow of Pembroke College, Oxford; Edwin Bailey Elliott, B.A., Fellow of Queen's College, Oxford; H. M. Jeffery, M.A., of Cheltenham; Charles Smith, M.A., Fellow of Sidney Sussex College, Cambridge; and Benjamin Williamson, M.A., Fellow and Tutor of Trinity College, Dublin.

The Auditor (Mr. Stirling) stated that he had examined the Treasurer's accounts, and found them perfectly correct.

The Chairman, on the recommendation of the Council, nominated Drs. Klein, Kronecker, and Zeuthen, for the honour of Foreign Membership.

Prof. Cayley then read his paper on "the Potentials of Polygons and Polyhedra."

Mr. Tucker (in the absence of J. J. Sylvester, Esq., F.R.S.) gave a sketch of the contents of two letters from M. Mannheim, on "Three and Seven Bar Motion."

The following presents were received:—

"Memoir on the Transformation of Elliptic Functions," by Prof. Cayley: from the Author (from the Phil. Trans., read Jan. 8, 1874).

Carte de Visite of Mr. J. Walmsley.

On the Potentials of Polygons and Polyhedra. By Prof. Cayley.

[Read December 10th, 1874.]

The problem of the attraction of polyhedra is treated of by Mehler, "Crelle," t. 66, pp. 375—381 (1866); but the results here obtained are exhibited under forms which are very different from his, and which give rise to further developments of the theory.

* Rule 36.—At no two successive Meetings shall the papers be entirely on Applied Mathematics.