



XXX. On the development of certain trigonometrical functions

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parts of the country, have added some new and very interesting minerals to our former catalogue. The following is a list of these :

Variolite.—This variety of compact felspar has been lately met with in the hornblende rock of Morne.

Anthracite.—A compact variety of this mineral, with a highly metallic lustre, occurs frequently in the grauwacké of Down.

White Carbonate of Lead—accompanies galena and the green phosphate of lead in the Newtonard's lead-mine, Down.

Colophonite.—This mineral occurs in quartz veins traversing siliceous slate near Glassdrummond, Morne. It is of a brownish yellow colour, and is crystallized in rhombic dodecahedrons, with very unequal angles, and having striæ parallel to the lesser axis of the rhomboid. The mineral has very much the appearance of cinnamon stone.

Sulphuret of Molybdena.—This mineral has been lately discovered in a siliceous slate, or, I believe, rather a chlorite slate, on the shore near the mountains of Morne. The crystalline form is a six-sided table, terminated by a low six-sided pyramid, of which the base angles are truncated. Neither this mineral nor the colophonite have, so far as I know, been before noticed in Ireland.

Yours, &c.

Belfast, Aug. 11, 1834.

JAMES BRYCE, JUN.

XXX. *On the Development of certain Trigonometrical Functions.* By J. R. YOUNG, Professor of Mathematics in the Royal College, Belfast.*

THE series given by analytical writers for the development of a circular arc in terms of its sine, cosine, and tangent are as follows :

$$\sin^{-1}x = x + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{3^3 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} +, \&c.$$

$$\cos^{-1}x = \frac{\pi}{2} - x + \frac{x^3}{1 \cdot 2 \cdot 3} - \frac{3x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} +, \&c.$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} +, \&c.$$

which series, as well as those for the development of the arc in terms of the other trigonometrical lines, are true only for the least of the arcs to which these trigonometrical lines belong. I am not aware that any explanation has ever been given of this want of generality; or that any one has made known how it comes to pass, that while the developments of

* Communicated by the Author.

$\sin x$, $\cos x$, &c., in terms of x are true in all cases, yet the developments of the inverse functions, which, *à priori*, we should expect to possess equal generality, turn out to be true only in one particular case. In allusion to this circumstance Lacroix remarks: "Les series qui expriment $\sin x$ et $\cos x$ par l'arc, sont plus générales que leurs inverses, qui expriment l'arc par le sinus ou le cosinus: les dernières ne conduisent qu'au plus petit des arcs qui ont le même sinus ou le même cosinus, tandis que les premières donnent le sinus ou le cosinus, quel que soit celui de ces arcs qu'on prenne pour x ."—*Lacroix, Calcul Diff. et Int.*, tom. iii. p. 620.

Now it is the object of this short paper to show that the series for $\sin^{-1}x$, $\cos^{-1}x$, &c., when properly investigated, possess the same generality as those for $\sin x$, $\cos x$, and that the defective forms above arise from an oversight committed in the analytical processes whence they are deduced.

If we turn to Lacroix, or to any other writer on the Calculus, we find the investigation of the development of $y = \sin^{-1}x$ conducted as follows:

$$\begin{aligned} y &= \sin^{-1}x \\ \frac{dy}{dx} &= (1-x^2)^{-\frac{1}{2}} \\ \frac{d^2y}{dx^2} &= x(1-x^2)^{-\frac{3}{2}} \\ \frac{d^3y}{dx^3} &= (1-x^2)^{-\frac{5}{2}} + 3x^2(-x^2)^{-\frac{5}{2}} \\ &\quad \&c. \qquad \&c. \end{aligned}$$

The values of these expressions for $x = 0$ are said to be

$$y = 0, \quad \frac{dy}{dx} = 1, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 1, \quad \&c.;$$

and hence, by Maclaurin's theorem, the first of the above developments is inferred. Now when $x = 0$ it is not necessarily true that $y = 0$; it is true only when the proposed arc is less than a quadrant. If the arc be greater than a quadrant, and terminate in the second quadrant, then the value of it for $x = 0$ can obviously be no other than $y = \pi$; if it terminate in the fourth quadrant, its value for $x = 0$ must be $y = 2\pi$; if in the fifth, $y = 3\pi$, and so on*. It must also be observed that when the arc terminates in the second quadrant,

* We are here considering only the positive arcs; the negative arcs having the same sign will terminate in the 3rd, 4th, 7th, &c., quadrants, going round the circle in the opposite direction; and the corresponding values of y , for $x = 0$, will obviously be $-\pi$, -2π , -3π , &c., the same as before, but with opposite signs.

$\frac{dy}{dx}$, and therefore the following coefficients, are negative; when it terminates in the fourth quadrant they are positive, as at first; when in the fifth negative, and so on. Hence the general expression for the development of $\sin^{-1}x$, terminating in the $k+1$ th quadrant, is

$$\sin^{-1}x = k\pi \pm \left\{ x + \frac{x^3}{1.2.3} + \frac{3^2 x^5}{1.2.3.4.5} + \frac{3^2 5^2 x^7}{1.2.3.4.5.6.7} + \&c. \right\}$$

the upper sign being used when $k+1$ is odd, and the lower when it is even, $k+1$ being either positive or negative.

The series for $y = \cos^{-1}x$ is inferred from the following conditions, which are said to have place when $x = 0$, viz.

$$y = \frac{\pi}{2}, \quad \frac{dy}{dx} = -1, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 1, \quad \&c.$$

But the first of these conditions is true only when the proposed arc, y , terminates in the first quadrant. If it terminate in the fourth, y must be $\frac{3\pi}{2}$, when $x = 0$; if it terminate in the fifth, y must be $\frac{5\pi}{2}$; if in the eighth, y must be $\frac{7\pi}{2}$, and so on. Moreover, in the fourth quadrant $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ &c., have signs contrary to those which they have in the first; in the fifth the signs are the same as in the first; in the eighth opposite, and so on. Hence the general expression for the development of $\cos^{-1}x$, is

$$\cos^{-1}x = \frac{(2k+1)\pm 1}{2} \cdot \frac{\pi}{2} \mp \left\{ x - \frac{x^3}{1.2.3} + \frac{3x^5}{1.2.3.4.5} - \&c. \right\}$$

the arc terminating in the $k+1$ th quadrant; the upper sign being used when $k+1$ is odd, and the lower when it is even. It is obvious that if the arc be negative this expression will take the minus sign.

By attending to similar considerations we shall find, for the development of $\tan^{-1}x$, the general expression

$$\tan^{-1}x = k\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} +, \quad \&c.,$$

the arc always terminating in the $2k+1$ th quadrant.

The development of $\tan^{-1}x$ is frequently deduced by the aid of a certain logarithmic formula, without the application of the calculus; the resulting form is, however, limited to the single case above mentioned; and this limitation arises, as in the former process, from an oversight in the investigation, although one of a very different kind. The investigation we

have in view is as follows. Setting out from the known formulas—(See Young's Diff. Calc., p. 30.)

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1} \sin x$$

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1} \sin x,$$

and taking the logarithms, we have

$$x\sqrt{-1} = \log (\cos x + \sqrt{-1} \sin x)$$

$$-x\sqrt{-1} = \log (\cos x - \sqrt{-1} \sin x);$$

therefore, by subtraction,

$$2x\sqrt{-1} = \log \frac{\cos x + \sqrt{-1} \sin x}{\cos x - \sqrt{-1} \sin x} = \log \frac{1 + \sqrt{-1} \tan x}{1 - \sqrt{-1} \tan x}.$$

$$\text{Now, } \log \frac{1+u}{1-u} = 2 \left\{ u + \frac{u^3}{3} + \frac{u^5}{5} + \frac{u^7}{7} +, \&c. \right\}$$

hence, substituting $\sqrt{-1} \tan x$ for u , we have

$$2x\sqrt{-1} = 2 \left\{ \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \frac{\tan^7 x}{7} +, \&c. \right\} \sqrt{-1}$$

$$\therefore x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \frac{\tan^7 x}{7} +, \&c.$$

The reason that we have obtained this defective form arises from an omission in the general expression for the logarithm of $\frac{1+u}{1-u}$, which omission would lead to no error if we were dealing with *real* quantities only. It was proved by Euler, that every number has, besides the logarithm usually considered, an infinite number of others, all imaginary*: thus, if we represent the usually received value of $\log A$ by $\text{Log } A$, then will the general expression for $\log A$ be

$$\log A = \text{Log } A + 2k\pi\sqrt{-1}.$$

When, therefore, we are dealing with imaginary quantities, we are not warranted in omitting the imaginary values of $\log A$, as is done in the foregoing process. Restoring, therefore, what has been improperly omitted, we have

$$\log \frac{1+u}{1-u} = 2 \left\{ u + \frac{u^3}{3} + \frac{u^5}{5} + \frac{u^7}{7} +, \&c. \right\} + 2k\pi\sqrt{-1}$$

* This may be proved as follows. By Young's Calculus, page 33,

$$\sqrt{-1} = e^{\frac{\pi}{2}\sqrt{-1}} \therefore -1 = e^{\pi\sqrt{-1}} \therefore 1 = e^{2k\pi\sqrt{-1}} \therefore A = A e^{2k\pi\sqrt{-1}} \\ \therefore \log A = \log A + 2k\pi\sqrt{-1}.$$

$$\therefore 2x\sqrt{-1} = 2 \left\{ \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \&c. \right\} \sqrt{-1} \\ + 2k\pi\sqrt{-1}$$

$$\therefore x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \&c., + k\pi$$

as before obtained.

Many other developments, besides this, are deduced in an incomplete form from neglecting imaginary logarithms. Take the following from among many that might be selected. Lacroix, at p. 136, vol. i. of his *Calculus*, develops the expression $(\sqrt{-1})^{\sqrt{-1}}$ as follows :

Substitute $\sqrt{-1}$ for u in the known formula

$$\log u = u - u^{-1} - \frac{1}{2}(u^2 - u^{-2}) + \frac{1}{3}(u^3 - u^{-3}) - \&c.,$$

and it becomes

$$\log \sqrt{-1} = \sqrt{-1} - \frac{1}{\sqrt{-1}} - \frac{1}{2}(-1+1) + \frac{1}{3}(-\sqrt{-1} \\ + \frac{1}{\sqrt{-1}}) - \&c.$$

$$= \frac{-2}{\sqrt{-1}} \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. \right\}$$

$$= \frac{-2}{\sqrt{-1}} \cdot \frac{\pi}{4} = \frac{\pi}{2} \sqrt{-1}$$

$$\therefore \sqrt{-1} \log \sqrt{-1} = -\frac{\pi}{2} \therefore (\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{\pi}{2}}.$$

This result is incomplete, for it is known that

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{4k+1}{2}\pi};$$

and this is the result which we should have obtained if the imaginary quantity $2k\pi\sqrt{-1}$ had not been improperly omitted; for we should then have had

$$\log \sqrt{-1} = \frac{\pi}{2} \sqrt{-1} + 2k\pi\sqrt{-1}$$

$$\therefore \sqrt{-1} \log \sqrt{-1} = -\frac{\pi}{2} - 2k\pi = -\frac{4k+1}{2}\pi.$$

$$\therefore (\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{4k+1}{2}\pi}.$$

These instances not only verify Euler's theory of imaginary

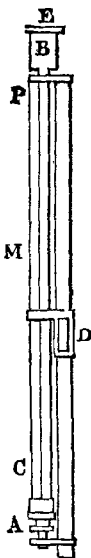
logarithms, but at the same time show their use in analytical investigations.

July 1834.

J. R. YOUNG.

XXXI. *Description of an Improvement in the Construction of Say's Instrument for measuring Specific Gravities. By W. H. M.**

A DESCRIPTION of this instrument as originally constructed, and which might with more propriety be called an instrument for measuring volumes, is given by its inventor, M. Say, a French officer of engineers, in one of the volumes of the *Annales de Chimie* for 1797 (xxiii, i.), and also by Mr. Faraday in his work on Chemical Manipulation. The annexed figure represents the instrument in its improved form. A B are two glass tubes, of 0.2 inch internal diameter, one 34 and the other 35 inches long, placed close together, and having their lower ends cemented into an iron cap, into the lower end of which an iron stop-cock is screwed. The upper end of the longer tube is cemented into the bottom of a cup B, having its rim ground truly plane, the capacity of which is a little more than half that of the tube. The tubes are fixed parallel to a graduated scale, carrying a sliding vernier D, provided with an index formed of two slips of brass, one before and the other behind the tubes, having their lower edges in a plane perpendicular to the scale. E is a piece of plate-glass, having its lower surface greased, and large enough to close the mouth of the cup B. The instrument may either be fixed permanently in a vertical position against a wall, like a barometer, or else may have a broad foot with three screws, by which it may be rendered vertical for use.



The substance to be examined, which may be any solid liquid or powder that is not volatile, is placed in a small cup, which goes into the cup B. The stop-cock at A is closed, and mercury is poured through a small funnel into the shorter tube till it rises to a mark on the longer tube at P; the mouth of the cup B is then closed, so as to be air-tight, by the plate of glass E. The stop-cock must now be opened, and the mercury permitted to escape in a small stream till its surface in the longer tube stands about 15 inches higher than in the shorter tube, when the stop-cock is to be closed.

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