On the Null Spaces of a One-System and its Associated Complexes.

By W. H. Young. Received November 3rd, 1898, and subsequently, in revised form, February 25th, 1899. Read November 10th, 1898.

A one-system lying in an odd space S_{2m-1} is known to be reducible in general to m one-vectors. Of these m one-vectors, r can be chosen to lie in any odd space S_{2r-1} of perfectly general position, the $S_{2m-1-2r}$ containing the remaining $(m-r)$ one-vectors being then determined, as also are the systems in the S_{2r-1} and $S_{2m-1-2r}$ respectively. A slightly different theorem holds good for even spaces.* I here occupy myself with the theory of the spaces for which these theorems require modification. Such spaces, which I have called the null spaces of the one-system, are of several species. An S_r, where r is less than m, may have any species from 1 to $\frac{r}{2}$ or $\frac{r+1}{2}$ inclusive, according as r is even or odd. An S_r of maximum species I call a *thoroughly* null space, since in this case none of the m one-vectors can be chosen to lie in it. The null lines of the one-system generate a linear complex, and conversely, given a linear complex, we may construct it by means of a one-system. The thoroughly null spaces are the *vollständige Raume* of the associated linear complex discussed by S. Kantor,† and generate, as he has shown, a linear $\infty^{(n-\frac{1}{2}r)(r+1)}$ complex. More generally, the null S_r's of species p generate a linear $\infty^{(n-r)(r+1)-\frac{1}{2}p'(p'+1)}$ complex, where p' is $2p$ or $(2p-1)$ according as r is even or odd.

The methods employed are those indicated in Grassmann's *Ausdehnungslehre*, an English exposition of which has been given by E. Lasker in papers on "The Geometrical Calculus," published in the *Proceedings* of this Society, Vol. XXVIII. All that is required for the present paper will, however, be found in the paper by me "On Flat-Space Coordinates" (*infra*, pp. 54–69).

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* *Cf.* "On Systems of One-Vectors in Space of n Dimensions," *Proc. Lond. Math. Soc.*, Vol. XXIX., Theorems VII. *b* and *c*.

† "Allgemeine Theorie der linearen Complexe," 1897, *Crelle's Journal.*

2. *Notation.*

Let $a_{12}, a_{13}, \ldots, a_{1,n+1}, a_{23}, a_{24}, \ldots, a_{2,n+1}, \ldots, a_{n,n+1}$ be a system of quantities, and let

$$a_{ij} = -a_{ji}.$$

We adopt the following notation

$$12 \equiv [12] \equiv a_{12},$$

$$[1234] \equiv 12\,.\,34-13\,.\,24+14\,.\,23,$$

$$\vdots$$

$$[123 \ldots 2r] = 12.34 \ldots [(2r-1)\,2r]-13.24 \ldots [(2r-1)\,2r]+\&c.,$$

where the rule for writing down the right-hand side is as follows:— Every possible interchange of numerals is made in the first term, every such interchange being accompanied by a change of sign; the right-hand side then consists of the sum of all terms so obtained.

It follows at once from the definition that

$$[12 \ldots 2r] = [12][34 \ldots 2r]-[13][24 \ldots 2r]+\ldots$$
$$\ldots+[1, 2r][23 \ldots 2r-1],$$

with a number of similar expansions.

It will be convenient to speak of the *dimensions of a square bracket;* thus r will be said to be the dimensions of the square bracket on the left-hand side of the above equation. It is evident that, if all the square brackets of any the same dimensions vanish, all those of higher dimensions also vanish.

When n is odd and equal to $2m-1$ the square of the square bracket o dimensions m is the skew determinant formed in the usual way from the quantities a_{ij}. When n is even the determinant is, of course, identically equal to zero. The square of any other square bracket, whether n be odd or even, is a minor of this determinant. From the known properties of determinants, we deduce a variety of theorems, *e.g.*, if, n being odd, we denote by b's the square brackets of dimensions $(m-1)$ in the a's, the square brackets of dimensions $(m-1)$ in the b's are proportional to the square brackets of dimensions *one* in the a's, *i.e.*, to the a's themselves. We have, in fact, the identical equation

$$[34 \ldots 2m]_b \equiv \{[12 \ldots 2m]_a\}^{m-2} a_{12},$$

where we have written subscripts to call attention to the letters used in forming the square brackets. This theorem will be of use in the sequel.

3. *Covariants of a System of One-Vectors.*

Consider any system of one-vectors, not necessarily reduced to a canonical form. Take every possible set of r non-intersecting one-vectors: each such set determines a $(2r-1)$-vector, defined, without ambiguity, by the $2r$-pyramid having those r one-vectors for non-intersecting edges. Proceeding thus, we obtain a $(2r-1)$-system, easily seen to be covariant with respect to the original one-system. Suppose the original one-system changed in any manner whatever to an equivalent one-system; then the laws of the composition and resolution of vectors show that at every step of the work the $(2r-1)$-system changes into an equivalent $(2r-1)$-system. Here r must be such that $(2r-1)$ is less than n. If $(2r-1)$ be equal to n, a similar process will give us a number of scalar quantities, whose sum remains invariable, which is therefore an invariant of the system.

Again, take any fixed q-vector, external to the system, and let q and r be such that $(2r+q)$ is less than n, and form $(2r+q)$-pyramids by joining up the q-vector to all the $(2r-1)$-vectors of the $(2r-1)$-system. We thus get a $(2r+q-1)$-system, which possesses also covariant character with respect to the original system. If $(2r+q)$ be equal to n, we have a system of scalar quantities whose sum is constant for the same q-vector and the same one-system.

4. *Coordinates of a Covariant System.*

There are two modes in which we find it convenient to change a one-system. In the first, we replace each one-vector by components along the edges of a fundamental $(n+1)$-pyramid. We thus get the system replaced by single one-vectors along the edges; the ratios of these to the one-vectors denoted by the edges themselves, taken in the order 12, 13, ..., 23, 24, ... always in ascending order of numerals, we shall call the coordinates of the one-system, and denote by the symbols a_{ij} of § 2. In the same way the coordinates of a $(2r-1)$-system are thus defined; replace the $(2r-1)$-system by $(2r-1)$-vectors in the S_{2r-1} faces of the fundamental pyramid; the coordinates of the system are defined to be the ratios of these vectors to the $(2r-1)$-vectors represented by the fundamental $2r$-pyramids in those faces, the vertices being taken in definite order, *e.g.*, $(12 \ldots 2r)$, so that the order of the numerals is always ascending.

Suppose, then, the one-system replaced in the first mode. Then we know the covariant $(2r-1)$-system is equivalent to that got by combining the one-vectors along every r non-intersecting edges of the fundamental pyramid. This gives us the coordinates of the $(2r-1)$-system.* For example, in the S_{2r-1} whose vertices are numbered $1, 2, \ldots, 2r$, the magnitude of the $(2r-1)$-vector bears to that of the corresponding fundamental $(2r-1)$-vector a ratio which is expressed by means of the square bracket $[12 \ldots 2r]$, and similarly for the other coordinates. That is, *the coordinates of the* $(2r-1)$-*system are the square brackets of dimensions r.* Since these coordinates depend only on $\frac{1}{2}n(n+1)$ quantities, while those of the general $(2r-1)$-system are $\frac{\overline{n+1}!}{2r!\,\overline{n+1-2r}!}$ in number, it is evident that these systems are of very special types, except when $r = m-1$, that is, except for the $(n-2)$-system.

5. *Degeneration of One-Systems.*

Now let us use the second mode of reduction, in which we replace the one-system by the minimum number, say k, of equivalent one-vectors; this we may call the reduction to a canonical form. Taking the k one-vectors r at a time, we get a system of $\frac{k!}{r!\,\overline{k-r}!}$ $(2r-1)$-vectors, forming a system of which the expressions already obtained are the coordinates.

It is thus evident that, when r is greater than k, the square brackets of dimensions r are all zero.

We can at once deduce the conditions that a given one-system should degenerate one or more times. We can, for example, write down the conditions that the one-system should be equivalent to k one-vectors, where k is any integer less than m.† In this case, and in this case only, the covariant system of $(2k+1)$-vectors does not exist. *Thus the necessary and sufficient conditions that a system should be equivalent to k one-vectors are that the square brackets of dimensions* $(k+1)$ *should vanish, and those of dimensions k should not vanish.* Further the $(2k-1)$-system is equivalent to a single vector, namely, that determined by the k one-vectors, and *the coordinates of the* S_{2k-1} *in which the k one-vectors lie are given by the square brackets of dimensions k.*

6. *Classification of Linear* S_{n-2} *Complexes.*

The equation to a linear S_{n-2} complex involves the $\frac{1}{2}n(n+1)$ coordinates of an S_{n-2}. These are connected by the well-known

* *Cf. infra*, § 16, p. 67.

† *Cf.* Lasker, *loc. cit.*

quadratic relations whose number is the number of combinations of $(n+1)$ things taken four at a time. The equation also contains $\frac{1}{2}n(n+1)$ arbitrary coefficients; these may be regarded as the homogeneous coordinates a_{ij} of a certain one-system defined by them. The equation expresses the fact that the S_{n-2}, whose coordinates it contains, is such that the moment* of the one-system about it vanishes. The nature of the complex depends then on the nature of the system of one-vectors. We are thus led to a classification of the S_{n-2} complexes, which may be said to degenerate† 1, 2, ..., k times, according as the auxiliary one-system is reducible to $(m-1)$, to $(m-2)$, ..., to $(m-k)$ one-vectors. These different classes of S_{n-2} complexes will accordingly be characterized by the vanishing of the square brackets of corresponding order.

7. *Degenerate S_{n-2} Complexes.*

The consideration of complexes in even space, and of degenerate complexes in general, is easily seen to be reducible to that of undegenerate complexes in odd space. Thus suppose the one-system equivalent to k one-vectors lying in an S_{2k-1}. The complex consists of all those S_{n-2}'s whose moment about the one-system is zero, and thus evidently includes all those S_{n-2}'s which contain the S_{2k-1}, or which meet it in an S_{2k-2}. Consider, however, an S_{n-2} which meets the S_{2k-1} in an S_{2k-3}; the necessary and sufficient condition that its moment about the one-system should vanish is that in the S_{2k-1} the moment of this S_{2k-3} should vanish, that is, that the S_{2k-3} of intersection should belong to the undegenerate complex in the S_{2k-1} determined by the k one-vectors. Thus we may generate the original complex (undegenerate if n be even and equal to $2k$), as follows:—*Describe the undegenerate S_{2k-3} complex of the S_{2k-1} which corresponds to the k one-vectors. Draw through its S_{2k-3}'s all possible S_{n-2}'s; further, if n be not equal to $2k$, complete the system by drawing all possible S_{n-2}'s through the S_{2k-1}, and through the S_{2k-2}'s contained in the S_{2k-1}; if n be equal to $2k$, the system has to be completed by taking all the S_{n-2}'s in the S_{2k-1}.*

8. *Classification of Linear Line Complexes.*

The results of the preceding two articles may be at once applied, by means of the principle of duality, to the theory of linear line complexes. In fact, we may map off such a complex unit by unit on to an S_{n-2} complex, corresponding line and S_{n-2} having the same

* See *infra*, §§ 5, 6, pp. 57–59.

† See note, p. 38.

coordinates, and the line complex and the dual S_{n-2} complex the same equation. Regarded as the equation to a line complex, the equation expresses the fact that the moment about every line of the complex of a certain $(n-2)$-system vanishes. We have only to modify the phraseology in the usual way. Thus, corresponding to an S_{n-2} complex degenerate k times, we have a line complex degenerate the like number of times.

In odd space $(n = 2m-1)$ corresponding to the auxiliary one-system of the S_{n-2} complex, lying in an $S_{2m-2k-1}$, we have an auxiliary $(n-2)$-system of the line complex, passing through an S_{2k-1}. It is usual to call such an S_{2k-1} the *centre* of the complex; its coordinates are the same as those of the $S_{2m-2k-1}$ of the auxiliary one-system of the S_{n-2} complex, and may therefore be at once written down.

In even space $(n = 2m)$, we have an auxiliary $(n-2)$-system of the line complex, passing through an S_{2k} centre, where k may be zero, in which case we have the undegenerate complex with its point centre.

Whether the space be odd or even we have the following result:—

*The conditions that a linear line complex should be degenerate k times are that all the square brackets of dimensions $(m-k+1)$ formed from the coefficients in the equation should vanish, and those of dimensions $(m-k)$ should not vanish, and the coordinates of the centre of such a degenerate complex are given by the square brackets of dimensions $(m-k)$. This is true for the undegenerate complex in even space $(n = 2m)$ if we put k equal to 0.**

9. *Degenerate Line Complexes and the Undegenerate Line Complex in Even Space.*

Consider a linear line complex degenerate k times, where, when $k = 0$, we understand the undegenerate line complex in even space. The complex has an $S_{n-2m+2k}$ centre. The auxiliary system of $(n-2)$-vectors then reduces to $(m-k)$ $(n-2)$-vectors, passing through the $S_{n-2m+2k}$ centre. Making use of the method of sections,† and cutting by an $S_{2m-2k-1}$, which does not meet the S_{2k-1}, we obtain a system of $(2m-2k-3)$-vectors lying in the $S_{2m-2k-1}$, whose moments about a unit vector in the S_{2k-1} are the original $(n-2)$-system. Form the

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* *Cf.* Segre, "Ricerche sulle omografie e sulle correlazione," *Memorie R. Acc. Torino*, 1885. It should be noted that in Segre's terminology "specialized" does not correspond exactly to "degenerate"; a line complex is specialized q times when it has an S_{q-1} centre. A line complex in even space is *always* specialized an odd number of times. The point of view from which we regard line complexes leads naturally to the adoption of language differing from that already in use.

† See *infra*, § 4, p. 57.

linear complex composed of the lines lying in the $S_{2m-2k-1}$, whose moment about this system vanishes. Every such line is evidently a line of the original complex. Conversely, by taking all possible $S_{2m-2k-1}$'s, we shall obtain all the lines of the original complex. It is, however, unnecessary to take more than one $S_{2m-2k-1}$. In fact we have merely to join up to the $S_{n-2m+2k}$ centre all the lines of the subsidiary line complex in the $S_{2m-2k-1}$. We thus get a complex of $S_{n-2m+2k+2}$'s, such that every line in every $S_{n-2m+2k+2}$ of it is a line of the original complex. The proof of this is evident, if we reflect that every line that meets the centre is a line of the complex, and therefore in every such $S_{n-2m+2k+2}$, we can construct a fundamental $(n-2m+2k+3)$-pyramid, every S_1 edge of which is a line of the complex, that is, is such that the moment about it of the auxiliary $(n-2)$-system vanishes. Any one-vector in the $S_{n-2m+2k+2}$ may be replaced by components along the edges of this pyramid; therefore the moment of the $(n-2)$-system about it is also zero. We may add that every line of the complex is obtained in this way, and obtained only once.

10. *The Covariant One-System of an $(n-2)$-System.*

There is a more picturesque and often more convenient method of passing from a linear line complex to a one-system than the method of duality.

We have seen that the covariant r-system of an undegenerate one-system in odd space is, in every case but one, special in character. The exception is when r is equal to $(n-2)$.

Every undegenerate $(n-2)$-system in odd space is, in fact, the covariant $(n-2)$-system of a certain one-system. We proceed to show how to construct the one-system, and to find its coordinates. Such an $(n-2)$-system is reducible to m $(n-2)$-vectors. We may conceive it accordingly represented by such a canonical set. Every $(m-1)$ of these vectors intersect in a straight line. Thus we obtain m straight lines, such that every $(m-1)$ of them determine the S_{n-2} of one of the canonical set.

Now suppose one-vectors whose magnitudes are $x_1, x_2, \ldots, x_m$ taken along these lines. If m equations hold of the form

$$x_2 x_3 \ldots x_m = A_1,$$

where A_1 is the ratio of the first $(n-2)$-vector to the $(n-2)$-vector obtained by taking unity for each x, then we shall have the required one-system. Evidently these equations are satisfied by

$$x_1 : x_2 : \ldots : x_m = \frac{1}{A_1} : \frac{1}{A_2} \ldots : \frac{1}{A_m}.$$

It is convenient to know the coordinates of the one-system when those of the $(n-2)$-system are given. Let them be denoted by a's, and those of the $(n-2)$-system by b's. Then, by § 4, we have to solve equations of the form

$$\lambda\,[12]_b \equiv \lambda b_{12} = [34 \dots 2m]_a.$$

By § 2, the solutions of these equations are of the form

$$[12]_a \{[12 \dots 2m]_a\}^{m-2} = [34 \dots 2m]_b \lambda^{m-1}.$$

In other words, the homogeneous coordinates of the one-system may be taken to be the square brackets of order $(m-1)$ in the b's.

This one-system may be called the covariant one-system of the $(n-2)$-system. Its covariant systems will also be covariant with respect to the $(n-2)$-system; and, without dwelling on the proof, we may assert that a covariant $(2r-1)$-system has for coordinates the square brackets of dimensions r in the a's or $(m-r)$ in the b's.

If the system be a degenerate one in odd space or degenerate or undegenerate in even space, the above requires modification; the one-system obtained is then a concomitant of the second type described in § 3, having an element of arbitrariness. Let us consider an $(n-2)$-system, degenerate k times, where, for $k=0$, we have the undegenerate complex in even space. The $(n-2)$-system is then equivalent to $(m-k)$ $(n-2)$-vectors all intersecting in an $S_{n-2m+2k}$. By the method of sections obtain a set of $(2m-2k-3)$-vectors in an $S_{2m-2k-1}$ not intersecting the $S_{n-2m+2k}$. This latter set is undegenerate and in odd space, and may therefore be treated by the method explained above. We thus get an undegenerate one-system in the $S_{2m-2k-1}$, which has this property: taking all but one of the one-vectors, and building up with these and with the complementary $(n-2m+2k)$-vector an $(n-2)$-vector, the system so obtained is the original set. The element of arbitrariness in this one-system is the choice of the $S_{2m-2k-1}$, which is only restrained not to cut the $S_{n-2m+2k}$, and is therefore any $S_{2m-2k-1}$ of perfectly general position.

11. *Construction of a Linear Line Complex.*

Anticipating the definition and discussion of §§ 12, 13, we may here insert simple constructions as resulting from the preceding article. *To construct an undegenerate linear line complex from its equation. Form the auxiliary $(n-2)$-system, and deduce the covariant one-system. The null-lines of the one-system are the lines of the complex.*

Next, to construct a linear line complex having an $S_{n-2m+2k}$ centre. As in § 9, *obtain a system of subsidiary* $(2m-2k-3)$*-vectors lying in an* $S_{2m-2k-1}$, *not cutting the centre. Form the covariant one-system of this subsidiary system. Take all the null lines of this one-system, and join up these null lines to the centre, thus forming a complex of* $S_{n-2m+2k+2}$*'s. Every line in every* $S_{n-2m+2k+2}$ *of this complex is a line of the original complex.*

Conversely, given a one-system, degenerate k times, and therefore equivalent to $(m-k)$ one-vectors, lying in an $S_{2m-2k-1}$, we can construct a linear complex, degenerate the same number of times, as follows :—Take an $S_{n-2m+2k}$, not intersecting the $S_{2m-2k-1}$, and join up to the null lines of the one-system in the $S_{2n-2k-1}$; all the lines in the $S_{n-2m+2k+2}$'s so obtained constitute the complex required.

12. *Null Spaces.*

We shall now confine our attention to odd space ($n = 2m-1$), and to undegenerate systems, unless the contrary is stated or obviously implied. The necessary modifications for even space, or when the system is degenerate, may easily be made.

In the paper on one-vectors it was shown that, in the canonical form, one of the m representative one-vectors may be made to act along any straight line of perfectly general position, and that, more generally, r of them can be made to lie in any S_{2r-1} of perfectly general position. It is, however, evident that for special positions of the S_{2r-1} this would not be possible.* In fact, when it is possible to choose r of the m one-vectors of the canonical form in our S_{2r-1}, the remaining $(m-r)$ one-vectors will define a $(2m-2r-1)$-vector, whose moment about the S_{2r-1} does not vanish, while all the other $(2m-2r-1)$-vectors, got by taking together $(m-r)$ one-vectors of this canonical form, have a zero moment about the S_{2r-1}. For the theorem to be true the S_{2r-1} must then not be such that the moment of the covariant $(2m-2r-1)$-system about it is zero. If this moment be zero, we shall call the space in question a *null space.*

It is evident from § 10 that the null spaces of a one-system may equally well be called the null spaces of the $(n-2)$-system with which it is associated, and *vice versa.*

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* Thus in Theorem v. (*b*) the construction fails if the chosen line through O be one of the ∞^3 which meet the S_3 determined by the second and third one-vectors. Since O is arbitrary, this shows that the construction fails for ∞^7 of the ∞^8 straight lines in S_5, which are cases of exception and, in accordance with the definition to be presently given, will be called *null lines.*

13. *Null Lines.*

Confining our attention first to null lines, we see that a null line may be defined in two equivalent ways:

(1) *As a line along which a one-vector of the canonical form cannot be made to act.*

(2) *As a line such that the moment of the covariant $(n-2)$-system about it vanishes.*

The equivalence of these two definitions follows from the mode in which we show that along a straight line of perfectly general position one of the one-vectors of the canonical form may be made to lie. The proof only fails when the arbitrary line lies in the fixed S_{n-1}, through the arbitrarily chosen point, and the S_{n-2} of the remaining one-vectors. Adopting a convenient term, we may say: the construction fails always, and fails only, when the line lies in the "polar" S_{n-1} of one of its points.

From this definition of polar it is evident that the polar of every point on such an exceptional line contains the line. Choosing some definite line through the arbitrary point along which one of the one-vectors should act, we have a fixed S_{n-2}, in which the remaining $(m-1)$ one-vectors lie. This S_{n-2}, lying in the polar S_{n-1}, cuts any null line through the arbitrary point, and through this second point one of the remaining one-vectors can be made to pass. Thus every null line, according to the first definition, is such that two of the one-vectors may be made to meet it; it at once follows that it is null according to the second definition. That the second definition involves the first has virtually been shown in the preceding article. For, if a one-vector could act along a line, the moment of the covariant $(n-2)$-system about it could not vanish.

It is easily seen that *the moment about any straight line of the covariant $(n-4)$-system, and of systems of lower order, can none of them vanish.* For, by § 4, if any one of these vanish, all the covariant systems of higher order also have zero moment round the line. Hence the covariant $(n-2)$-system has zero moment round the line, so that the line is a null line. Choosing two of the one-vectors to meet it, a third cannot do so, for three such would lie in S_4, and be reducible to two. Now the moment of the $(n-4)$-system about a line meeting two of the one-vectors is evidently not zero. Thus the theorem is proved.

14. *Thoroughly Null Spaces.*

We shall use the expression *thoroughly null space* to denote a space which has no lines in it other than null lines. We easily see that such spaces exist by the following construction :—

THEOREM 1.—*If $(r+1)$ points be taken, one on each of as many distinct one-vectors of any canonical set of m one-vectors, these always determine a thoroughly null S_r.*

For they are the vertices of an $(r+1)$-pyramid, of which every edge is a null line. Consider then any other line in the S_r. A unit one-vector along it may be replaced by components along the edges of the $(r+1)$-pyramid, and its moment about the covariant $(n-2)$-system is the sum of the moments of these components. But the moment of each component is zero, for its line of action is a null line; therefore the moment of the line in question is zero, and the line is also a null line, as was to be proved.

THEOREM 2.—*If S_r be a thoroughly null space, the polar of every point in it contains the whole S_r.*

For the polar contains all the null lines through its pole.

THEOREM 3.—*If S_r be a thoroughly null space, we may always arrange that $(r+1)$ of the m one-vectors should meet it; more cannot meet it.*

For, through any point A of the S_r we may make one of the one-vectors pass. The remaining $(m-1)$ one-vectors then lie in the S_{n-1} polar of A; they also lie in the polar of any other point B on the line of action of the first one-vector. The polar of A contains the null line S_r, and the polar of B does not. Thus the S_{n-2}, in which the $(m-1)$ remaining one-vectors lie, does not contain the S_r, but meets it in an S_{r-1}, which is, of course, thoroughly null. Repeating this process r times, we get r one-vectors passing through as many points of the S_r, and the remaining $(m-r)$ lying in a space which intersects the S_r in a point. Through this point one of the remaining one-vectors may be made to act. This demonstrates the theorem.*

It is to be remarked that here $r+1 \leqslant m$.

It will appear subsequently that there are no thoroughly null spaces of dimensions higher than $(m-1)$.

THEOREM 4.—*A thoroughly null S_r is such that the moment about it of the covariant $(n-2r)$-system vanishes.*

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* It is otherwise obvious that an S_r cannot meet more than $(r+1)$ of the one-vectors. If it met $(r+2)$, these would reduce to $(r+1)$.

This follows from Theorem 3. For a thoroughly null S_r meets $(r+1)$ of the m one-vectors, and therefore meets the spaces determined by every $(m-r)$ of them.

THEOREM 5.—*Conversely an S_r such that the moment about it of the covariant $(n-2r)$-system vanishes is thoroughly null.*

The proof of this theorem is contained in that of Theorem 7, § 15, of which it is a special case.

15. *On the Species of Null Spaces.*

An S_{2r-1} or an S_{2r} of perfectly general position can be made, as we know, to contain r of the one-vectors, and, evidently, no more. We have seen, on the other hand, that spaces exist which cannot be made to contain even one of the one-vectors. We are thus led to a classification of spaces, with respect to a given one-system, according to the number of one-vectors of the canonical form which they can be made to contain.

We shall say that an S_{2r-1}, or an S_{2r}, is a null space of species p if the maximum number of one-vectors which it can be made to contain is $(r-p)$.

If the dimensions of the space in question be less than $(m-1)$, the last article shows that p can attain the maximum r, the space being then what we call thoroughly null. The maximum value of p when the space is of dimensions higher than $(m-1)$ is given by the following theorem:—

THEOREM 1.—*In every S_{n-r} where r is less than m, at least $(m-r)$ of the one-vectors can be made to lie.*

Since the one-system can be replaced by a one-vector through any chosen point, and $(m-1)$ in any S_{n-1} not containing the point, the theorem is obviously true when $r=1$. Suppose it proved for all values of r up to $(k-1)$: we proceed to prove it by induction, when r is equal to k.

Take any S_{n-k}, and through it pass an S_{n-k+1}; then $(m-k+1)$ of the one-vectors can be chosen in this S_{n-k+1}. These determine an $S_{2m-2k+1}$ in the S_{n-k+1}, which is cut by the S_{n-k} (unless the latter contains it) in an S_{2m-2k}. In this S_{2m-2k} of $S_{2m-2k+1}$ we may, by hypothesis, put $(m-k)$ of the $(m-k+1)$ one-vectors. Thus $(m-k)$ of the one-vectors can be put in the S_{n-k}, as was to be proved.

Thus the maximum value of p for an S_{n-2r} is the same as for an S_{2r-1}; viz., it is r. Similarly, it is r for an S_{n-2r-1}; that is, the same as for an S_{2r}.

THEOREM 2.—*If, in an S_{n-r} where r is less than m, $(m-r)$ of the one-vectors, and no more, can be made to lie, all the remaining r one-vectors can be made to meet it.*

For the remaining r one-vectors lie in an S_{2r-1}, which is cut by the S_{n-r} in an S_{r-1}. Since none of the one-vectors can be chosen in this S_{r-1}, it is thoroughly null. Hence, by Theorem 3 of the last article, all these r one-vectors can be made to meet it.

THEOREM 3.—*If, in an S_{n-r} where r is less than m, $(m-r+k)$ of the one-vectors, and no more, can be made to lie, then $(r-2k)$, and no more, can be made to meet it.*

The remaining $(r-k)$ one-vectors determine an $S_{2r-2k-1}$, which is met by the S_{n-r} in an S_{r-2k-1}. As, by hypothesis, none of the $(r-k)$ one-vectors can be made to lie in this, the S_{r-2k-1} is a thoroughly null-space for the system of $(r-k)$ one-vectors in the $S_{2r-2k-1}$. Therefore $(r-2k)$ of these one-vectors may be made to meet it. The remaining k one-vectors will not meet it.

THEOREM 4.—*If, in an S_r where r is less than m, k of the one-vectors, and no more, can be made to lie, then, of the remaining $(m-k)$ one-vectors, $(r-2k+1)$ may be made to meet the S_r, and no more.*

For the $(m-k)$ one-vectors lie in an S_{n-2k}, intersecting the S_r in an S_{r-2k}, which is, by the hypothesis, thoroughly null for the system in the S_{n-2k}. Hence $(r-2k+1)$ of these $(m-k)$ one-vectors, and no more, can be made to meet the S_{r-2k}, which proves the theorem.

From the last three theorems it follows that every null space may be constructed by means of a suitable choice of the canonical set of m one-vectors, as follows :—

THEOREM 5.—*To construct an S_r of species p, take a suitable canonical set of m one-vectors, choose 2p points if r be odd, and $(2p+1)$ if r be even, on as many different one-vectors, and complete the space by taking sufficient of the remaining one-vectors to give an S_r.*

We have still to show that every space constructed in this way is of species p ; that is, we have to show that a space so constructed has the maximum number of one-vectors in it, and could not, by another choice of the canonical form, be made to contain more. This follows from our next theorem.

THEOREM 6.—*If S_{2r-1} be of species p, the moment about it of the covariant $(n-2r-2p+2)$-system vanishes, as does that of any higher and no lower system.*

If S_{2r} be of species p, the moment about it of the covariant $(n-2r-2p)$-system vanishes, as does that of any higher and no lower system.

These follow from Theorem 5.

The covariant properties possessed by these spaces indicate that a space which when constructed with a particular set of one-vectors contains λ and meets μ in the manner explained could not contain more than λ, and therefore meet less than μ, by means of any other canonical form, for then we should have different covariant systems with zero moment about the space. In fact, if a different choice of the canonical form could give the numbers λ' and μ', we must have, first of all,

$$2\lambda+\mu = 2\lambda'+\mu' = \text{dimensions of the space.}$$

$(\lambda+\mu)$ will therefore necessarily alter, and with it the covariant system whose moment vanishes.

THEOREM 7.—*If an S_{2r-1} be such that the moment of the covariant $(n-2r-2p+2)$-system about it vanishes, and not that of any lower system, the S_{2r-1} is of species p.*

And, if an S_{2r} be such that the moment of the covariant $(n-2r-2p)$-system about it vanishes, and not that of any lower system, the S_{2r} is of species p.

Any other species is, in fact, inconsistent with the Theorem 6.

Summing up, we have proved the identity of two definitions of species, one respecting the number of one-vectors which can be made to lie in a space, and the other respecting the covariant systems which have zero moment about the space. We have also given a geometrical construction for a null space of any species.

16. *Polarity.*

We have hitherto only defined the polar of a point. Next to define the polar of a straight line.

It may be proved that the polar S_{n-1}'s of every point on a straight line intersect in an S_{n-2}; this we call the polar S_{n-2} of the straight line. If the line be not a null line the theorem on which the definition depends is obvious, the S_{n-2} being that determined by the remaining one-vectors when one has been chosen to act along the line. Next, take a null line, and draw two one-vectors to meet it. These determine an S_3 containing the line, and the remaining one-vectors determine an S_{n-4}, not intersecting the S_3. Without disturbing the S_{n-4}, we can move the first pair of one-vectors about in their S_3, so that one of them intersects the null line in any desired point of it. Thus the polar of any point of the null line contains the S_{n-2} determined by this S_{n-4} and the null line itself. Hence all the polars of points on a null line intersect in an S_{n-2} containing the null line.

A similar proof may be used to prove the existence and properties of the polar S_{n-2r} of an S_{2r-1}. For let the S_{2r-1} be of species p. Draw $(r-p)$ one-vectors to lie in it and $2p$ to meet it. These determine an $S_{2r+2p-1}$ containing the whole S_{2r-1}. The $2p$ points lie, as is seen in § 15, in a thoroughly null S_{2p-1}. The polar of any point in this S_{2p-1} contains of course the S_{2p-1} and also the $(r-p)$ one-vectors first chosen; hence it contains the whole S_{2r-1}. Thus without disturbing the S_{2p-1} we can move the other one-vectors about so that one of them passes through any chosen point of the S_{2r-1}. The S_{2p-1} lies therefore in the polar of every point of the S_{2r-1}. Further, without disturbing the remaining $(m-r-p)$ one-vectors, we may move the $(r+p)$ about in their $S_{2r+2p-1}$, so that one of them passes through any chosen point of the S_{2r-1}. Thus the polar of every point of the S_{2r-1} contains the S_{n-2r} determined by the $S_{n-2r-2p}$ of the one-vectors not meeting the S_{2r-1} and the thoroughly null S_{2p-1}. This we may call the polar S_{n-2r} of the S_{2r-1}. *Mutatis mutandis*, the proof holds for the S_{n-2r-1} polar of an S_{2r}.

COR. 1.—*An S_{2r-1} and its polar S_{n-2r} are of the same species p, and intersect in a thoroughly null S_{2p-1}.*

Similarly, for an even space S_{2r} of species p, we may show the theorem holds, provided $p>0$.

COR. 2.—The thoroughly null S_{m-1}'s are their own polars.

17. *Coordinates of Polars of thoroughly Null and other Spaces.*

Given the coordinates of a point, those of the S_{n-1} polar may be at once written down. We have merely to take the sum of the moments in every S_{n-1} face of the components of the point about the components of the system of $(n-2)$-vectors, *e.g.*,

$$p_{12\ldots n} = p_1\,[23\ldots n]-p_2\,[13\ldots n]+\&c.$$

Given the coordinates of a line, those of its polar may be found as follows:—Assume the line is such that one of the one-vectors can be made to act along it, and let λ be the magnitude, and $p_{12}, p_{13}, \ldots$ the coordinates of the one-vector so chosen. Then

$$a_{12}-\lambda p_{12} : a_{13}-\lambda p_{13} : \&c.,$$

are the coordinates of a system of one-vectors which is degenerate once; therefore the square bracket of dimensions m of these quantities vanishes. Expanding in powers of λ, all the coefficients disappear except the constant term and the term involving the first power of λ. Thus we have the following equation defining λ:—

$$[12 \dots 2m] = \lambda \{p_{12}[34 \dots 2m] - p_{13}[24 \dots 2m] + \&c.\}.$$

This fails, it will be noticed, if the coefficient of λ vanishes. λ is then infinite; the line must therefore be a null line of the system. We may, however, deduce the result in this case also.

With this value of λ,

$$a_{12} - \lambda p_{12} : a_{13} - \lambda p_{13} : \&c.,$$

are the coordinates of the degenerate system. The S_{n-2} in which it lies is the required polar. Following, therefore, the rule of § 5, we form the square brackets of dimensions $(m-1)$; these are the coordinates required. If we go through the work, we find there is a great simplication, the coefficients of all powers of λ, except the first and the absolute term, vanishing. Thus, for instance,

$$p_{34\dots 2m} = [345 \dots 2m] - \lambda \{p_{34}[56 \dots 2m] - p_{35}[46 \dots 2m] + \&c.\}.$$

When the line is a null line the absolute term may be neglected in comparison to that involving λ, and the coordinate of the polar S_{n-2} of a null line may be taken to be the coefficient of λ itself.

We can see this more easily still geometrically. For, as the polar of a null line contains the null line, the polar S_{n-2} is that of the moment about the null line of the covariant $(n-2)$-system; the coordinate can therefore be written down in the usual way. In a similar way, the polar spaces of all thoroughly null spaces can be written down at once. Thus, to find the polar of a thoroughly null S_2. This is the space determined by the thoroughly null S_2 and the remaining $(m-3)$ one-vectors of the canonical form, when three have been made to meet it. That is, it is the space of the moment about the S_2 of the covariant $(n-6)$-system. More generally the polar of a thoroughly null S_r is that determined by the S_r, and the $(m-r-1)$ one-vectors which do not meet it, and is therefore the space of the vector representing the moment about the S_r of the covariant $(n-2r-2)$-system.*

18. *The Equations to the Complexes of Null Spaces.*

First consider the null lines. Their characteristic property is that the moment about them of the covariant $(n-2)$-system vanishes.

* See end of § 18. The indications are sufficient to enable the reader to find the coordinates of the polar of any space whatever. The results, being less simple, are not given here.

This moment is a scalar quantity; there is, therefore, only one condition that a line should be a null line; viz., it is

$$p_{12}[34 \dots 2m] - p_{13}[24 \dots 2m] + \&c. = 0,$$

or, in the notation of § 2, $\Sigma b_{ij} p_{ij} = 0.$

In the general case of a null S_{2r-1} of species p the characteristic property that the moment of the covariant $(n-2r-2p+2)$-system about it vanishes leads to several equations. In fact the moment of this system about a unit $(2r-1)$-vector in an S_{2r-1} gives rise, in the general case, to a system of $(n-2p+2)$-vectors; this system must, in our case, be in equilibrium. We thus have as many equations as such a system has coordinates, viz., the number of coordinates of a $(2p-3)$-system; that is, the number of $(2p-2)$-pyramidal faces in an $(n+1)$-pyramid. The coefficients in these equations are the square brackets of dimensions $(m-r-p+1)$.

In the case of a null S_{2r} of species p, the number of equations is the number of $(2p-1)$ pyramidal faces, and the coefficients are the square brackets of dimensions $(m-r-p)$.

It will be noted that the number of equations depends only on the species of the space, and not on its dimensions. The equations are not, however, all independent; they are linear equations, having syzygies between them.

To determine the dimensions of the entity formed by the S_r's of a given species, we must either determine the syzygies, and the syzygies between the syzygies, and so on, or else adopt another method. This we shall do in two subsequent articles (§§ 23, 24).

It will be noticed that, denoting by

$$\Theta = 0$$

the equation to the complex of null lines, Θ is the coefficient of λ in the equation of the preceding article determining λ; and that the coordinates of the polar S_{n-2} of a null line are

$$p_{34 \dots 2m} : p_{24 \dots 2m} : \&c. = \frac{\partial \Theta}{\partial a_{12}} : \frac{\partial \Theta}{\partial a_{13}} : \&c.$$

By Euler's theorem we then have

$$a_{12} p_{34 \dots 2m} + a_{13} p_{24 \dots 2m} + \&c. = 0,$$

the equation to the complex of null S_{n-2}'s of species 1, which verifies our result.

In a similar way, if

$$\Theta_{34\ldots 2m}=0,\quad \Theta_{24\ldots 2m}=0,\quad \&c.,$$

be the equations to the complex of null S_3's, where

$$\Theta_{34\ldots 2m}=p_{3450}[78\ldots 2m]-p_{3507}[48\ldots 2m]+\&c.,$$

the coordinates of the polar S_{n-4} of a null S_3 are

$$p_{56\ldots 2m}:p_{46\ldots 2m}:\&c.=\frac{\partial\Theta_{3456\ldots 2m}}{\partial a_{34}}:\frac{\partial\Theta_{3456\ldots 2m}}{\partial a_{35}}:\&c.$$

$$=\frac{\partial\Theta_{1256\ldots 2m}}{\partial a_{12}}:\frac{\partial\Theta_{1316\ldots 2m}}{\partial a_{13}}:\&c.,$$

and a variety of equivalent ratios.

Euler's theorem gives us then the equations to the complex of null S_{n-4}'s of species 2.

It is easy to generalize from the above and write down symbolically the coordinates of the polar of any thoroughly null space.

19. *Associated Complexes of a given Linear Complex.*

Suppose now the linear complex to be given, and let the coefficients in its equation be the quantities b. Then the preceding shows that there are a series of associated complexes, which may be written down as follows, making use of the known relations connecting the a's and the b's.

Linear Complex. $\quad \Sigma b_{ij}p_{ij}=0.$

Complex of S_2's. $\quad \Theta_{2345\ldots 2m}=0,\quad \Theta_{1345\ldots 2m}=0,\ \&c.,$

where $\quad \Theta_{2345\ldots 2m}\equiv p_{234}[1234]_b-p_{235}[1235]_b+\&c.$

Complex of S_3's. $\quad \Theta_{3456\ldots 2m}=0,\quad \Theta_{2456\ldots 2m}=0,\ \&c.,$

where $\quad \Theta_{3456\ldots 2m}\equiv p_{3450}[123456]_b-p_{3507}[123567]_b+\&c.$

&c., &c.,

the coefficients corresponding to an S_r complex being always the square brackets of dimensions r.

These we shall call the *associated complexes of the given linear complex.* The original complex is defined by one equation; the associated complexes are each of them defined by several equations, with, moreover, syzygies between them. The relation between the linear line complex and an associated S_r complex is this: *taking the lines of the original complex, we build up all possible S_r's; the S_r's so obtained constitute the associated complex in question.*

21. *On the Associated Complexes of a Degenerate Complex.*

Suppose the complex degenerate k times, so that it has an $S_{n-2m+2k}$ centre. The auxiliary $(n-2)$-system consists then of the system formed by joining the centre up to the $(2m-2k-3)$-vectors of a $(2m-2k-3)$-system lying in any $S_{2m-2k-1}$, which does not meet the $S_{n-2m+2k}$; this $(2m-2k-3)$-system being a covariant system of an undegenerate one-system in the $S_{2m-2k-1}$. We have to find the loci of the thoroughly null spaces of this $(n-2)$-system. We shall employ the method of duality. Corresponding to the degenerate $(n-2)$-system, we have a degenerate one-system lying in an $S_{2m-2k-1}$, having the same coordinates as those of the $(n-2)$-system. We can, at once, write down the coordinates of the covariant $(2r-1)$-system, where r may have any value from 1 to $(m-k)$ inclusive; they are the square brackets of dimensions r. Dually these are the coordinates of the covariant $(n-2r)$-system of the original $(n-2)$-system. That is to say, *the coefficients in the equations to the associated S_r complex, where r may have any value from 1 to $(m-k)$ inclusive, are the square brackets of dimensions r.*

22. *On the Arrangement of the Null Lines in the Null Spaces.*

The null lines in a null space, exception being of course made of the case in which the space is thoroughly null, form an entity whose dimensions are one less than the dimensions of the line space peculiar to the space. In fact, for a line to be a null line constitutes a single condition, viz., it must belong to a certain linear complex. The null lines of a space form therefore what we may call the "section" of the complex by this space. Considering for definiteness an S_{2r-1} of species p (where $2r \leqslant m$), let us consider the nature of the section for all values of p from 0 to $(r-1)$. When p is zero, that is, when the space is ordinary, r of the one-vectors can be made to lie in it, the system determined by these r one-vectors being a determinate one. The section in this case is composed therefore of the null lines of an undegenerate one-system of the S_{2r-1}; in other words, it is an undegenerate linear complex of that space.

Next suppose the S_{2r-1} to be of species $p>0$. In this case $(r-p)$

of the one-vectors may be made to lie in it, and $2p$ to meet it. These $2p$ determine by their intersection with the S_{2r-1} a determinate S_{2p-1}, which is thoroughly null. Every line in the S_{2r-1} which meets this S_{2p-1} is thoroughly null. For, take a point on such a line not in the S_{2p-1}; one of the $(r-p)$ one-vectors may be made to pass through it, and one of the $2p$ to pass through the point where it meets the S_{2p-1}, which shows that the line is a null line. Taking a definite set of $(r-p)$ one-vectors, these determine an $S_{2r-2p-1}$, and form an undegenerate one-system in it. The null lines of this system are null lines of the original one-system. Join up every such line to the S_{2p-1}; we thus get S_{2p+1}'s every line of which is a null line of the original one-system, and therefore a line of the linear line complex which it defines. It will be noted that, in virtue of § 9, the result we have arrived at is as follows :—

The section of the linear line complex of a one-system by a null space of species p is a linear line complex degenerate p times.

23. *Enumeration of the thoroughly Null Spaces.*

Let $f(n, r)$ denote the number* of thoroughly null S_r's for the general undegenerate one-system in S_n.

Then $\{f(n, r)-n+r\}$ is the number of such S_r's which pass through an arbitrary point. But all the thoroughly null S_r's which pass through a given point necessarily lie in the polar S_{n-1} of that point; and, taking any S_{n-2} of the polar S_{n-1} not passing through the pole, each such S_r will give a thoroughly null S_{r-1} of the S_{n-2}; and, *vice versa*, each thoroughly null S_{r-1} of that S_{n-2}, joined up to the pole, gives one of the null S_r's passing through the pole. Hence

$$f(n, r)-n+r = f(n-2, r-1).$$

Or, putting $n = 2m-1$, we may write

$$\phi(m, r) \qquad -\phi(m-1, r-1) = 2m-1-r;$$

therefore

$$\phi(m-1, r-1)-\phi(m-2, r-2) = 2(m-1)-1-r+1,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\phi(m-r+2, 2)-\phi(m-r+1, 1) = 2(m-r+2)-1-2;$$

further

$$\phi(m-r+1, 1) \qquad\qquad = 2(2m-2r)-1.$$

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* We shall say that the number of S_r's is x when there are ∞^x of them.

These give at once

$$\phi(m, r) = (n-r)(r+1) - \tfrac{1}{2}r(r+1),$$

which is the number of the dimensions of the entity formed by the thoroughly null S_r's.*

As the total number of S_r's in S_n is $(r+1)(n-r)$, we see that the numerous equations obtained in § 18 are equivalent to $\frac{1}{2}r(r+1)$ only. Thus, for instance, the thoroughly null S_2's, the equations to which are $(n+1)$ in number, are only bound by three independent conditions.

24. *Enumeration of Null Spaces of any Species.*

We have seen that a null S_{2r-1} of species p intersects its polar in a thoroughly null S_{2p-1}; further, all the S_{2r-1}'s which pass through a thoroughly null S_{2p-1} and lie in the polar S_{n-2p} are of species p at least, and there are no others. Now in such an S_{2r-1} there is only one thoroughly null S_{2p-1} which has this unique position with regard to it. Hence the number of null S_{2r-1}'s of species p is equal to the number of S_{2r-1}'s which pass through a thoroughly null S_{2p-1}, and lie in its polar, *plus* the number of thoroughly null S_{2p-1}'s. Writing $2p-1 = p'$ and $2r-1 = r'$ for brevity, the required number is

$$(r'-p')(n-p'-1-r') + (n-p')(p'+1) - \tfrac{1}{2}p'(p'+1)$$
$$= (n-r')(r'+1) - \tfrac{1}{2}p'(p'+1),$$

which is $p(2p-1)$ less than the number of general S_{2r-1}'s. The equations to be satisfied by an S_{2r-1} of species p are therefore equivalent to $p(2p-1)$ independent conditions. The above reasoning holds for an S_{2r}, only that we have to write $p' = 2p$, $r' = 2r$ in accordance with § 15.

Combining this result with the rule of § 18, we see that *the null S_r's of species p generate a linear* $\infty^{(n-r)(r+1)-\frac{1}{2}p'(p'+1)}$ *complex*, where p' is $2p$ or $(2p-1)$ according as r is even or odd. These complexes may be said, in an extended sense, to be "associated" with the complex of null lines. The S_r's are not built up of complex lines, but the coefficients in the equations are still square brackets formed from the coefficients b_{ik}. Thus, for instance, the equation to the complex of null S_3's of species 1 is

$$p_{1234}[1234]_b - p_{1345}[1345]_b + \&c. = 0.$$

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* *Cf.* S. Kantor, *loc. cit.*