



## I. On the law of probability for a system of correlated variables

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I. *On the Law of Probability for a System of Correlated Variables.* By S. H. BURBURY, F.R.S.\*

1. **I**N the works of Karl Pearson, Yule, and other writers on the theory of evolution, the term correlation has acquired a special meaning. For the purposes of this paper I define it as follows:—Let  $x$  and  $y$  be two quantities, each of which varies continuously between its own limits, and let them be independent variables, in the general sense, that if either be given, the other is not thereby determined, but may vary continuously through a finite range of values. That is, the system of  $x$  and  $y$  has two degrees of freedom.

The chance that  $x$  shall lie between  $x'$  and  $x' + dx$ , that is the number of cases in which, out of a very great number of cases, it so lies, is a function of  $x'$ , say  $f_1(x')dx$ . The corresponding chance for  $y$  lying between  $y'$  and  $y' + dy$  shall be  $f_2(y')dy$ . It may be that, notwithstanding the system having two degrees of freedom,  $f_1(x')$  is a function of  $y$  as well as of  $x'$ , and  $f_2(y')$  is a function of  $x$  as well as of  $y'$ , so that  $\frac{d}{dy}f_1(x) \neq 0$ , and  $\frac{d}{dx}f_2(y) \neq 0$ . If this is the case,  $x$  and  $y$  are correlated. The chance that simultaneously  $x$  shall lie between  $x_1$  and  $x_1 + dx$ , and  $y$  between  $y_1$  and  $y_1 + dy$  is of the general form  $\phi(x_1y_1)dx dy$ . If

$$\frac{d}{dy}f_1(x) = 0, \text{ and } \frac{d}{dx}f_2(y) = 0, \phi(xy) = f_1(x)f_2(y),$$

and  $x$  and  $y$  are not correlated. If they are correlated,

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$\phi(xy)$  is not expressible as the product of two functions, one of  $x$  only, the other of  $y$  only.

2. It follows from this definition, that if  $x$  and  $y$  are both functions of any the same variables,  $x$  and  $y$  are correlated. If for instance (1)  $f(x, t) = 0$ , and (2)  $\phi(y, t) = 0$ , so that  $x$  and  $y$  are both functions of  $t$ ,  $f_1(x)$  of art. 1 is by (1) a function of  $t$  as well as of  $x$ , and therefore by (2) a function of  $y$  as well as of  $x$ ; and therefore  $x$  and  $y$  are, by the definition, correlated. Generally, if  $n$  variables  $x_1 \dots x_n$  are connected by equations of condition less than  $n$  in number, correlation exists between them by virtue of those equations. It follows also that if  $x$  and  $y$  are correlated, any functions of  $x$  and  $y$ , as  $f(x)$  and  $\phi(y)$ , are correlated with each other.

3. *Of the mean product of two correlated variables.*—The most general definition of mean product is this: There being  $N$  values of  $x$ , and  $N$  values of  $y$ , we assign to every  $x$  some one of the  $N$  values of  $y$  as a companion factor, by this means forming  $N$  products each of an  $x$  and one  $y$ . Let them be

$$x_1y_1, x_2y_2, \dots x_Ny_N. \quad \text{Then I define } \overline{xy} = \frac{x_1y_1 + \dots + x_Ny_N}{N^2}.$$

This selection of products might be effected in any one of  $N$  different ways. Practically, if  $x$  and  $y$  are both functions of a third variable  $t$ , we might take every  $x$  with the value of  $y$  for the same  $t$ , so that  $\overline{xy} = \int xy dt$ . Similarly, if  $x$  and  $y$  are functions of two other variables  $z$  and  $t$ , we should define  $\overline{xy} = \iint xy dz dt$ , and so on.

4. The square of the mean product, so defined, of two correlated variables, cannot be greater than the product of their mean squares. For, taking the general definition of mean product above given, there are, as there stated,  $N$  different ways in which  $N$   $x$ 's and  $N$   $y$ 's may be arranged to form  $N$  products. Of these there must be some one way for which  $\overline{xy}$  is not less than for any other. And for this one, and therefore for every other way,

$$\overline{x^2} = \frac{\sum x^2}{N}, \quad \overline{y^2} = \frac{\sum y^2}{N}, \quad \overline{xy} = \frac{\sum xy}{N},$$

and

$$\overline{x^2} \cdot \overline{y^2} - (\overline{xy})^2 = \frac{1}{N^2} \{ (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + \&c. \\ + (x_py_q - x_qy_p)^2 \},$$

which is  $\geq 0$ .

5. Generally, if two variables are correlated, their mean product differs from zero, even though the means of each separately, or of one of them, are zero. For the mean product is  $\overline{xy} = \iint \phi(xy)xy \, dx \, dy$ , and if  $x$  and  $y$  are correlated, so that  $\phi(xy)$  cannot be expressed as  $f_1(x)f_2(y)$ , this expression for  $\overline{xy}$  is not in general zero. If  $\phi(xy) = f_1(x)f_2(y)$ , then  $\overline{xy} = \iint \phi(xy)xy \, dx \, dy = \int f_1(x)x \, dx \cdot \int f_2(y)y \, dy$ , and since  $\bar{x} = \int f_1(x)x \, dx$ , and  $\bar{y} = \int f_2(y)y \, dy$ , the mean product is the product of the means, and is zero if either  $\bar{x}$  or  $\bar{y}$  is zero. As an example of this theorem let  $x$  and  $y$  be two vibrators, having the same period, but different phases, so that we may have  $x = A \sin nt$ ,  $y = B \sin (nt + \alpha)$ . Then  $\bar{x} = 0$ ,  $\bar{y} = 0$ , but  $\overline{xy} = \frac{1}{2}AB \cos \alpha$ . Also in this case  $\overline{x^2} = \frac{1}{2}A^2$ ,  $\overline{y^2} = \frac{1}{2}B^2$ . And  $\overline{xy}$  may change, while  $\overline{x^2}$  and  $\overline{y^2}$  remain unchanged. Similarly, any two variables  $x$  and  $y$  may, with given numerical values, be more, or may be less, likely to have the same sign, than, with the same numerical values, to have opposite signs. If the chance of their having the same sign be  $\phi_1(xy)$ , and the chance of their having opposite signs be  $\phi_2(xy)$ , then

$$\overline{xy} = \iint (\phi_1(xy) - \phi_2(xy))xy \, dx \, dy.$$

But the mean squares  $\overline{x^2}$ ,  $\overline{y^2}$  are independent of the difference  $\phi_1(xy) - \phi_2(xy)$ , and may therefore be constant while  $\overline{xy}$  changes.

#### *Of very small Correlations.*

6. It may be that  $f(x')$  of art. 1, although it is a function of  $y$  as well as of  $x'$ , yet is very little affected by change in  $y$ ; that is  $\frac{df(x')}{dy}$  may be very small or negligible. Similarly  $\frac{df_2(y')}{dx}$  may be very small or negligible. In the same case  $\phi(xy)$  may differ inappreciably from a product of the form  $f_1(x)f_2(y)$ , so that for some purposes we may without appreciable error treat  $x$  and  $y$  as not correlated. For instance, in the kinetic theory of gases, if  $x_1 \dots x_n$  are the vector velocities,  $m_1 \dots m_n$  the masses, of  $n$  molecules,  $n$  being a very great number,  $x_1 \dots x_n$  are, strictly speaking, correlated by virtue of the relation  $\sum m x^2 = 2T$  where  $T$  is the kinetic energy, supposed constant. For if  $x_n$  be given we alter the limits of integration for  $x_1 \dots x_{n-1}$ , so that the chance of  $x_1$  having a

given value depends on  $x_n$  as well as on  $x_1$ . But  $n$  being very great, these correlations, *e.g.*  $\frac{df(x_1)}{dx_n}$ , &c., are inappreciable, and in the ordinary theory are generally treated as non-existent.

7. The definition above given of correlation is purely algebraic. Generally, if  $x$  and  $y$  be two of the variables which determine the state of a material system the parts of which mutually act on each other, the time differential coefficients,  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ , are correlated by virtue of that mutual action.

For  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  are functions of the same variables, and therefore correlated. And if the system be defined by  $n$  generalized coordinates  $q_1 \dots q_n$ , and their corresponding velocities  $\dot{q}_1 \dots \dot{q}_n$ , the products  $\dot{q}_1 \dot{q}_2$  &c. appear in the expression for the kinetic energy, and therefore the mean product  $\dot{q}_1 \dot{q}_2$  is not generally zero, and whether or not the means  $\overline{\dot{q}_1}$  and  $\overline{\dot{q}_2}$  are separately zero,  $\dot{q}_1$  and  $\dot{q}_2$  are correlated.

8. Since correlations may, or may not, be negligible we may suppose that  $x_1 \dots x_n$ , or the things to which they relate, have at every instant positions in space, and that the correlation between any two of them, as  $x_p$  and  $x_q$ , is or is not negligible, according to the distance which at the instant separates them from each other. But this localization is not essential to the general statement that correlations may or may not be negligible.

#### *A General Problem stated.*

9. Let  $s_1 \dots s_n$  be  $n$  quantities, which, until otherwise stated, shall be each of zero dimensions, each of which varies continuously between assigned limits. I assume them to be in general correlated with each other, but that such correlations may as regards any  $s$ , as  $s_p$ , be negligible for some or for most of the others. The chance that they shall respectively lie

$s_1$  between  $r_1$  and  $r_1 + dr_1$ ,

$s_2$     „     $r_2$     „     $r_2 + dr_2$ ,

•   •   •   •   •   •

$s_n$  between  $r_n$  and  $r_n + dr_n$ ,

shall be denoted by  $\phi(r_1 \dots r_n) dr_1 \dots dr_n$ , so that

$$\iint \dots \phi(r_1 \dots r_n) dr_1 \dots dr_n = 1$$

between the given limits; and I assume the function  $\phi(s_1 \dots s_n)$  to be finite and continuous for all values of  $s_1 \dots s_n$  within the given limits.

It may be the case that a certain function of  $s_1 \dots s_n$ , say  $\psi(s_1 \dots s_n)$ , is necessarily constant, and that the variations of  $s_1 \dots s_n$  are subject to that condition. Or there may be more than one such constant function. If so, the chance  $\phi(s_1 \dots s_n) ds_1 \dots ds_n$  must be understood as subject to the constancy of  $\psi$ , or other such functions as aforesaid.

10. From this we can express the mean values of  $s_1 \dots s_n$ , of their squares, or binary products, or other functions, but it will not be necessary to go beyond powers and products of the second degree. Such mean values will be denoted as usual by a bar. Thus

$$\left. \begin{array}{l} \text{Also} \quad \bar{s}_1 = \iint \dots \phi(s_1 \dots s_n) s_1 ds_1 \dots ds_n, \text{ \&c.} \\ \bar{s}_1^2 = \iint \dots \phi(s_1 \dots s_n) s_1^2 ds_1 \dots ds_n, \text{ \&c.,} \\ \text{and} \quad \bar{s_p s_q} = \iint \dots \phi(s_1 \dots s_n) s_p s_q ds_1 \dots ds_n, \text{ \&c.} \end{array} \right\} \dots \quad (1)$$

11. The object of the first part of this paper is to prove that,  $s_1 \dots s_n$  being as above stated correlated *inter se*, then for very great values of  $n$ ,  $\phi(s_1 \dots s_n)$  necessarily has the form  $\phi(s_1 \dots s_n) = A e^{-Q}$ , in which  $A$  is a constant, and  $Q$  is a homogeneous quadratic function of  $s_1 \dots s_n$ , involving both their squares, and (as a consequence of the correlation) their products, namely

$$Q = \frac{1}{2} \alpha_1 s_1^2 + \beta_{12} s_1 s_2 + \frac{1}{2} \alpha_2 s_2^2 + \dots + \frac{1}{2} \alpha_n s_n^2 \dots \quad (2)$$

and the coefficients  $\alpha \beta$  are functions of the mean values  $\bar{s}_1 \dots \bar{s}_n$ , of the mean squares  $\bar{s}_1^2 \dots \bar{s}_n^2$ , and of the mean products  $\bar{s}_1 s_2 \dots \bar{s}_p s_q$ , &c.

This is a well known result for the case in which the quadratic function  $Q$  is incapable of becoming negative. It is necessary, however, shortly to give the proof, which is done in Part I. In Part II. I propose to show *inter alia* why it is necessary for  $Q$  to be positive, and I shall then apply the theorem to certain physical problems.

## PART I.

*Proposition I.*

12. Introducing  $\theta_1, \theta_2, \dots, \theta_n$  as auxiliary variables,

$$\phi(r_1 \dots r_n) = \iint_{-\infty}^{+\infty} \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\sum \overline{s-r} \theta \sqrt{-1}}. \quad (3)$$

multiplied by a constant independent of  $r_1 \dots r_n$ . Here

$$\sum \overline{s-r} \theta = \overline{s_1-r_1} \theta_1 + \overline{s_2-r_2} \theta_2 + \dots + \overline{s_n-r_n} \theta_n,$$

and the limits of integration for each  $\theta$  are  $\pm \infty$ . Let the right-hand member of the equation (3) be denoted by S.

And in S substitute for  $e^{\overline{s_n-r_n} \theta_n \sqrt{-1}}$  its equivalent

$$\cos \overline{s_n-r_n} \theta_n + \sqrt{-1} \sin \overline{s_n-r_n} \theta_n,$$

and integrate according to  $\theta_n$  from  $\theta_n = A$  to  $\theta_n = -A$ . That reduces S to

$$\iint \dots d\theta_1 d\theta_2 \dots d\theta_{n-1} \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n \\ \times e^{(\overline{s_1-r_1} \theta_1 + \dots + \overline{s_{n-1}-r_{n-1}} \theta_{n-1}) \sqrt{-1}} \dots 2 \frac{\sin \overline{s_n-r_n} A}{\overline{s_n-r_n}}, \quad (4)$$

the imaginary part evidently disappearing. This has to be integrated now for  $s_n$ . Let

$$(\overline{s_n-r_n})A = x, \text{ or } s_n = \frac{x}{A} + r_n, \text{ and } \frac{ds_n}{\overline{s_n-r_n}} = \frac{dx}{x}.$$

Then make A infinite. The limits of  $x$  thus become  $\pm \infty$ , and S =

$$\iint \dots d\theta_1 \dots d\theta_{n-1} \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_{n-1} \\ \times e^{(\overline{s_1-r_1} \theta_1 + \dots + \overline{s_{n-1}-r_{n-1}} \theta_{n-1}) \sqrt{-1}} \int_{-\infty}^{\infty} \frac{2 \sin x}{x} dx, \quad (5)$$

in which  $\phi(s_1 \dots s_n)$  is now by virtue of the equation  $(\overline{s_n-r_n})A = x$  a function of  $s_1 \dots s_{n-1}, x$ .

13. Now,  $\int_0^\infty \frac{\sin x}{x} dx$  is evidently positive, and  $\int_\pi^\infty \frac{\sin x}{x} dx$  is evidently negative. Therefore there exists an angle  $z$  between zero and  $\pi$  such that  $\int_z^\infty \frac{\sin x}{x} dx = 0$ , and similarly

$$\int_{-\infty}^{-z} \frac{\sin x}{x} dx = 0. \quad \text{But } \int_{-z}^z \frac{\sin x}{x} dx = \int_{-\infty}^\infty \frac{\sin x}{x} dx$$

is a determinate positive quantity, which shall be denoted by  $Z$ .

Again,  $\int_\pi^\infty \frac{\sin x}{x} dx$  is negative, and  $\int_{2\pi}^\infty \frac{\sin x}{x} dx$  is positive. Therefore there exists an angle  $z_1$  between  $\pi$  and  $2\pi$ , such that  $\int_{z_1}^\infty \frac{\sin x}{x} dx = 0$ , and therefore also  $\int_z^{z_1} \frac{\sin x}{x} dx = 0$ . Similarly there is an angle  $z_2$  between  $2\pi$  and  $3\pi$  such that  $\int_{z_1}^{z_2} \frac{\sin x}{x} dx = 0$ . And the range of integration from  $z$  to  $\infty$ , or from  $-z$  to  $-\infty$ , may thus be divided into parts  $z_1 - z$ ,  $z_2 - z_1$ , &c., such that  $\int_{z_{q-1}}^{z_q} \frac{\sin x}{x} dx = 0$  for all integral values of  $q$  greater than unity.

14. It can now be shown that

$$\int_{z_{q-1}}^{z_q} \frac{\sin x}{x} \phi(s_1 \dots s_n) dx = 0,$$

$\phi(s_1 \dots s_n)$  being by virtue of the equation  $(s_n - r_n)A = x$  a function of  $x$ .

For we may suppose  $x$  to increase from  $-\infty$  to  $+\infty$  by successive increments each not greater than  $2\pi$ . Then for any such increment of  $x$ , the increment of  $s_n$  is  $\leq \frac{2\pi}{A}$ , that is, it is infinitesimal. Also since  $\phi(s_1 \dots s_n)$  is finite and continuous for all possible values of  $s_1 \dots s_n$ ,  $\frac{d\phi(s_1 \dots s_n)}{ds_n}$  cannot be infinite. It follows that corresponding to the



increment ( $\leq 2\pi$ ) of  $x$ , the increment of  $\phi(s_1 \dots s_n)$  is infinitesimal. Therefore in the integral

$$\int_{z_{q-1}}^{z_q} \phi(s_1 \dots s_n) \frac{\sin x}{x} dx,$$

$z_q - z_{q-1}$  being less than  $2\pi$ ,  $\phi(s_1 \dots s_n)$  may be regarded as constant, and therefore

$$\int_{z_{q-1}}^{z_q} \phi(s_1 \dots s_n) \frac{\sin x}{x} dx = 0$$

for each integral value of  $q$  greater than unity. Therefore

$$\int_{-x}^x \phi(s_1 \dots s_n) \frac{\sin x}{x} dx = \int_{-z}^z \phi(s_1 \dots s_n) \frac{\sin x}{x} dx.$$

But in this integral, since  $z$  is less than  $\pi$ , the range of  $x$  is less than  $2\pi$ , and  $\phi(s_1 \dots s_n)$  may be treated as constant, having the value which it has when  $x=0$ , that is when

$$(s_n - r_n)A = 0, \text{ or } s_n = r_n, \text{ and } \phi(s_1 \dots s_n) = \phi(s_1 \dots s_{n-1} r_n).$$

We have then

$$\begin{aligned} & \iint \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma_1^n s - r \theta \sqrt{-1}} \\ &= Z \iint \dots d\theta_1 \dots d\theta_{n-1} \iint \dots \phi(s_1 \dots s_{n-1} r_n) ds_1 \dots ds_{n-1} e^{\Sigma_1^{n-1} s - r \theta \sqrt{-1}}. \end{aligned} \quad (6)$$

15. As the result of the two integrations for  $\theta_n$  and  $s_n$ ,  $\theta_n$  has disappeared, and  $s_n$  has been replaced by  $r_n$ . And by successive double integrations in this way

$$\iint \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma_1^n s - r \theta \sqrt{-1}}$$

is reduced to  $Z^n \phi(r_1 \dots r_n)$ ; or

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma_1^n s - r \theta \sqrt{-1}}, \quad (7)$$

and  $\frac{1}{Z^n}$  is a constant independent of  $r_1 \dots r_n$ .

Proposition I. is thus proved.

16. It comes next in order to prove that ( $n$  being very great) we may in evaluating the above integral neglect powers and products of  $\theta_1 \dots \theta_n$  above the second degree.

$$\text{Let } X = \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) e^{\Sigma s \theta \sqrt{-1}},$$

so that

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n X e^{-\Sigma r \theta \sqrt{-1}}. \quad (8)$$

Then

$$X = \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\cos \Sigma s \theta + \sqrt{-1} \sin \Sigma s \theta) = \alpha + \beta \sqrt{-1},$$

if  $\alpha$  denote the real, and  $\beta \sqrt{-1}$  the imaginary part of the last expression for  $X$ . Since  $\iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n = 1$ , evidently if every  $\theta = 0$ ,  $\alpha = 1$ , and  $\beta = 0$ , and  $\alpha^2 + \beta^2 = 1$ . And we can now prove

### Proposition II.

That if any  $\theta$  differs from zero,  $\alpha^2 + \beta^2$  contains the product of  $n$  factors, each of which, unless  $\theta = 0$ , is numerically, and generally in a finite ratio, less than unity.

For

$$\alpha^2 + \beta^2 = \left\{ \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \cos(s_1 \theta_1 + \dots + s_n \theta_n) \right\}^2 + \left\{ \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \sin(s_1 \theta_1 + \dots + s_n \theta_n) \right\}^2, \quad (9)$$

that is, the sum of the squares of the integrals. Or replacing the square of each integral by the product of two similar integrals between the same limits,  $\alpha^2 + \beta^2$

$$= \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \cos(\Sigma s \theta) \times \iint \dots ds'_1 \dots ds'_n \phi(s'_1 \dots s'_n) \cos(\Sigma s' \theta) \\ + \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \sin(\Sigma s \theta) \times \iint \dots ds'_1 \dots ds'_n \phi(s'_1 \dots s'_n) \sin \Sigma s' \theta. \quad (10)$$

$$= \iint \dots ds_1 \dots ds_n ds'_1 \dots ds'_n \phi(s_1 \dots s_n) \phi(s'_1 \dots s'_n) \cos(\overline{s_1 - s'_1} \theta_1 \\ + \overline{s_2 - s'_2} \theta_2 + \dots + \overline{s_n - s'_n} \theta_n) \quad (11)$$

17. Again,

$$\cos(\overline{s_1 - s'_1} \theta_1 + \overline{s_2 - s'_2} \theta_2 + \dots + \overline{s_n - s'_n} \theta_n)$$

consists, when expanded, of the term

$$\cos \overline{s_1 - s'_1} \theta_1 \cdot \cos \overline{s_2 - s'_2} \theta_2 \dots \cos \overline{s_n - s'_n} \theta_n,$$

and other terms each of which contains one or more of the factors  $\sin \overline{s_1 - s'_1} \theta_1$ ,  $\sin \overline{s_2 - s'_2} \theta_2$ , &c., each in the first degree.

Each of these factors, *e.g.*  $\sin(s_p - s_p')\theta_p$ , when integrated between the same limits for  $s_p$  and for  $s_p'$ , gives the result zero. Therefore every term vanishes in the integration for  $s \dots s'$  except the product of the cosines, and

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \iint \dots ds_1' \dots ds_n' \phi(s_1 \dots s_n) \phi(s_1' \dots s_n') \cos(\overline{s_1 - s_1'} \theta_1 + \\ & \quad \dots + \overline{s_n - s_n'} \theta_n) \\ = & \iint \dots ds_1 \dots ds_n \iint \dots ds_1' \dots ds_n' \phi(s_1 \dots s_n) \phi(s_1' \dots s_n') \cos \overline{s_1' - s_1'} \theta_1 \cos \overline{s_2 - s_2'} \theta_2 \\ & \quad \dots \cos \overline{s_n - s_n'} \theta_n. \end{aligned}$$

As  $n$  becomes indefinitely great, this product becomes indefinitely small, unless each of the factors  $\cos \overline{s - s'} \theta$  is equal to or nearly equal to unity. That condition is satisfied for  $\theta = 0$ , but not if  $\theta$  is any multiple of  $\pi$ , because if  $\theta = 0$ ,  $\cos \overline{s - s'} \theta = 1$  for all values of  $s - s'$ . But if  $\theta$  is a multiple of  $\pi$ , only for a particular value of  $\overline{s - s'}$ .

18. If this condition be not satisfied,  $\alpha^2 + \beta^2$ , since it contains the product of  $n$  cosines, which are *not* in general nearly equal to unity, is,  $n$  being very great, indefinitely small. Therefore  $\alpha$  and  $\beta$ , and therefore  $X$ , are indefinitely small. We might fix limits between which  $\theta_1 \dots \theta_n$  respectively must lie, say

$$\begin{array}{llll} \theta_1 & \text{between} & q_1 & \text{and} & -q_1 \\ \theta_2 & & q_2 & & -q_2 \\ \&c. & \theta_n & & q_n & & -q_n, \end{array}$$

in which  $q_1 \dots q_n$  are so small that all powers and products of them above the second degree may be neglected. Then unless  $\theta_1 \dots \theta_n$  lie within these limits,  $X$  is indefinitely small, and therefore also  $Xe^{-\Sigma r \theta \sqrt{-1}}$  indefinitely small. If  $\theta_1 \dots \theta_n$  do lie within these limits, we may in evaluating  $X$ , and therefore in evaluating

$$Xe^{-\Sigma r \theta \sqrt{-1}}, \text{ or } \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma \overline{s - s'} r \theta \sqrt{-1}},$$

neglect powers and products of  $\theta_1 \dots \theta_n$  above the second degree.

19. In the expression

$$\begin{aligned} \phi(r_1 \dots r_n) &= \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n X e^{-\Sigma r \theta \sqrt{-1}} \\ &= \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n \iint ds_1 \dots ds_n \phi(s_1 \dots s_n) e^{\Sigma \overline{s - s'} r \theta \sqrt{-1}}, \quad (12) \end{aligned}$$

expand the exponential, neglecting in accordance with the last article powers and products of  $\theta_1 \dots \theta_n$  above the second degree. Then  $e^{\overline{\Sigma s - r\theta} \sqrt{-1}}$  is replaced by

$$1 + \overline{\Sigma s - r\theta} \sqrt{-1} + \frac{1}{2} (\overline{\Sigma s - r\theta} \sqrt{-1})^2. \quad (13)$$

To this expression we may now apply art. 10, and so obtain

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) e^{\overline{\Sigma s - r\theta} \sqrt{-1}} \\ &= \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \\ &+ \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \overline{\Sigma_1^n s - r\theta} \sqrt{-1} \\ &- \frac{1}{2} \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\overline{\Sigma_1^n s - r\theta})^2 \\ &= 1 + \overline{\Sigma_1^n (s - r)} \theta \sqrt{-1} \\ &- \frac{1}{2} \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\overline{\Sigma_1^n s - r\theta})^2. \quad (14) \end{aligned}$$

20. Now,

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\overline{\Sigma_1^n s - r\theta})^2 \\ &= \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \overline{\Sigma_{p=1}^{p=n} (s_p^2 - 2s_p r_p + r_p^2)} \theta_p^2 \end{aligned}$$

for all positive integral values of  $p$

$$+ 2 \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \overline{\Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (s_p s_q + r_p r_q - s_p r_q - s_q r_p)} \theta_p \theta_q,$$

the last term including all pairs of unequal integral values of  $p$  and  $q$

$$\begin{aligned} &= \overline{\Sigma_{p=1}^{p=n} (s_p^2 - 2s_p r_p + r_p^2)} \theta_p^2 \\ &+ 2 \overline{\Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (s_p s_q + r_p r_q - s_p r_q - s_q r_p)} \theta_p \theta_q. \quad (15) \end{aligned}$$

by art. 10.

Also  $(\overline{\Sigma (s - r)} \theta)^2$

$$\begin{aligned} &= \overline{\Sigma_{p=1}^{p=n} (s_p^2 - 2s_p r_p + r_p^2)} \theta_p^2 \\ &+ 2 \overline{\Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (s_p s_q + r_p r_q - s_p r_q - s_q r_p)} \theta_p \theta_q. \quad (16) \end{aligned}$$

which differs from (15) only in  $\overline{s_p^2}$  being replaced by  $\frac{2}{s_p}$  for every  $p$ , and  $\overline{s_p s_q}$  by  $\overline{s_p s_q}$  for every  $p$  and  $q$ . Whence

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (1 + \Sigma \overline{s} - r\theta \sqrt{-1} - \tfrac{1}{2}(\Sigma \overline{s} - r\theta)^2) \\ &= 1 + \Sigma \overline{s} - r\theta \sqrt{-1} - \tfrac{1}{2}(\Sigma \overline{s} - r\theta)^2 \\ & \quad - \tfrac{1}{2} \Sigma_{p=1}^{p=n} (\overline{s_p^2} - \frac{2}{s_p}) \theta_p^2 - \Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (\overline{s_p s_q} - \overline{s_p} \overline{s_q}) \theta_p \theta_q, \end{aligned}$$

and therefore finally

$$\begin{aligned} \phi(r_1 \dots r_n) &= \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n e^{\Sigma(\overline{s} - r)\theta \sqrt{-1}} \\ & \quad \times e^{-\frac{1}{2} \Sigma (\overline{s^2} - \frac{2}{s}) \theta^2 + \Sigma \Sigma (\overline{s s'} - \overline{s} \overline{s'}) \theta \theta'}, \dots \dots (17) \end{aligned}$$

because in restoring the exponential we may again neglect powers and products of  $\theta_1 \dots \theta_n$  above the second degree.

21. I now simplify this expression by assuming every  $\overline{s}$  to be zero, and will point out in art. 37 how this becomes important. Our equation then becomes

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \int_{-\infty}^{+\infty} \dots d\theta_1 \dots d\theta_n e^{-(\frac{1}{2} \Sigma \overline{s^2} \theta^2 + \Sigma \Sigma \overline{s s'} \theta \theta')} e^{-\Sigma r \theta \sqrt{-1}}.$$

Let us now write

$$\begin{aligned} \overline{s_1^2} &= a_1, \\ \overline{s_2^2} &= a_2, \\ &\&c., \end{aligned}$$

and  $\overline{s_p s_q} = b_{pq} = b_{qp}$  for every  $p$  and  $q$ ; and let the index so obtained, or

$$\tfrac{1}{2} a_1 \theta_1^2 + b_{12} \theta_1 \theta_2 + \tfrac{1}{2} a_2 \theta_2^2 + \&c. + \tfrac{1}{2} a_n \theta_n^2 = Q_\theta,$$

so that

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \int_{-\infty}^{+\infty} \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} e^{-\Sigma r \theta \sqrt{-1}}. \quad (18)$$

22. Before integrating for  $\theta_1 \dots \theta_n$ , it is necessary to introduce the conjugate functions  $u_1 \dots u_n$ .

The coefficients  $\overline{a}$ ,  $b$  are a property of the given system, being determinate if  $s_1 \dots s_n$  are given in form. Let us then

write, using  $r_1 \dots r_n$  as any values of  $s_1 \dots s_n$ ,

$$\begin{aligned} r_1 &= a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c., \\ r_2 &= b_{21} u_1 + a_2 u_2 + b_{23} u_3 + \&c., \\ \&c. &= \&c., \end{aligned}$$

from which follow algebraically

$$\left. \begin{aligned} u_1 &= \frac{D_{11}}{D} r_1 + \frac{D_{12}}{D} r_2 + \frac{D_{13}}{D} r_3 + \&c., \\ u_2 &= \frac{D_{21}}{D} r_1 + \frac{D_{22}}{D} r_2 + \&c., \\ \&c. &= \&c., \end{aligned} \right\} \quad (19)$$

in which  $D$  is the determinant of the quadratic function  $Q_\theta$ , namely,

$$D = \begin{vmatrix} a_1 & b_{12} & b_{13} & \dots \\ b_{21} & a_2 & b_{23} & \dots \\ b_{31} & b_{32} & a_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad \text{or} \quad D = \Sigma \pm \frac{dr_1}{du_1} \frac{dr_1}{du_2} \dots,$$

and  $D_{11}$ ,  $D_{22}$ , &c. are its coaxial minors obtained by erasing the first, respective second, or  $n$ th row and column. Also  $D_{pq}$  is the anaxial minor obtained by erasing the  $p$ th row and  $q$ th column, or *vice versa*. Since, by definition,  $b_{pq} = b_{qp}$  for every  $p$  and  $q$ , it follows that  $D_{pq} = D_{qp}$  for every  $p$  and  $q$ . The signs of the anaxial minors are so taken that

$$D = a_1 D_{11} + b_{12} D_{12} + b_{13} D_{13} + \&c.$$

23. Again, let

$$\frac{D_{11}}{D} = \alpha_1, \quad \frac{D_{22}}{D} = \alpha_2, \quad \&c.,$$

and

$$\frac{D_{12}}{D} = \beta_{12} \dots \dots \frac{D_{pq}}{D} = \beta_{pq}.$$

The  $r$ 's and  $u$ 's are then connected by the symmetrical systems

$$\left. \begin{aligned} r_1 &= a_1 u_1 + b_{12} u_2 + \&c., \\ r_2 &= b_{21} u_1 + a_2 u_2 + \&c., \\ \&c., \end{aligned} \right\} \quad \dots \quad (19a)$$

and

$$\left. \begin{aligned} u_1 &= \alpha_1 r_1 + \beta_{12} r_2 + \&c., \\ u_2 &= \beta_{21} r_1 + \alpha_2 r_2 + \&c., \\ \&c. &= \&c. \end{aligned} \right\} \quad \dots \quad (19b)$$

Also if  $\Delta$  denote the determinant

$$\Delta = \begin{vmatrix} \alpha_1 & \beta_{12} & \beta_{13} & \dots \\ \beta_{21} & \alpha_2 & \beta_{23} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad . \quad . \quad . \quad (20)$$

$$\text{or} \quad \Delta = \Sigma \pm \frac{du_1}{dr_1} \frac{du_1}{dr_2} \dots,$$

then by a known proposition  $\Delta = \frac{1}{D}$ . And as

$$\alpha_1 = \frac{D_{11}}{D}, \quad \beta_{12} = \frac{D_{12}}{D}, \quad \&c.,$$

so reciprocally

$$a_1 = \frac{\Delta_{11}}{\Delta}, \quad b_{12} = \frac{\Delta_{12}}{\Delta}, \quad \&c.$$

These results may be confirmed as follows :—If we integrate  $r_1^2 e^{-Q}$  for all the variables  $r$  except  $r_1$  between limits  $\pm \infty$ , we obtain  $\overline{r_1^2} = \frac{\Delta_{11}}{\Delta}$ . But  $\overline{r_1^2} = a_1$  by definition. Therefore

$a_1 = \frac{\Delta_{11}}{\Delta}$ . Similarly  $\overline{r_1 r_2} = b_{12} = \frac{\Delta_{12}}{\Delta}$ , and so on. In the same way, expressing  $Q$  in terms of the  $u$ 's,

$$\iint \dots e^{-Q} u_1^2 du_2 \dots du_n = \frac{D_{11}}{D} = \overline{u_1^2},$$

$$\iint \dots e^{-Q} u_1 u_2 du_3 \dots du_n = \frac{D_{12}}{D} = \overline{u_1 u_2} \quad \&c.$$

24. The integration of  $e^{-Q_\theta} e^{-\Sigma r_\theta \sqrt{-1}}$  for  $\theta_1 \dots \theta_n$  between limits  $\pm \infty$  is given by Todhunter (Cambridge Phil. Trans. vol. xi. 1871, p. 219) on the assumption that the coefficients in  $Q_\theta$  are such as to make  $Q_\theta$  necessarily positive. The necessary and sufficient condition for which is that  $D$  and every coaxial minor of  $D$  is positive.

Let

$$\iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \cdot e^{-\Sigma r_\theta \sqrt{-1}} = U.$$

Then replacing  $e^{-\Sigma r_\theta \sqrt{-1}}$  by  $\cos \Sigma r_\theta - \sqrt{-1} \sin \Sigma r_\theta$ , we have, since the imaginary term disappears in the integration between  $+\infty$  and  $-\infty$ ,

$$U = \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \cos \Sigma r_\theta.$$

Then

$$\begin{aligned}\frac{dU}{du_1} &= - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \frac{d}{du_1}(r_1\theta_1 + \dots + r_n\theta_n), \\ &= - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \left( \theta_1 \frac{dr_1}{du_1} + \dots + \theta_n \frac{dr_n}{du_1} \right), \\ &= - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) (a_1\theta_1 + b_{12}\theta_2 \dots \&c.),\end{aligned}$$

or

$$\frac{dU}{du_1} = - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \frac{dQ_\theta}{d\theta_1} \dots \dots \dots (21)$$

Again integrating  $e^{-Q_\theta} \cos(r_1\theta_1 + \dots + r_n\theta_n)$  for  $\theta_1$  by parts, remembering that  $Q_\theta$  being positive the integrated term vanishes at both limits when  $\theta_1$  is infinite, we have

$$\begin{aligned}\int_{-\infty}^{\infty} d\theta_1 e^{-Q_\theta} \cos(r_1\theta_1 + \dots + r_n\theta_n) \\ = \frac{1}{r_1} \int_{-\infty}^{\infty} d\theta_1 e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \frac{dQ_\theta}{d\theta_1}.\end{aligned} \quad (22)$$

Integrate both sides of this for  $\theta_2 \dots \theta_n$ . That gives by (21) and (22)

$$U = -\frac{1}{r_1} \frac{dU}{du_1} \quad \text{or} \quad \frac{dU}{du_1} = -r_1 U.$$

Similarly

$$\frac{dU}{du_2} = -r_2 U, \&c.,$$

and therefore, since

$$u_1 \frac{dr_1}{du_1} + u_2 \frac{dr_2}{du_1} + \&c. = r_1,$$

$$U = A e^{-\frac{1}{2}(r_1 u_1 + r_2 u_2 + \dots + r_n u_n)} = A e^{-\frac{1}{2} \Sigma r u}, \quad \dots \dots (23)$$

where  $A$  is constant.

Let

$$Q = \frac{1}{2} \Sigma r u,$$

or

$$\left. \begin{aligned}Q_u &= \frac{1}{2} a_1 u_1^2 + b_{12} u_1 u_2 + \dots + \frac{1}{2} a_2 u_2^2 + \&c., \\ Q_r &= \frac{1}{2} \alpha_1 r_1^2 + \beta_{12} r_1 r_2 + \dots + \frac{1}{2} \alpha_2 r_2^2 + \&c.\end{aligned} \right\} \dots \dots (24)$$

We have thus three equivalent forms of the index  $Q$ . Now  $Q_u$  is the same function of  $u_1 \dots u_n$  that  $Q_\theta$  is of  $\theta_1 \dots \theta_n$ . This is Todhunter's result modified only by a change in notation, he using  $a$  and  $2b$  where I have used  $\frac{1}{2}a$  and  $b$ . The proposition stated in art. 11 is thus proved.

25. It is convenient at this point to consider the relations between the coefficients  $a$ ,  $b$  and  $\alpha$ ,  $\beta$ , and between them and



the  $r$ 's and  $u$ 's, as regards dimensions. In  $e^{-Q}$ ,  $Q$  must have zero dimensions, that is  $r_1 u_1$  has zero dimensions, or those of  $u$  are the reciprocal of those of  $r$ . Therefore  $\alpha, \beta$  must have dimensions the reciprocal of those of  $r^2$ . If, therefore, for instance  $r$  has dimensions  $\frac{L}{T}$ ,  $u$  has  $\frac{T}{L}$ ,  $\alpha, \beta$  &c. have  $\frac{L^2}{T^2}$ , or if  $r$  has  $\frac{T}{L}$ ,  $u$  has  $\frac{L}{T}$ . Also if  $r$  has  $\frac{L}{T}$ ,  $D$  has  $\left(\frac{L^2}{T^2}\right)^n$ , and  $\frac{D_{11}}{D}$  &c., or  $\alpha, \beta$ , have  $\frac{T^2}{L^2}$ .

## PART II.

26. We have thus proved that  $\phi(r_1 \dots r_n)$ , which represents the chance of the variables  $s_1 \dots s_n$  having the values  $r_1 \dots r_n$ , or values infinitely near thereto, has the form  $\phi(r_1 \dots r_n) = C e^{-Q}$ , and  $Q$  is expressed as a quadratic function of  $r_1 \dots r_n$ , containing products as well as squares of the variables, and with coefficients as above stated. Also expressing  $\phi(r_1 \dots r_n)$  in terms of  $u_1 \dots u_n$ , we may write

$$\begin{aligned} \phi(r_1 \dots r_n) dr_1 \dots dr_n &= \psi(u_1 \dots u_n) du_1 \dots du_n \Sigma \pm \frac{dr_1}{du_1} \frac{dr_1}{du_2} \\ &= \psi(u_1 \dots u_n) du_1 \dots du_n D, \end{aligned}$$

or since  $D = \frac{1}{\Delta}$ ,

$$\psi(u_1 \dots u_n) du_1 \dots du_n = \phi(r_1 \dots r_n) dr_1 \dots dr_n \Delta.$$

Whether we use  $r_1 \dots r_n$  or  $u_1 \dots u_n$  for independent variable is a question of convenience.

Since  $e^{-Q}$  does not contain the time explicitly, we are in effect assuming that it is independent of the time, and therefore that if the system represented by  $r_1 \dots r_n$  is in motion, such motion is stationary. And  $\phi(r_1 \dots r_n) dr_1 \dots dr_n$ , or  $\psi(u_1 \dots u_n) du_1 \dots du_n$ , represents the time during which, on average of any sufficiently long time in that stationary motion, the variables lie within the limits  $r_1 \dots r_1 + dr_1$ , &c., or  $u_1 \dots u_1 + du_1$ , &c.

27. Since  $Q$  contains products of the variables, the law of distribution of the values of  $s_1 \dots s_n$  is not generally of the form  $e^{-(m_1 s_1^2 + m_2 s_2^2 + \dots + m_n s_n^2)}$ , with  $m_1 \dots m_n$  constants, and cannot possibly have that form, unless firstly every  $\bar{s} = 0$ , and secondly there is no correlation, for if  $Q = \Sigma m s^2$ ,  $e^{-Q}$  is the product of  $n$  factors each containing only one of the variables  $s_1 \dots s_n$ . Therefore, by definition, there is no correlation.

Every proof of the theorem in its ordinary form, *i. e.* with  $Q = \sum ms^2$ , either is based necessarily on the assumption, express or implied, that there is no correlation, or else it would prove the absence of correlation as a necessary fact, that is, that no physical system in which the motions of the parts are correlated can exist in stationary motion.

28. The theorem as above stated fails, or the proof fails, if  $Q$  can become negative; and therefore the proof fails if  $D$ , or as the case may be  $\Delta$ , becomes negative. It fails also, and this is important, if one of the variables on which  $Q$  depends becomes discontinuous, which may happen by the variation of external conditions.

### *The Law of Maximum Probability.*

29. Since  $Q$  contains products as well as squares of the variables  $r_1 \dots r_n$  or  $u_1 \dots u_n$ , we can effectively make  $e^{-Q}$  maximum, subject to the constants of the system, and the kinetic energy may be one of such constants. By making  $e^{-Q}$  maximum we obtain the most probable, or normal state of the system, subject to the constants.

30. Whether we should use  $r_1 \dots r_n$ , or  $u_1 \dots u_n$  for independent variables is, as above stated, a question of convenience.

The constant kinetic energy has dimensions  $\frac{ML^2}{T^2}$ . It is therefore convenient to use the  $r$ 's, if  $r$  has dimensions  $\frac{L}{T}$ , and  $u$  has  $\frac{T}{L}$ ; and the  $u$ 's, if  $u$  has dimensions  $\frac{L}{T}$ . I will assume then that the kinetic energy, or  $2nE$ ,

$$= m_1 u_1^2 + m_2 u_2^2 + \dots + m_n u_n^2$$

and

$$Q = \frac{1}{2} a_1 u_1^2 + b_{12} u_1 u_2 + \dots + \frac{1}{2} a_2 u_2^2 + \&c.$$

The kinetic energy,  $\sum mu^2$ , if expressed in terms of  $r_1 \dots r_n$ , would be a quadratic function of  $r_1 \dots r_n$ , with coefficients functions of  $\alpha, \beta$  of (20).

31. I assume also that  $E$  is either constant, or varies very slowly with the time, while  $u_1 \dots u_n$  in general vary very rapidly, so that they may go through cycles of changes while  $E$  is sensibly constant.  $E$  belongs to the class of variables which Max Planck calls "langsam veränderlich," while  $s_1 \dots s_n$  belong to the class "schnell veränderlich."

Also I assume that the  $b$ , or correlation coefficients, are

functions of *inter alia* a quantity  $v$ , which, like  $E$ , is “*langsam veränderlich*.” And that they possess the property that as  $v$  diminishes every  $b^2$  increases *cæteris paribus*. That is, the  $b$  coefficients may, as mentioned in art. 8, be functions of the instantaneous distance between the variables to which they relate, *e. g.*,  $b_{pq}$  may be a function of  $\rho$ , the distance which at the instant separates  $u_p$  and  $u_q$ , or the things to which  $u_p$  and  $u_q$  relate, from each other. If then the distances  $\rho$  diminish by the diminution of  $v$ , which will generally be the case if  $v$  denote the volume of the system, the condition that every  $\frac{dv^2}{dv}$  is negative will be satisfied, if every  $\frac{db^2}{d\rho}$  is negative.

32. Since the two functions  $\phi(r_1 \dots r_n)$  and  $Ce^{-Q}$ , or let us say  $\phi$  and  $F$ , are equal to one another throughout a certain range of values of  $v$ , for all of which  $D$  is positive, it follows that at every point or value of  $v$  within that range  $\frac{d\phi}{dv} = \frac{dF}{dv}$ , and all the derived coefficients of  $\phi$ , as  $\frac{d^2\phi}{dv^2}$ , &c., are respectively equal to the corresponding derived coefficients of  $F$ . Therefore, by Taylor's theorem,  $\phi = F$  for all values of  $v$  for which that theorem can be legitimately applied, with initial  $v$  within the given range. If, however, when a certain value of  $v$ , say  $v = V$ , is reached, a discontinuous change takes place in  $v$ , Taylor's theorem will in general at that point fail, and the equation  $\phi = F$  will cease to be true.

33. Since  $e^{-Q}$  is maximum,

$$\frac{dQ}{du_1} \partial u_1 + \frac{dQ}{du_2} \partial u_2 + \&c. = 0,$$

and since  $E$  is constant,

$$m_1 u_1 \partial u_1 + m_2 u_2 \partial u_2 + \&c. = 0.$$

Whence, if  $\lambda$  be the indeterminate multiplier,

$$\frac{dQ}{du_1} = \lambda m_1 u_1, \quad \frac{dQ}{du_2} = \lambda m_2 u_2 \quad \&c.,$$

or in the notation of (24)

$$\left. \begin{aligned} a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c. &= \lambda m_1 u_1 \\ b_{12} u_1 + a_2 u_2 + b_{23} u_3 + \&c. &= \lambda m_2 u_2 \\ \&c. \end{aligned} \right\} \quad . \quad . \quad (A)$$

and  $\lambda$  is given by the determinantal equation of the  $n$ th degree,

$$\left. \begin{array}{cccc} \left( \frac{a_1}{m_1} - \lambda \right) & \frac{b_{12}}{m_1} & \frac{b_{13}}{m_1} & \dots \\ \frac{b_{21}}{m_1} & \left( \frac{a_2}{m_2} - \lambda \right) & \frac{b_{23}}{m_2} & \dots \\ \frac{b_{31}}{m_3} & \frac{b_{32}}{m_3} & \left( \frac{a_3}{m_3} - \lambda \right) & \dots \\ \dots & \dots & \dots & \dots \end{array} \right\} = 0 \quad \dots \quad (B)$$

If  $\lambda$  be any real root of equation (B), its substitution in (A) determines the ratios  $\frac{u_2}{u_1}, \frac{u_3}{u_1}$ , &c. in a system of values of  $u_1 \dots u_n$  for which  $Q$  is minimum, given E.

Also

$$Q = \frac{1}{2} \sum u \frac{dQ}{du} = \frac{1}{2} \lambda \sum m u^2 = \lambda n E,$$

and this, with the ratios  $\frac{u_2}{u_1}, \frac{u_3}{u_1}$ , &c., determines an actual set of values of  $u_1 \dots u_n$  which make  $Q$  minimum, given E.

34. Now expanding equation (B), we have

$$\lambda^n - \Delta_1 \lambda^{n-1} + \Delta_2 \lambda^{n-2} - \dots \pm \Delta_n = 0; \quad \dots \quad (C)$$

in which  $\Delta_1 = \sum \frac{a}{m}$ , and is the sum of the roots,  $\Delta_n$  is the determinant B when  $\lambda=0$ , and differs from D only by the factor  $\frac{1}{m_1 m_2 \dots m_n}$ , and is also the product of the  $n$  roots, and has the + or - sign prefixed, according as  $n$  is even or odd.  $\Delta_{n-1}$  is the sum of the coaxial minors of D, having  $(n-1)^2$  constituents, each divided by the product of  $(n-1)$   $m$ 's, and is equal to the sum of the products of the roots taken  $n-1$  together, and has the opposite sign to that of  $\Delta_n$  prefixed, and so on down to  $\Delta_2$ , which is the sum of the coaxial minors of D having  $2^2$  constituents, each divided by the product of two  $m$ 's, as  $\frac{a_1 a_2 - b_{12}^2}{m_1 m_2}$ , and is equal to the sum of the products of all pairs of the roots, and always has the positive sign prefixed.

It thus appears that the determinant D divided by

$m_1 m_2 \dots m_n$  is the product of  $n$  factors, which are the  $n$  roots of equation (B); also that the sum of the coaxial minors of  $D$  each of  $t^2$  constituents, and each divided by the proper factors  $m$ ,  $t$  in number, is equal to the sum of the products of the  $n$  factors of  $\Delta_n$  taken  $t$  together, whatever number  $t$  may be less than  $n$ . For  $\Delta_{n-1}$  is by the theory of equations the sum of the products of the  $n$  roots taken  $n-1$  together. And it is also the sum of the coaxial minors of  $D$  each of  $(n-1)^2$  constituents divided as aforesaid, &c. It is possible to construct a determinant which shall possess this property, but its demonstration would be too long for the present paper.

Since  $Q = \lambda n E$ , and  $E$  is positive,  $Q$  has the same sign as  $\lambda$ ; and since we are limited to positive values of  $Q$ , we are limited to positive values of  $\lambda$ . That is, equation (B) can have for our purpose no negative roots. But  $D$  is proportional to the product of the  $n$  roots of (B). Therefore we are limited to positive values of  $D$ .

35. Every maximum value of  $e^{-Q}$  determines a state of stable equilibrium for the system, that is, if  $u_1 \dots u_n$  are changing rapidly, a state of stationary motion. There may be many such states corresponding to different real roots of (B). We may call the state corresponding to any particular real root  $\lambda$ , the state  $\lambda$ . If  $\lambda_1$  and  $\lambda_2$  be two real roots, and  $\lambda_2 > \lambda_1$ , then the state  $\lambda_1$  is more probable than the state  $\lambda_2$  in the ratio  $e^{(\lambda_2 - \lambda_1)nE}$ . If  $\lambda_2 - \lambda_1$  is not nearly zero, and  $n$  very great, then the state  $\lambda_1$  is more probable than the state  $\lambda_2$  in a very high ratio, and the more so as  $nE$  increases. It follows that very great values of  $\lambda$  are in a very high degree improbable, and may be neglected. The state corresponding to the *least root* of  $B$  is the most probable of all the states of stable equilibrium—I define it to be the *normal state*.

36. I think the method thus investigated is applicable to determine the normal state of any material system whose parts mutually influence each other, and therefore become correlated. And is not this the case with almost all material systems in nature? A rare gas is perhaps the only known system to which the assumption of no correlation has been or can be legitimately applied. Further, a system of mutually acting, and therefore correlated, parts is a living system. On the other hand, if  $Q$  be reduced to a sum of squares, as Boltzmann's  $H$  theorem professes to prove, it would be, if left to itself, a dead system, for which no further change is possible.

## On the Equipartition of Energy.

37. In the first equation of (19)

$$u_1 = \frac{D_{11}}{D} r_1 + \frac{D_{12}}{D} r_2 + \&c.,$$

multiply both sides by  $r_1$ , and take mean values of both sides, remembering that since  $r_1 \dots r_n$  are possible values of  $s_1 \dots s_n$ ,  $\overline{r_1^2} = \overline{s_1^2} = a_1$ . Similarly  $\overline{r_2^2} = a_2$  &c. and  $\overline{r_p r_q} = b_{pq}$  &c. That gives

$$\begin{aligned} \overline{r_1 u_1} &= \frac{a_1 D_{11} + b_{12} D_{12} + b_{13} D_{13} + \&c.}{D} \\ &= \frac{D}{D} = 1. \end{aligned}$$

Similarly  $\overline{r_2 u_2} = 1$  and  $\overline{r_1 u_1} = \overline{r_2 u_2}$  &c. =  $\overline{r_n u_n}$ . Note that this result would fail if  $\overline{s_1}, \overline{s_2}$  &c. were not zero, for then  $\overline{r_1^2} \neq a_1$  &c.

Evidently  $\overline{r_1 u_1} = \overline{u_1 \frac{dQ}{du_1}}$  &c. and  $\overline{r_1 u_1} = \overline{r_1 \frac{dQ}{dr_1}}$  &c. Whence the law of equipartition of energy takes the general form

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} = \&c.$$

Let us now apply the results of making  $Q$  minimum above investigated. We have, since

$$Q = \lambda n E, \quad \frac{dQ}{du} = \lambda n \frac{dE}{du}, \quad \text{or} \quad u \frac{dQ}{du} = \lambda n u \frac{dE}{du},$$

for each  $u$ . Whence, if we may assume that

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} \quad \&c.$$

for the normal or most probable state, we have

$$\overline{u_1 \frac{dE}{du_1}} = \overline{u_2 \frac{dE}{du_2}} \quad \&c.$$

And if  $2nE = \Sigma m u^2$ ,  $\overline{m_1 u_1^2} = \overline{m_2 u_2^2}$  &c. It may perhaps be objected that the law

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} \quad \&c.$$

is proved only when the means are taken over *all* values of the variables, consistent with the constancy of  $E$ , and may

fail if confined to those values which make  $Q$  minimum. But  $E$  is constant for all possible values of  $u_1 \dots u_n$ , and therefore for the values that make  $Q$  minimum.

It appears then that the law of equipartition of energy is not necessarily dependent on Maxwell's law of the distribution of velocities,  $e^{-\frac{1}{2}\sum mu^2}$ , but depends on the conditions (1) that every variable  $s$  has zero for its mean, and (2) that  $2nE$  has the form  $\sum mu^2$ .

*Application of the Method to Gases.*

38. I assume now that the system represents a gas, or two or more gases uniformly mixed, and that  $u_1 \dots u_n$  are the vector velocities,  $m_1 \dots m_n$  the masses, of the molecules. For the dimensions of  $u_1 \dots u_n$  and the other variables in this case see art. 25.

I assume, further, that each of the gases forming the mixture, or the single gas if there be only one, is homogeneous as regards the constitution of its molecules, and that the mixture is homogeneous as regards the proportions in which different gases are mixed. Also that  $E$  represents its temperature, and is uniform throughout. And that  $v$  represents the volume containing a given number of molecules, and has, up to a certain point hereafter to be defined, the same value at all points. The definitions of *temperature at a point* and *density at a point* present no difficulty in a system of this homogeneous character.

Since the number of molecules in volume  $v$  is proportional to  $v$ ,  $\sum mu^2$  for the molecules in volume  $v$  is proportional to  $vE$ ; and instead of  $Q = \lambda nE$  it is convenient to write  $Q = \lambda vE$ , using  $v$  to express so much of the space occupied by the gas as contains  $n$  molecules.

I proceed to prove that the velocities of any two molecules if sufficiently near to each other, say at distance less than  $\rho$  from each other, will be appreciably correlated, if at sufficiently great distance inappreciably. And I assume  $-b_{pq}$  to be, for any distance  $\rho$ , proportional to the force  $R$  which at that distance the two molecules denoted by  $p$  and  $q$  exert on each other, and that for very small values of  $\rho$  such force is repulsive, and  $-b_{pq}$  negative. This is consistent with art. 8.

39. In Clausius' Virial equation, let  $P$  denote pressure and  $v$  volume,  $u$  the vector velocity,  $m$  the mass of a molecule,  $R$  the force which acts between two molecules at distance  $\rho$  from each other,  $R$  being positive when the force is repulsive. Then Clausius' equation is

$$\frac{3}{2}Pv = \sum \frac{1}{2}mu^2 + \frac{1}{2}\sum\sum R\rho.$$

If the gas be rare enough, the last or Virial term is negligible, because  $R$  is negligible for all but an infinitely small proportion of the pairs of molecules at every instant. For such a gas then

$$\frac{3}{2} P v = \frac{1}{2} \Sigma m u^2 = n E$$

very approximately. That is, Boyle's law is fulfilled very approximately. This is the case for air and other gases at ordinary pressures. For such gases, since  $R$  is, for nearly all pairs, negligible, the correlation coefficients  $b$  are, for nearly all pairs, negligible, and Maxwell's law holds. But as the density of the gas increases, the resistance to compression increases beyond—and ultimately very far beyond—what it would be under Boyle's law (see Lord Kelvin's paper "On the Problem of a Spherical Gaseous Nebula," p. 260 note). For a sufficiently dense gas therefore the Virial term can no longer be neglected, as a considerable part of the pressure  $P$ , in  $\frac{3}{2} P v$ , is due to it. For the same reason the correlation coefficients  $b$ , which are proportional to the  $R$ 's, can no longer be neglected.  $Q$  becomes a complete quadratic function containing products as well as squares of the velocities.

40. Experimentally, by reason of the homogeneity, any portion of the gas, if containing a sufficiently great number of molecules, has the same properties as any other portion, or as the whole. Analytically the same result appears thus: About any point  $O$  in the gas as centre, suppose a sphere of radius  $c$  containing  $N$  molecules, and another concentric sphere of radius  $c + c'$ , containing  $N + N'$  molecules, and both  $c$  and  $c'$  much greater than the radius of correlation. Let  $Q_N$  be the value of  $Q$  for the  $N$  molecules in the  $c$  sphere,  $Q_{N+N'}$  its value for the  $N + N'$  molecules within the sphere  $c + c'$ . Then, if  $N$  be very great,

$$\frac{Q_N}{Q_{N+N'}} = \frac{N}{N + N'}.$$

Therefore  $\frac{Q}{NE}$ , or  $\lambda$ , is independent of  $N$ . It is true the number of roots of equation B, as applied to the  $N$  molecules, is  $N$ , and as applied to the  $N + N'$  molecules is  $N + N'$ , or  $N'$  new roots are introduced by taking in the additional  $N'$  molecules. The statement that  $\lambda$  is independent of  $N$  means therefore that all the roots of (B), as applied to the  $N$  molecules, are also roots of (B) as applied to the  $N + N'$  molecules. And in particular the least root of (B) is the same in the two cases. The correlation coefficients between the  $N'$  new variables and the original  $N$  are, except in an infinitely small proportion of the cases, evanescent. Therefore by art. 35 the normal state



is the same in the two cases ; and therefore the system of  $N$  molecules in the sphere  $c$  has the same physical properties as the system of  $N+N'$  molecules in the sphere  $c+c'$ .

41. I have not up to this point assumed either sign for the correlation coefficients. That must depend on the physical relations of the system. If the forces between two molecules be repulsive at sufficiently small distances, as must be the case if discrete molecules are to exist permanently, let  $\chi$  denote the potential at any point of all such forces. Then, according

to Boltzmann's law,  $e^{-2h\chi}$ , where  $\frac{3}{2h}$  is mean kinetic energy,

must be a maximum, or  $\bar{\chi}$  must be a minimum, given the total energy, when the motion is stationary. That will be the case if molecules very near each other move on average in the same direction, so that very near approaches involving high potentials are rare. That is, if,  $m_p$  and  $m_q$  are neighbouring molecules, the scalar product  $u_p u_q$  is on average positive.

The molecules of the dense gas tend to move, not as in the ordinary rare gas each independently of all its neighbours, but *in streams*. And this tendency increases as the density increases.

Of two motions of a system with the same total energy,  $\chi+T$ , that one is the more probable for which  $h\chi$  is the less. Let the potential of mutual action of two molecules  $m$  and  $m'$ , when distant  $r$  from each other, be  $f(r)$ . We might assume  $f(r) = \frac{1}{r^q}$ , where  $q$  is positive and not less than unity, or  $f(r) = \frac{1}{r} e^{-\kappa r}$ , where  $\kappa$  is positive. In either case if  $x < r$

$$f(r+x) + f(r-x) > 2f(r) \quad \text{and} \quad \frac{d}{dx} \{f(r+x) + f(r-x)\}$$

is positive. It follows that if  $r$  is constant on average of time, the mean value of  $f(r)$  for the same time is least when  $m$  and  $m'$  have no relative velocity, or  $x=0$ , and increases as the kinetic energy of their relative velocity increases. If therefore they have any common velocity, *i. e.* stream motion, the mean potential  $\chi$  or  $\Sigma f(r)$  is less than it would be if with the same total kinetic energy they had no stream motion.

I think the existence of the streams is thus proved. It is a plausible theory that, of all possible stationary motions which a material system may have, that one is the most probable, and therefore will be the actual motion, in which the mean value of  $e^{-2h\chi}$  is maximum, or  $\bar{\chi}$  is minimum. That is analogous to the theorem, that of all

positions which a material system may have in statical equilibrium, that one will be chosen for which  $\chi$  is minimum. In fact, by making  $h$  infinite, the first theorem is reduced to the statical theorem. If this theorem be not accepted in all its generality, still it must be accepted for a gas, for which in fact Boltzmann proved it. It amounts merely to saying that the system chooses, for given total energy, the motion of *least resistance*.

42. It is evident that, as  $v$  diminishes, and therefore the coefficients of products in  $Q$  increase in absolute magnitude,  $Q$  generally diminishes. For the only terms in  $Q$  that can be negative are the terms containing products. And such terms can always be made negative by making the correlation coefficient, e. g.  $b_{pq}$ , negative if  $u_p$  and  $u_q$  have the same sign, or by making it positive if they have opposite signs. As we have already seen, there are, if the system represents a gas, physical reasons why when  $b_{pq}$  is not inappreciable,  $u_p$  and  $u_q$  should in general have the same sign, and therefore why, if  $Q$  is to be as small as possible,  $b_{pq}$  should be negative.

But, at all events when  $v$  is very great, and therefore the correlation coefficients very small,  $\frac{D_{pq}}{D}$ , which is equal to  $\frac{\overline{u_p u_q}}{D}$ , has the opposite sign to  $b_{pq}$ . For  $D_{pq} = -b_{pq}$  multiplied by a coaxial minor of  $D$  which is positive, plus terms containing products of more than one  $b$ , which,  $v$  being great enough, are negligible compared with the term containing only  $b_{pq}$ . Therefore  $D_{pq}$  is generally of the opposite sign to  $\overline{r_p r_q}$ , that is to  $b_{pq}$ . On the other hand,  $\beta_{pq}$  of (20) is equal to  $\overline{u_p u_q}$ , and is positive.

Since, further,  $Q$  diminishes as  $v$  diminishes, and  $Q = \lambda n E$ , it follows that  $\lambda$  diminishes as  $v$  diminishes, or if  $\lambda$  be the least root of  $B$ ,  $\frac{d\lambda}{dv}$  is positive. Also, since as  $v$  diminishes the correlation terms in  $Q$  assume relatively more importance,  $\frac{d\lambda}{dv}$  generally increases as  $v$  diminishes, and  $\frac{d^2\lambda}{dv^2}$  is negative.

43. Having thus explained what I consider to be the general form of motion of the dense gas, I now assume that it is being compressed with constant temperature, that is  $v$  diminishes,  $E$  remaining constant.

I make an hypothesis, namely this:—That “the critical volume”  $V$  at which a gas under compression undergoes liquefaction, is the volume at which  $D=0$ , and that in such liquefaction  $v$ , the volume, changes discontinuously. And

that this is the discontinuous change in  $v$  at which the equation  $\phi(r_1 \dots r_n) = Ce^{-Q}$  ceases to be true, expressed above analytically as the failure of Taylor's Theorem. It will appear that this hypothesis leads to results agreeing with experiment in the liquefaction of gases under pressure.

44. *On possible variations of density between different parts of the system, as a consequence of the diminution of  $v$ .*

If for the whole gas, or for any separate portion of it,  $v$  varies continuously,  $\lambda$  the least real root of equation (B), as applied to that portion, will in general vary continuously as a function of  $v$ .

Let us suppose the volume  $v$  of the gas to be divided into  $N$  parts, each equal to  $\frac{v}{N}$ . Let  $v'$  be a volume less than  $v$  divided into  $N$  parts each equal to  $\frac{v'}{N}$ , and let  $\lambda'$  and  $\frac{d\lambda'}{dv}$  denote the values of those functions for  $v'$ . Suppose that by the compression  $v$  is reduced to  $v - \partial v$ . This may happen in either of two ways. Suppose  $\partial v = \frac{v - v'}{N}$ . Then, event A, as  $v$  diminishes by  $\partial v$ , each of the  $N$  parts into which  $v$  is divided diminishes in the same ratio, so that the density, however varying with the time, remains constant in space throughout the whole volume  $v$ . Or, event B, as  $v$  is diminished by  $\partial v$ , some one of the  $N$  parts into which  $v$  is divided, is diminished by  $\partial v$ , and therefore, since  $\partial v = \frac{v - v'}{N}$ , is reduced to  $\frac{v'}{N}$ , the other parts remaining unchanged in volume.

In event A,  $\lambda$  varies continuously throughout, and becomes for each diminution  $\partial v$  of  $v$ ,  $\lambda - \frac{d\lambda}{dv} \partial v$ . The chance of this happening is therefore, for each  $\partial v$ , represented by

$$e^{-(\lambda - \frac{d\lambda}{dv} \partial v) v E}.$$

In event B, the chance is the product of two chances, (1) that for  $\frac{v}{N}$ ,  $\lambda$  shall become  $\lambda'$ , and (2) that for  $v(1 - \frac{1}{N})$ ,  $\lambda$  shall be unchanged. The chance is therefore

$$e^{-\lambda' \frac{vE}{N}} e^{-\lambda(1 - \frac{1}{N}) v E}.$$

45. If  $P_A$  denote the probability of event A,  $P_B$  that of event B, we have

$$P_A = e^{-(\lambda - \frac{d\lambda}{dv}) \frac{vE}{N}},$$

$$P_B = e^{-\lambda' \frac{vE}{N}} e^{-\lambda(1 - \frac{1}{N})vE},$$

and  $P_B/P_A = e^{(\frac{\lambda - \lambda'}{v - v'} - \frac{d\lambda}{dv}) \frac{vE}{N}}$ .

Event B is the more probable when  $\lambda - \lambda'$  is greater than

$$(v - v') \frac{d\lambda}{dv}.$$

Now  $\lambda' = \lambda - (v - v') \frac{d\lambda}{dv} + \frac{1}{2}(v - v')^2 \frac{d^2\lambda}{dv^2} + \&c.,$

and by making  $v - v'$  very small we may reject all higher powers of it. And since  $\frac{d^2\lambda}{dv^2}$  is negative (art. 42)  $\frac{\lambda - \lambda'}{v - v'} \geq \frac{d\lambda}{dv}$

—that is  $P_B \geq P_A$ . Further, when  $v' = V$ ,  $\lambda' = 0$ , because  $\lambda'$  is the least root of (B), and must therefore vanish when D, the product of the roots, becomes zero. Therefore ultimately

$$\frac{\lambda}{v - v'} > \frac{d\lambda}{dv}.$$

46. It thus appears that, consistently with the theory of probabilities, inequalities of density *may arise* at any time. But so long as the system remains gas, they will be dispersed by diffusion as fast as they arise. The uniformity of density is therefore stable. But as  $v$  approaches  $V$ , the portion of gas which assumes the greater density becomes liquid, and is not dispersed by diffusion. Liquefaction will therefore take place of separate portions of gas successively, other portions remaining as gas unaffected, until, with continuing compression, they undergo liquefaction in their turn. And the proportion of gas liquefied for a diminution,  $\partial v$ , of the volume of gas for the time being is  $\frac{\partial v}{v - V}$ .

47. It is true that we cannot define *a priori* what particular part of  $v$  will as  $\frac{v}{N}$  bear the whole loss of volume  $\partial v$ . In the same way, when a gas liquefies under pressure, we know that as the diminishing volume approaches the critical volume, some portion, though we cannot define what specific portion, will be liquefied, the rest remaining as gas unaffected. In

fact the result above obtained agrees in this respect with the results obtained experimentally in the liquefaction of gases under compression.

### *Change of E.*

48. As  $E$  increases with  $v$  constant, the axial constituents of the determinant  $D$  increase, while the anaxial constituents which depend on  $v$  remain generally constant. The effect of that is to increase  $D$  and all its coaxial minors. If therefore  $E$  be great enough, no amount of compression will reduce  $D$  to zero, and liquefaction cannot take place.

## II. *Electrical Oscillations in Coupled Circuits.* By E. TAYLOR JONES, D.Sc., *Professor of Physics in the University College of North Wales, Bangor* \*.

[Plate I.]

IT is well known that in a system consisting of two circuits containing capacity, self-inductance, and sufficiently great mutual inductance, each circuit has two natural periods of electrical oscillation. The problem of determining the constants of the two oscillations has been considered by Oberbeck †, M. Wien ‡, and Drude §. Drude showed that in addition to having different periods the two oscillations also have in general different damping coefficients, and calculated their values; he further showed how to calculate the potential at the terminals of the secondary circuit at any time after the application of a given potential-difference to those of the primary.

The present paper deals mainly with the case in which the oscillations are started by breaking a current in the primary. The expression for the secondary potential is deduced for this case by Drude's method, and compared with measurements of photographs obtained by means of the short-period electrometer (or "electrostatic oscillograph," as it has been called) described by the author ||. The instrument shows that the course of the variation of the secondary potential is much simpler when the oscillations are started

\* Communicated by the Author.

† Oberbeck, *Wied. Ann.* lv. p. 623 (1895).

‡ M. Wien, *Wied. Ann.* lxi. p. 151 (1897).

§ Drude, *Ann. der Physik*, xiii. p. 512 (1904).

|| E. T. Jones, *Phil. Mag.* August, 1907.