



I. On the theory of capillary action, and the depression of the mercury in the tubes of barometers

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[NEW SERIES.]

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- I. *On the Theory of Capillary Action, and the Depression of the Mercury in the Tubes of Barometers.* By J. IVORY, Esq. M.A. F.R.S.*

THE elevation and depression of liquids in capillary tubes, that is, tubes of a very small bore resembling a hair, has long occupied the attention of natural philosophers. If we would acquire a precise notion of the present state of this branch of science, we must consider it in two separate points of view. In the first place we are in possession of a mathematical theory which quadrates exactly with the phenomena; but, in the second place, when we direct our attention to the physical foundation of this theory, we find as many opinions as there are different inquirers, and no one account is entirely free from difficulties. My present intention is to make some observations on the first part of the subject, without touching at all on the second, which is reserved for future discussion.

The mathematical theory of the capillary phenomena is not complicated, being derived from two independent principles. The first of these is the fact, that the surface of the same liquid always makes the same angle with the surfaces of solids immersed in it, when the matter of the solids is the same. In the case of glass and water, and generally whenever a solid is *wetted* by a liquid, the two surfaces touch one another, or make an infinitely small angle. In glass and mercury, the two surfaces are inclined to one another in an angle of about 42° .

* Communicated by the Author.

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The second principle is the equation of the curve surface of the liquid under the influence of the capillary force. When a liquid is elevated above the general level by capillary action, the surface is always concave upward; on the contrary, it is always convex upward, when the liquid is depressed below the level. In either case, the distance of any point in the curve surface from the level, is proportional to the sum of the curvatures estimated in any two directions at right angles to one another. To speak more precisely, let a normal be drawn to the curve surface at any point; make two planes perpendicular to one another, pass by the normal and intersect the curve surface; put R and R' for the radii of curvature of the two sections, and y for the distance of the point from the general level; the equation of the surface in the capillary space is,

$$y = \beta \left\{ \frac{1}{R} + \frac{1}{R'} \right\},$$

β being a quantity to be determined by experiment. It is a property well known to mathematicians, that the sum of the curvatures is always the same, provided the two sections be at right angles to one another.

A little reflection will show that the two principles we have mentioned fully ascertain every circumstance relating to the capillary phenomena. The equation determines the position of every point in the curve surface above or below the general level; and the other principle limits the extent of the same surface in the capillary space, by making known the inclination of its extreme boundary to the surface of the immersed solid.

It is evident that the equation cannot be verified by direct observation. It was first suggested by the resemblance of the surface of liquids under capillary action to the class of elastic curves treated of in geometry. In reality, there seems to be no circumstance accompanying the greater or less elevation or depression, except a variation of curvature; so that the attention of the inquirer is naturally directed to examine the relation of these two things. There is no other way of proving that the equation accords with nature, but by comparing the mathematical deductions from it with the results obtained by careful and accurate experiments. The most general classification of the capillary phenomena we owe to Dr. Jurin, who makes the quantity of the displaced fluid, which is the proper measure of the capillary force, proportional in all cases to the length of the line of common section of the surfaces of the fluid and solid immersed in it. I do not here allude to the physical cause assigned by Dr. Jurin, which will not bear examination;

amination; but to the law of the phænomena, considered as a general fact allowed to be consonant to experience. Now one of the most curious points in Laplace's theory of capillary action*, is a demonstration, deduced from the equation of the curve surface, which proves that the volume of elevated fluid, or the space made void by the capillary force in the case of a depression, is proportional to the interior periphery of any cylindrical or prismatic tube immersed in the fluid. And it is correct to affirm generally, that what is called Dr. Jurin's theory, is no other than a mathematical consequence flowing from the two principles we have laid down.

There is one set of facts very proper to bring the exact agreement of the mathematical theory with nature to a severe trial. We allude to the depression of the mercury in the tubes of barometers of various diameters. The accuracy requisite in modern philosophical pursuits has drawn the attention of experimentalists to determine the quantity of the depression, in order to derive the true height of the mercury, from the observed height. In tubes from about one tenth of an inch in diameter, to seven or eight tenths, the convex curvature of the surface and the depression are found to vary very quickly and notably; and the comparison of the theory with such a series of connected experiments, cannot but furnish a delicate test of its exactness. Here, however, a difficulty occurs. The mathematical determination of the depression is a problem of great difficulty, which does not yield to the methods of investigation usually employed by analysts, and which has not yet been solved in a satisfactory manner. The remainder of this article is an attempt to overcome this difficulty.

I put y for the vertical ordinate of a point in the convex surface of the mercury, or the distance below the general level; r for the distance of the same point from the axis of the tube; and z for the sine of the inclination of the vertical section of the curve surface to r , or to the horizon: according to what is taught in geometry, $\frac{dr}{dz}$ is the radius of curvature of the vertical section; and $\frac{r}{z}$, the radius of curvature of the section at right angles to the vertical section: and hence we obtain from the principles laid down,

$$\frac{dz}{dr} + \frac{z}{r} = 4\beta^2 y,$$

$$\frac{dy}{dr} = \frac{z}{\sqrt{1-z^2}},$$

* *Suppl. à la Théorie de l'Action Capillaire*, p. 10.

$4\beta^2$ being a quantity to be found by experiment in tubes of any given matter.

In the glass tubes of which barometers are made, $4\beta^2$ is very nearly equal to 4.96, and β to 7; and the value of z at the surface of such tubes, or the sine of the inclination of the mercury to the horizon, may be taken equal to 0.735. These numbers, which are very convenient, accord nearly with the results of the best experiments, or at least approach to them within the limits of the errors to which all such experimental determinations are liable.

Instead of the curve surface actually formed in the tube, I shall now consider another similar surface, of which the linear dimensions are increased fourteen times, or in the proportion of 2β to 1. If y represent the vertical ordinate of this new curve, and y stand for $2\beta r$, we shall have these equations, viz.

$$\frac{dz}{dx} + \frac{z}{x} = y$$

$$\frac{dy}{dx} = \frac{z}{\sqrt{1-z^2}}.$$

Exterminate y , and put $t = \frac{x}{2} = \beta r$; then

$$\frac{d dz}{dt^2} + \frac{dz}{t dt} - \frac{z}{t^2} = \frac{4z}{\sqrt{1-z^2}}.$$

It may be proper to observe here, that the depression is the value of the vertical ordinate, when z and x vanish together, in which case also $\frac{z}{x} = \frac{dz}{dx}$. Hence the depression is equal to $\frac{2z}{x}$ or $\frac{2dz}{dx}$, that is, to $\frac{z}{t}$ or $\frac{dz}{dt}$, when z and t are both evanescent.

Next assume, $z = q t c^{\int \omega dt}$,

c being the base of Napier's logarithms, and q a constant quantity, which is no other than the depression. Substitute the values of z and its fluxions in the last equation, leaving untouched the radical on the right-hand side; then,

$$\omega^2 + 3 \frac{\omega}{t} + \frac{d\omega}{dt} = \frac{4}{\sqrt{1-z^2}}$$

$$\frac{dz}{dt} = z \left(\omega + \frac{1}{t} \right).$$

In the foregoing operations z and ω are considered as functions of t : but I shall now suppose that z is a function of t , and ω a function of the independent variables z and t . In the last equation we must therefore write $\frac{d\omega}{dt} + \frac{d\omega}{dz} \cdot \frac{dz}{dt}$ for

$$\frac{d\omega}{dt};$$

$\frac{d\omega}{dt}$; which being done, and the value of $\frac{dz}{dt}$ substituted, we shall get this equation in partial fluxions, viz.

$$\omega^3 + \omega \frac{d\omega}{dz} z + \frac{d\omega}{dz} \cdot \frac{z}{t} + 3 \frac{\omega}{t} + \frac{d\omega}{dt} = \frac{4}{\sqrt{1-z^2}}, \quad (1)$$

to which we must join the value of $\frac{dz}{dt}$, viz.

$$\frac{dz}{dt} = z \left(\frac{1}{t} + \omega \right). \quad (2)$$

I next write the equation (1) in this manner,

$$\frac{1}{t} \cdot \frac{d\omega}{dz} z^3 + \frac{d\omega}{dt} + \frac{d\omega}{dz} \frac{z^2}{2z} = \frac{4}{\sqrt{1-z^2}};$$

and assume, $\omega = g + Az^2 + Bz^4 + \&c.$

$g, A, B, \&c.$ being functions of t . Substitute this value of ω , and expand the radical on the right-hand side of the equation; then, by equating the coefficients of the like powers of z , we shall get,

$$\begin{aligned} g^3 + 3 \frac{g}{t} + \frac{dg}{dt} &= 4 \\ 4gA + 5 \frac{A}{t} + \frac{dA}{dt} &= 2 \\ 6gB + 7 \frac{B}{t} + \frac{dB}{dt} &= \frac{3}{2} - 3A^2 \\ &\&c. \end{aligned} \quad (3)$$

These equations give us,

$$\begin{aligned} g &= t - \frac{t^3}{6} + \&c. \\ A &= \frac{t}{3} - \frac{t^3}{6} + \&c. \\ B &= \frac{3}{16} t - \frac{7}{48} t^3 + \&c. \\ &\&c. \end{aligned}$$

There is another consequence of the first of the equations (3) which it is necessary to notice. Put

$$\lambda = c \int g^{\frac{3}{2}} dt;$$

then we readily derive from the equation mentioned,

$$\frac{d\lambda}{dt^2} + 3 \frac{d\lambda}{t dt} = 4\lambda;$$

and by integrating in a series, without introducing any arbitrary constants, which the present purpose does not require, we shall get,

$$\lambda = 1 + \frac{t^2}{2} + \frac{t^4}{2^2 \cdot 3} + \frac{t^6}{2^2 \cdot 3^2 \cdot 4} + \frac{t^8}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5} + \&c.$$

If

If l be the diameter of the tube, then $t = \beta r = \frac{7}{2} \cdot l$, and $t^2 = 12.25 l^2$: thus we have,

$$\begin{aligned} \lambda = 1 &+ 6.125 l^2 \\ &+ 12.505 l^4 \\ &+ 12.756 l^6 \\ &+ 7.819 l^8 \\ &+ 3.193 l^{10} \\ &+ 0.931 l^{12} \\ &+ \&c. \end{aligned} \quad (A)$$

In all tubes a few terms of this series will give the value of λ with sufficient exactness.

Substitute the value of ω that has been found in the equation (2); then

$$\frac{dz}{z dt} = \frac{1}{t} + g + Az^2 + Bz^4 + \&c.$$

The quantities g , A , B , &c. evidently vanish when t is equal to zero; let us then inquire what are their values when t is infinitely great. When t is infinite, the equation (1) becomes simply,

$$\frac{d \cdot \omega^2 z^2}{2 z dz} = \frac{4}{\sqrt{1-z^2}};$$

hence,

$$\omega = \frac{2 \sqrt{2-2\sqrt{1-z^2}}}{z},$$

or

$$\omega = 2 + \frac{z^2}{4} + \frac{7 \cdot z^4}{64} + \&c.$$

Thus it appears that while t increases from zero to be infinitely great, g , A , B , &c. increase from zero to the finite quantities 2 , $\frac{1}{4}$, $\frac{7}{64}$, &c. The terms in the values of ω , and of $\frac{dz}{dt}$, which are omitted, do not, therefore, affect the approximation except in a very limited degree. The same conclusion may be obtained another way; namely, by solving the equations (3) in serieses of the descending powers of t . If this be done, the parts of g , A , B , &c. independent of $\frac{1}{t}$, will agree with the values above assigned.

Put $2a$ for the value of ω , or of $\frac{dz}{dt}$, when t is infinite; then,

$$a = \frac{\sqrt{2-2\sqrt{1-z^2}}}{z} = 1 + \frac{z^2}{8} + \frac{7z^4}{128} + \&c.$$

Subtract ag from both sides of the equation before found, then,

$$\frac{dz}{z dt} - ag = \frac{1}{t} - (a-1)g + Az^2 + Bz^4 + \&c.:$$

and,

and, by making,

$$A' = A - \frac{\rho}{8} = \frac{5}{24} t - \frac{7}{48} t^3 + \&c.$$

$$B' = B - \frac{7\rho}{128} = \frac{17}{128} t - \frac{35}{256} t^3 + \&c.$$

we get, $\frac{dz}{zdt} - a\rho = \frac{1}{t} + A'z^2 + B'z^4 + \&c.$

The left side of this equation is equal to zero when t is infinite; for $a\rho$ then becomes $2a$. From this it follows that A' , B' , &c. are evanescent both when t is equal to zero and when it is infinite. The terms on the right side, except the first, do not therefore increase indefinitely, but always remain inconsiderable.

Assume, $\frac{dz}{az} = \frac{d\sigma}{\sigma}$:

then, $\sigma = z - \frac{z^3}{16} - \&c.$

$$z = \sigma + \frac{\sigma^3}{16} + \&c.$$

$$a = 1 + \frac{\sigma^2}{8} + \frac{9\sigma^4}{128} + \&c.:$$

then, by dividing all the terms of the foregoing equation by a , and introducing σ for z , we shall get,

$$\frac{d\sigma}{\sigma dt} - \rho = \frac{1}{a} \cdot \left\{ \frac{1}{t} + A'\sigma^2 + \left(B' + \frac{A'}{8} \right) \sigma^4 + \&c. \right.$$

Next put, $\frac{d\sigma}{\sigma dt} - \rho = \frac{ds}{sdt} \because \sigma = sc \int \frac{dt}{t} = \lambda.s.$

Let a' stand for the same function of s that a is of σ , or of λs : then, making,

$$A'' = A' = \frac{5}{24} t - \frac{7}{48} t^3 + \&c.$$

$$B'' = B' + \frac{A'}{8} = \frac{61}{384} t - \frac{119}{768} t^3 + \&c.$$

we shall get,

$$\frac{a'ds}{sdt} = \frac{a'}{a} \cdot \left\{ \frac{1}{t} + A''\lambda^2.s^2 + B''\lambda^4.s^4 + \&c. \right.$$

The fraction $\frac{a'}{a}$ must next be reduced to a series of the powers of s . Now,

$$\frac{a'}{a} = \frac{1 + \frac{1}{8}s^2 + \frac{9}{128}s^4 + \&c.}{1 + \frac{1}{8}\lambda^2s^2 + \frac{9}{128}\lambda^4s^4 + \&c.}:$$

and,

8 Mr. Ivory on the Theory of Capillary Action, and
and, by division,

$$\frac{a'}{a} = 1 - t P. s^2 - t P'. s^4 - \&c.$$

$$P = \frac{\lambda^2 - 1}{8t} = \frac{t}{8} + \frac{5}{96} t^3 + \&c.$$

$$P' = \frac{7\lambda^4 + 2\lambda^2 - 9}{128t} = \frac{t}{8} + \frac{41}{384} t^3 + \&c.$$

The series for $\frac{a'}{a}$ must next be substituted, and, after having multiplied and reduced, we shall get,

$$\frac{a' ds}{s dt} = \frac{1}{t} + A''' s^2 + B''' s^4 + \&c.$$

$$A''' = A'' \lambda^2 - P = \frac{t}{12} + \frac{t^3}{96} + \&c.$$

$$B''' = B'' \lambda^4 - P' - A'' \lambda^2. Pt = \frac{13}{384} t + \frac{23}{768} t^3 + \&c.$$

The principle of all this analysis lies in this, That, the left side of the equation being equal to zero when t is infinite, all the terms on the right side except the first must be evanescent both when t is zero and when it is infinite. These terms, therefore, remain always inconsiderable, and of no value except in some tubes of small diameter. In order to exhibit the equation in the most simple form possible, I finally put,

$$\frac{a' ds}{s} = \frac{du}{u},$$

$$u = s + \frac{s^3}{16} + \&c.$$

$$s = u - \frac{u^3}{16} - \&c.:$$

and, by introducing u for s ,

$$\frac{du}{u dt} = \frac{1}{t} + A^{iv} u^2 + B^{iv} u^4 + \&c.$$

$$A^{iv} = A''' = \frac{t}{12} + \frac{t^3}{96} + \&c.$$

$$B^{iv} = B''' - \frac{A'''}{8} = \frac{3}{128} t + \frac{11}{384} t^3 + \&c.$$

We must now integrate, and for this purpose write the last equation in this manner,

$$\frac{d \cdot \frac{u}{t}}{dt} = \frac{A^{iv}}{t} u^3 + \frac{B^{iv}}{t} u^5 + \&c.:$$

then assume, $\frac{u}{t} = q + Q u^3 + Q' u^5 \&c.$

q being

q being a constant, and Q, Q' &c. functions of t . By differentiating,

$$\begin{aligned}\frac{d \cdot \frac{u}{t}}{dt} &= \frac{dQ}{dt} u^3 + \frac{dQ'}{dt} u^5 \\ &+ \frac{du}{u dt} (3 Q u^3 + 5 Q' u^5 + \&c.)\end{aligned}$$

Substitute the value of $\frac{d \cdot \frac{u}{t}}{u dt}$, and equate the coefficients of the two equal quantities, then

$$\begin{aligned}\frac{d \cdot Q t^3}{t^3 dt} &= \frac{A^{iv}}{t}, \\ \frac{d \cdot Q' t^5}{t^5 dt} &= \frac{B^{iv}}{t} - 3 A^{iv} Q;\end{aligned}$$

and hence,

$$Q = \frac{t}{48} + \frac{t^3}{576} + \&c.$$

$$Q' = \frac{t}{256} + \frac{3 t^3}{1024} + \&c.$$

These coefficients being known, we have the value of the constant q , which it is most convenient to arrange according to the powers of t , viz.

$$\begin{aligned}q &= \frac{u}{t} - t \left(\frac{u^3}{48} + \frac{u^5}{256} \right) \\ &- t^3 \left(\frac{u^3}{576} + \frac{3 u^5}{1024} \right).\end{aligned}$$

It has already been noticed that the depression is the value of $\frac{z}{t}$ when z and t are both evanescent. But it is obvious that

the vanishing fractions, $\frac{z}{t}, \frac{\sigma}{t}, \frac{s}{t}, \frac{u}{t}$ have all the same limit:

wherefore q is the depression. It does not however belong to the surface of the mercury in the tube, but to a surface increased fourteen times in its linear dimensions. To get the real depression we must therefore divide by 2β . Now

$t = \beta r = \frac{\beta}{2} l$, and $t^2 = 12 \cdot 25 l^2$; and we thus obtain the following formula for the real depression,

$$\begin{aligned}q &= \frac{u}{49 l} - l \left(\frac{u^3}{192} + \frac{u^5}{1024} \right) \\ &- l^3 \left(\frac{12 \cdot 25}{2304} u^3 + \frac{36 \cdot 75}{4096} u^5 \right).\end{aligned}$$

The symbol u stands for a known function of s ; but in practice it will be most convenient to have q expressed immediately

10 Mr. Ivory on the Theory of Capillary Action, &c.

in terms of s . The necessary operations being performed, we finally obtain,

$$q = \frac{1}{49l} \times \left(s + \frac{s^3}{16} + \frac{5s^5}{256} + \frac{119s^7}{12288} + \frac{393s^9}{65536} \right) - l \times \left(\frac{s^3}{192} + \frac{s^5}{512} \right). \quad (\text{B})$$

The term multiplied by l^3 being omitted, as it does not affect the value of q in the fifth decimal place. In order to find s , we must compute σ , for which purpose we readily obtain this series,

$$\sigma = z - \frac{z^3}{16} - \frac{z^5}{128} - \frac{35z^7}{12288} - \frac{137z^9}{98304} + \&c.:$$

and z being 0.735, we get $\sigma = 0.70805$. Then

$$s = \frac{.70805}{\lambda},$$

λ being computed by the formula (A).

As the diameter l decreases, s approaches indefinitely to σ , and u to z ; so that, for tubes of an extremely small bore, the depression is,

$$q = \frac{z}{49l} = \frac{.015}{l}.$$

On the other hand, the diameter of the tube increasing, s decreases very rapidly; and when the diameter is half an inch, or greater, all the powers of s are inconsiderable, and we have simply,

$$q = \frac{s}{49l} = \frac{.70805}{49l \cdot \lambda} = \frac{.01445}{l \cdot \lambda}.$$

The first term of q in the formula (B) being the expansion of a known function, it may be continued to any degree of approximation; but the terms set down are sufficient even for very small tubes. Suppose $l = \frac{1}{20}$, thus $\lambda = 1.01539$, $s = .69736$, and

$$q = \frac{.72274}{2.45} - \frac{.002}{20} = .29490.$$

Next, let $l = \frac{1}{2}$, then $\lambda = 1.5460$, $s = .1997$, and

$$q = \frac{.1997}{24.5} = 0.0815.$$

I have subjoined a Table of the depressions in tubes of various bores, and have added the experimental determinations published by Lord Charles Cavendish; by which it will be seen that the agreement of the theory with experiment is very satisfactory.

Depression of the Mercury in the Tubes of Barometers.

Diameter. inches.	Depression. inches	Observed by Ld. Cavendish.
0·05	0·29490	
·10	·14026	0·140
·15	·08628	·092
·20	·05811	·067
·25	·04077	·050
·30	·02919	·036
·35	·02110	·025
·40	·01534	·015
·45	·01117	
·50	·00815*	·007
·60	·00431*	·005
·70	·00228	
·80	·00119	

Dec. 14, 1827.

J. IVORY.

II. *Heights of some of the principal Beds of Ingleborough Hill and Moughton Fell, Yorkshire.* By JOHN NIXON, Esq.

To the Editors of the Philosophical Magazine and Annals.

Gentlemen,

IN the xith volume of the Annals of Philosophy, I proposed to the geologist the measurement, by one barometer only, of the heights and dip of strata known to be nearly horizontal, and furnished at the same time the requisite instructions and tables. From the numerous measurements of this description made by me in different parts of the north-west of Yorkshire, I beg to transmit you a selection of such as may serve to determine the thicknesses and dip of the principal beds of Ingleborough Hill and Moughton Fell.

Having carefully measured by trigonometry the heights of the summits and other elevated parts of the hills, as well as of different places at their bases, barometrical observations were

* In the Supplement to the Encyclopædia, Article FLUIDS, there is an arithmetical blunder in each of these numbers, as will appear by having recourse to the formula. I was obliged to notice this point on a former occasion, in order to stop the triumph of an antagonist, who has stuck to my skirts for a period of at least six years, and who seems to have great confidence in the efficacy of the adage, *If one way will not do, another will.*