theoretically to determine the optical effect of any transparent system, however complicated, on all beams of light passing across it in a given path, straight or curved.

On the Form of Hyperelliptic Integrals of the First Order, which are expressible as the Sum of two Elliptic Integrals. By W. BURNSIDE. Received March 7th, 1892. Read March 10th, 1892.

The reduction of the integral

$$\int \frac{(a+bx)\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda)}}$$

to the sum of two elliptic integrals is due originally to Jacobi.

The transformation is most simply exhibited in the following form.

If $$z = x+\frac{\kappa\lambda}{x},$$

then
$$(x-1)(x-\kappa\lambda) = x\,(z-1-\kappa\lambda),$$
$$(x-\kappa)(x-\lambda) = x\,(z-\kappa-\lambda),$$
$$x-\sqrt{\kappa\lambda} = \sqrt{x}\,\sqrt{z-2\sqrt{\kappa\lambda}},$$
$$x+\sqrt{\kappa\lambda} = \sqrt{x}\,\sqrt{z+2\sqrt{\kappa\lambda}},$$
$$(x-\sqrt{\kappa\lambda})(x+\sqrt{\kappa\lambda})\,dx = x^2dz,$$

and hence

$$\frac{(x+\sqrt{\kappa\lambda})\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda)}} = \frac{dz}{\sqrt{(z-2\sqrt{\kappa\lambda})(z-1-\kappa\lambda)(z-\kappa-\lambda)}},$$

and

$$\frac{(x-\sqrt{\kappa\lambda})\,dx}{\sqrt{x\,(x-1)(x-\kappa)(\kappa-\lambda)(x-\kappa\lambda)}} = \frac{dz}{\sqrt{(z+2\sqrt{\kappa\lambda})(z-1-\kappa\lambda)(z-\kappa-\lambda)}}.$$

It has also been shown by Herr Königsberger (*Borchardt's*

Journal, Vol. LXVII.) that if a hyperelliptic integral of the first order is reducible to the sum of two elliptic integrals by a quadratic substitution, it can always, by a linear substitution, be expressed in the above form.

I discuss in this paper the corresponding question for rational substitutions of a higher order than the second. For the case of a cubic transformation the problem is treated completely, the general form of the integral, and the two substitutions that are necessary to transform it, being given explicitly. For the case of the quartic and quintic transformations the equations of transformation are explicitly obtained with the general forms of the integrals to which they apply.

The general theory of hyperelliptic integrals of the first order shows that when one integral of the first species belonging to the equation

$$y^2 = f_6(x),$$

a sextic function, can be found with only two independent periods, a second with the same property always exists; and that, if suitable multiples of these are chosen as the normal integrals, the four periods of each are expressible in the form

$$w_1 \ldots\ldots 1, \quad 0, \quad \omega, \quad \frac{1}{n},$$

$$w_2 \ldots\ldots 0, \quad 1, \quad \frac{1}{n}, \quad \omega',$$

where n is an integer.

It follows from this that a double θ-function, whose arguments are any two independent integrals of the first species, can in this case, by a transformation of the n^{th} order, be expressed as a product of two simple θ-functions, and hence that the normal integrals can, by transformations of the n^{th} order, be reduced to elliptic integrals. (See a paper by the late Mme. S. Kowalevski, *Acta Mathematica*, Vol. IV.) The transformations will generally be distinct; that they are the same for the above-quoted case of the quadratic substitution results from the fact that for that case the equation

$$s^2 = f_6(x)$$

is rationally transformable into itself.

These considerations suggest starting with the elliptic integral, and determining the form of the substitution which will transform it into a hyperelliptic integral of the first order. If the elliptic integral be of the first species, *i.e.*, everywhere finite, the hyperelliptic integral will necessarily be so also.

Suppose, then, that the elliptic differential

$$\frac{dz}{\sqrt{z-\alpha \,.\, z-\beta \,.\, z-\gamma \,.\, z-\delta}}$$

is transformed by the substitution of the n^{th} order

$$z = \frac{U}{V},$$

where U, V are functions of x, either both of the n^{th} degree, or one of the n^{th} and one of the $n-1^{\text{th}}$; and for the present let n be odd. If the result is a hyperelliptic differential of the first order,

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V)$$

must be the product of a sextic function of x, and a perfect square; and this can only be the case when three of the factors are products of linear functions, and perfect squares, while the fourth is the product of a cubic function and a square.

The substitution may therefore be written generally in the form

$$z-\alpha : z-\beta : z-\gamma : z-\delta$$

$$:: (x-x_1)\,X_1^2 : (x-x_2)\,X_2^2 : (x-x_3)\,X_3^2 : (x-x_4)(x-x_5)(x-x_6)\,X_4^2,$$

where X_1, X_2, X_3 are rational integral functions of x of degree $\frac{n-1}{2}$, and X_4 is of degree $\frac{n-3}{2}$.

By linear substitutions, performed on z and x, this may always be written in the standard form

$$z : z-1 : 1 : z-z_0$$

$$:: xX_1^2 : (x-1)\,X_2^2 : X_3^2 : (x-\kappa)(x-\lambda)(x-\mu)\,X_4^2,$$

the coefficients of the leading terms in X_1, X_2, X_4 being unity (or all the same).

The relations between the constants in this equation are mos easily obtained by a consideration of the corresponding Riemann's surface.

The surface is n-sheeted and simply connected.

At each of the points $z = 0, 1, \infty, n-1$ of the sheets are connected in pairs by branch-points, and at $z = z_0$, $n-3$ sheets are so connected in pairs; this gives $2n-3$ branch-points, so that there must be one more to make the surface simply connected. Hence the equation of transformation can be written more completely

$$z : z-1 : 1 : z-z_0 : z-z_1$$

$$:: xX_1^2 : (x-1)\,X_2^2 : X_3^2 : (x-\kappa)(x-\lambda)(x-\mu)\,X_4^2 : X_5\,(x-x_1)^2,$$

where X_5 is a rational integral function with x^{n-2} for its leading term.

Now when z_0 and z_1 are given, there is only a finite number of ways of connecting the sheets of the surface together, and, for each way of connecting, x is a perfectly definite function of z, since it is a one-valued function on the Riemann's surface, with given values at three given points. Hence the constants in the above equation are algebraic functions of z_0 and z_1; or, more conveniently, of two arbitrary parameters.

When κ, λ, μ are so determined as functions of two arbitrary parameters,

$$\int \frac{(a+bx)\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\mu)}}$$

is a hyperelliptic integral of the first order, which, by transformations of the n^{th} order, can be expressed as the sum of two elliptic integrals, and the method by which it has been obtained shows that it can only differ by a linear transformation from the most general integral so expressible. The above transformation only changes the hyperelliptic integral into an elliptic integral for a particular value of the ratio a/b; the general theory shows that another transformation of the n^{th} order must exist, which will apply to another value of the ratio.

The differential coefficient dz/dx vanishes only at the finite-branch points of the surface; hence

$$\frac{dz}{dx} = \frac{X_1 X_2 X_4\,(x-x_1)}{X_3^3} \times \text{constant};$$

and therefore

$$\frac{dz}{\sqrt{z\,(z-1)(z-z_0)}} = \frac{\text{constant} \times (x-x_1)\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\mu)}},$$

determining the particular value of the ratio a/b to which the given substitution applies.

Cubic Transformation.

The equation for the cubic transformation involves only one arbitrary parameter, and so differs from the general case of any odd number; for in this case there can only be four branch-points on the surface, and of these three can be chosen arbitrarily. The coefficients in the hyperelliptic integral will, however, still involve two arbitraries, for the equation connecting the two differentials now becomes

$$\frac{dz}{\sqrt{z(z-1)(z-z_0)}} = \frac{\text{constant} \times (x-x_1)\,dx}{\sqrt{x(x-1)(xX_1^2 - z_0 X_3^2)}},$$

where z_0 is one arbitrary, and the coefficients of X_1 and X_3 involve another.

The equation of transformation is here the same as for the cubic transformation of an elliptic integral; as, however, it is to be used for a different purpose, it is convenient to put it in a slightly altered form.

I shall therefore write

$$z : z-1 : 1$$

$$:: x(x+2t+t^2)^2 : (x-1)(x-t^2)^2 : \left[(1+2t)\,x+t^2\right]^3.$$

This gives

$$\frac{dz}{dx} = \frac{(x+2t+t^2)(x-t^2)\left[(1+2t)\,x-(2t+t^2)\right]}{\left[(1+2t)\,x+t^2\right]^3},$$

and hence

$$\frac{dz}{\sqrt{z(z-1)(z-z_0)}} = \frac{\left[(1+2t)\,x-(2t+t^2)\right]dx}{\sqrt{x(x-1)\left[x(x+2t+t^2)^2 - z_0\left\{(1+2t)\,x+t^2\right\}^2\right]}}.$$

The general form of hyperelliptic integral, therefore, which can be expressed, by cubic transformations, as the sum of two elliptic integrals, is

$$\int \frac{(a+bx)\,dx}{\sqrt{x(x-1)\left[x(x+2t+t^2)^2 - z_0\left\{(1+2t)\,x+t^2\right\}^2\right]}},$$

where t and z_0 are arbitrary.

It is worth observing that, from the properties of the equation for the cubic transformation of elliptic integrals, this may be expressed in the form

$$\int \frac{(a+bx)\,dx}{\sqrt{x\,(x-1)(x-\mathrm{sn}^2 u)\left[x-\mathrm{sn}^2\left(u+\frac{2\omega}{3}\right)\right]\left[x-\mathrm{sn}^2\left(u+\frac{4\omega}{3}\right)\right]}},$$

2ω being any complete period of the elliptic function.

As regards the second transformation, it seemed natural to suppose that the two cubics $x\,(x-1)$ and $xX_1^2-z_0X_3$, into which the sextic under the square root naturally divides itself in the above analysis, would interchange parts, and I have found that such is the case.

This is perhaps most easily verified as follows. The equation for the second substitution

$$z = \frac{U}{V}$$

is, by supposition, to have roots 0, 1, ∞ for a particular value of z; hence, by a linear transformation of z, it can be thrown into the form

$$z = \frac{x^3+ax+b}{x\,(x-1)},$$

and it is to be shown that a and b can be so determined that this equation is consistent with

$$z-z_1 = \frac{(x-x_1)(x-x_1')^2}{x\,(x-1)},$$

$$z-z_2 = \frac{(x-x_2)(x-x_2')^2}{x\,(x-1)},$$

$$z-z_3 = \frac{(x-x_3)(x-x_3')^2}{x\,(x-1)},$$

where $$(x-x_1)(x-x_2)(x-x_3) \equiv xX_1^2-z_0X_3^2.$$

Now $$(x^3+ax+b)\,(x_1^2-x_1)-(x^2-x)(x_1^3+ax_1+b) = 0,$$

after casting out the factor $x-x_1$, has equal roots when

$$x_1^4+(2a+4b)\,x_1^3+(a^2-6b)\,x_1^2+(2ab+4b)\,x_1+b^2 = 0.$$

It is then to be shown that this equation is the same as

$$(x-x_0)\left[x\,(x+2t+t^2)^2-z_0\left\{(1+2t)\,x+t^2\right\}^2\right] = 0.$$

A comparison of the coefficients, which is somewhat tedious but in no way difficult, shows that the four equations of conditions, to which the identity of these equations leads, can be satisfied by the following values of a, b and x_0, viz.:—

$$a = t^2+2t(1-z_0)-z_0,$$
$$b = -z_0t^2,$$
$$x_0 = z_0.$$

The second substitution is therefore

$$z = \frac{x^3+x\left[t^2+2t(1-z_0)-z_0\right]-z_0t^2}{x(x-1)}.$$

Differentiating and simplifying, this leads to

$$\frac{dz}{dx} = \frac{(x+t)\left[x^3-(2+t)x^2+z_0\{(2t+1)x-t\}\right]}{x^2(x-1)^2},$$

and, since again dz/dx only vanishes at the branch-points, this must be

$$\frac{dz}{dx} = \frac{(x+t)(x-x_1')(x-x_2')(x-x_3')}{x^2(x-1)^2}.$$

Hence, finally,

$$\frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)}} = \frac{(x+t)\,dx}{\sqrt{x(x-1)(x-x_1)(x-x_2)(x-x_3)}}$$
$$= \frac{(x+t)\,dx}{\sqrt{x(x-1)\left[x(x+2t+t^2)^2-z_0\{(1+2t)x+t^2\}^2\right]}}.$$

Collecting the results that have been obtained for this case, they may be stated thus:—

The integrals of the first species, belonging to the equation

$$s^2 = x(x-1)\left[x(x+2t+t^2)^2-z_0\{(1+2t)x+t^2\}^2\right]$$

can always be transformed into the sum of two elliptic integrals by the substitutions

$$z = \frac{x(x+2t+t^2)^2}{\left[(1+2t)x+t^2\right]^2}$$

and
$$z = \frac{x^3+t^2x+2t-z_0\left[x(1+2t)+t^2\right]}{x(x-1)},$$

and any form for which such a reduction can be effected by a cubic substitution can be transformed to the given one by a linear substitution.

Quintic Transformation.

For this case the equation for the quintic transformation of elliptic integrals is of no assistance, as it corresponds to an essentially different form of Riemann's surface.

The problem of finding the equation of transformation may be stated thus:—

To determine p, q, p', q' as functions of two parameters, so that

$$x(x^2+px+q)^2-(x-1)(x^2+p'x+q')^2$$

shall be a perfect square.

By a linear integral substitution for x, this may be thrown into the more symmetrical form

$$(x+\alpha)(x^2+Ax+1)^2-(x+\beta)(x^2+Bx+1)^2 = \text{perfect square.}$$

Taking $(\alpha-\beta)(ux^2+vx+1)^2$ for the perfect square, and equating coefficients, there results

$$2\frac{A-B}{\alpha-\beta} = u^2-1,$$

$$\frac{A\alpha-B\beta}{\alpha-\beta} = v,$$

$$\frac{A^2-B^2}{\alpha-\beta} = 2v(u-1),$$

$$\frac{A^2\alpha-B^2\beta}{\alpha-\beta} = v^2-(u-1)^2.$$

The solution of these equations presents no difficulty, and the only irrational quantity which appears is the square root of $v^2-(u+1)^2$. Writing w for this, the values at once obtained are

$$A+B = \frac{4v}{u+1}, \qquad \alpha+\beta = \frac{4v}{(u+1)^2},$$

$$A-B = 2w\frac{u-1}{u+1}, \qquad \alpha-\beta = \frac{4w}{(u+1)^2}.$$

The coefficients A, B, α, β may be expressed rationally in terms of two parameters, conveniently, by taking:

$$u+1 = \frac{2}{st}, \qquad v = \frac{1}{s^2}+\frac{1}{t^2},$$

giving

$$w = \frac{1}{t^2}-\frac{1}{s^2},$$

and immediately leading to

$$\alpha = s^2, \qquad A = 2\frac{t}{s}+\frac{1}{t^2}-\frac{1}{s^2},$$

$$\beta = t^2, \qquad B = 2\frac{s}{t}+\frac{1}{s^2}-\frac{1}{t^2}.$$

The equation of transformation can then be written

$$z : z-1 : 1 :: (x+s^2)\left[x^2+\left(2\frac{t}{s}+\frac{1}{t^2}-\frac{1}{s^2}\right)x+1\right]^2$$

$$: (x+t^2)\left[x^2+\left(2\frac{s}{t}+\frac{1}{s^2}-\frac{1}{t^2}\right)x+1\right]^2$$

$$: (s^2-t^2)\left[\left(\frac{2}{st}-1\right)x^2+\left(\frac{1}{s^2}+\frac{1}{t^2}\right)x+1\right]^2.$$

On differentiating and simplifying, this gives

$$\frac{-1}{(z-1)^2}\frac{dz}{dx} = (s^2-t^2)\frac{x^2+\left(2\frac{t}{s}+\frac{1}{t^2}-\frac{1}{s^2}\right)x+1}{(x+t^2)^2\left[x^2+\left(2\frac{s}{t}+\frac{1}{s^2}-\frac{1}{t^2}\right)x+1\right]^3}$$

$$\times\left[\left(\frac{2}{ts}-1\right)x^2+\left(\frac{1}{s^2}+\frac{1}{t^2}\right)x+1\right]$$

$$\times\left[\left(\frac{2}{ts}-1\right)x^2+\left(\frac{1}{s^2}+\frac{1}{t^2}\right)(2ts-1)\,x+4ts-3\right].$$

The values of x at the remaining two branch-points are therefore given by

$$\left(\frac{2}{ts}-1\right)x^2+\left(\frac{1}{t^2}+\frac{1}{s^2}\right)(2ts-1)\,x+4ts-3 = 0;$$

and, if r is a root of this equation, the equation of transformation may be written

$$z : z-1 : 1 : z-z_r$$

$$:: (x+s^2)\times\text{square} : (x+t^2)\times\text{square} : (s^2-t^2)\times\text{square}$$
$$: (x-r)^2\,(x^3+ax^2+bx+c),$$

where a, b, c are rationally expressible in terms of r, s, t.

It may be noticed that the coefficients which are rational in s and t are also rational in ts and t/s, and that, writing

$$ts = \tau, \quad \frac{t}{s} = \sigma,$$

the equation defining r takes the simpler form

$$(2-\tau)\,r^2+\left(\sigma+\frac{1}{\sigma}\right)(2\tau-1)\,r+\tau\,(4\tau-3) = 0,$$

which is quadratic in each variable; but I have not been able to determine whether the three quantities r, σ, τ can be expressed as rational functions of two suitable chosen ones.

The hyperelliptic integral which this transformation immediately leads to is

$$\int\frac{(mx+n)\,dx}{\sqrt{(x+s^2)(x+t^2)(x^3+ax^2+bx+c)}},$$

where a, b, c are the determinate algebraical functions of s and t just introduced, and to this form every hyperelliptic integral can be brought which is reducible to two elliptic integrals by quintic transformations.

Transformations of Even Order.

When the transformation is of even order the product

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V)$$

may be the product of a sextic and a perfect square in two ways. Either (i.) the first two factors may be perfect squares, while the third and fourth consist of the products of a quadratic and quartic, each with a square; or (ii.) the first factor may be a perfect square, and each of the others the product of a quadratic and a square.

The first case is clearly equivalent to the combination of a quadratic transformation of the elliptic integral into another elliptic integral, and then a transformation of this by a substitution of degree $\frac{n}{2}$ into a hyperelliptic integral; and hence is not to be considered an independent transformation of order n. The second case need therefore alone be treated.

Considerations exactly parallel with those for a transformation of uneven order give, for the form of the equation of transformation,

$$z : z-1 : 1 : z-z_0 : z-z_1$$

$$:: X_1^2 : (x-1)(x-\kappa)\, X_2^2 : xX_3^2 : (x-\lambda)(x-\mu)\, X_4^2 : X_5\,(x-x_1)^2,$$

where X_1 is an integral function of x of degree $n/2$; X_2, X_3, X_4 are integral functions of degree $\frac{1}{2}n-1$, and X_5 is of degree $n-2$; the coefficients of the leading terms of X_1, X_2, X_4, X_5 are all unity (or the same), and the constants are all algebraical functions of two arbitraries. Supposing the coefficients so determined, then

$$\frac{dz}{dx} = \text{constant} \times \frac{X_1 X_2 X_4\,(x-x_1)}{x^2 X_3^3},$$

and
$$\frac{dz}{\sqrt{z\,(z-1)(z-z_0)}} = \frac{\text{constant} \times (x-x_1)\, dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\mu)}}.$$

Quartic Transformation.

In this case X_1 is a quadratic, and X_2, X_3, X_4 are linear factors. As in the case of the quintic transformation, the coefficients are most easily determined as functions of two arbitraries by performing a linear transformation on x, so as to make the forms more symmetrical. In particular, I suppose the variable so taken that the roots of $X_1=0$ are zero and infinity, and for convenience write $1/(1-z)$ for z. The equation of transformation takes the form

$$z : z-1 : 1 :: (x+2ax+a')(x+\alpha)^2 : (x^2+2bx+b')(x+\beta)^2 : Cx^2.$$

This gives
$$a'\alpha^2-b'\beta^2 = 0,$$
$$a-b = \beta-a,$$
$$a\alpha^2-b\beta^2 = b'\beta-a'a,$$
$$4\,(a\alpha-b\beta)+a'-b'+\alpha^2-\beta^2 = C,$$

or
$$a' = v\beta^2, \quad b' = v\alpha^2, \quad a = \beta\,\frac{v\alpha+\beta}{\alpha+\beta}, \quad b = \alpha\,\frac{\alpha+v\beta}{\alpha+\beta},$$

$$C = \frac{(1-v)(\alpha-\beta)^3}{\alpha+\beta},$$

where v is a new arbitrary.

Writing now αx for x and u for β/α, the equation becomes

$$z : z-1 : 1$$

$$:: (x+1)^2\left(x^2+2u\frac{u+v}{u+1}x+vu^2\right) : (x+u)^2\left(x^2+2\frac{uv+1}{u+1}x+v\right)$$

$$: \frac{(1-v)(1-u)^3}{u+1}x^3.$$

On differentiating and reducing, this leads to

$$\frac{dz}{dx} = 2\frac{u+1}{(v-1)(u-1)^3}\,\frac{(x+1)(x+u)\left(x^2+\dfrac{(v-1)u}{u+1}x-vu\right)}{x^3}.$$

The values of x at the other two branch-points are therefore given by

$$x^2+\frac{(v-1)u}{u+1}x-uv = 0.$$

If s, t are the roots of this equation,

$$u = \frac{s+t-st}{1-s-t},$$

$$v = -\frac{st(1-s-t)}{s+t-st},$$

and all the coefficients of the equation of transformation are rational in s and t. When they are so expressed, the equation in its complete form is

$$z : \left(x^2+2\frac{s^2+st+t^2}{1-s-t}x-\frac{st(s+t-st)}{1-s-t}\right)(x+1)^2$$

$$:: z-1 : \left(x^2+2(1-s-t)x-\frac{st(1-s-t)}{s+t-st}\right)\left(x+\frac{s+t-st}{1-s-t}\right)^2$$

$$:: 1 : \frac{(s+t)(1-2s-2t+st)^3}{(s+t-st)(1-s-t)^2}x^3$$

$$:: z-z_s : \left(x^2+2\frac{1-t+t^2}{1-s-t}x-\frac{t(s+t-st)}{s(1-s-t)}\right)(x-s)^2$$

$$:: z-z_t : \left(x^2+2\frac{1-s+s^2}{1-s-t}x-\frac{s(s+t-st)}{t(1-s-t)}\right)(x-t)^2.$$

The sextic function $f(x)$, such that $\int (a+bx)\,dx/\sqrt{f(x)}$ is reducible to elliptic integrals by a quartic transformation, may then be written in the form

$$\left[x^2+2(s^2+st+t^2)x-stC\right]\left[x^2+2(1-s+s^2)x-\frac{s}{t}C\right]$$
$$\times\left[x^2+2(1-t+t^2)x-\frac{t}{s}C\right],$$

where $$C\equiv(s+t-st)(1-s-t).$$

For transformations of a higher order than the fifth, the relation between z and x becomes very much more complicated, as it appears to be then impossible to express it in a form in which the coefficients are rational functions of the parameters. The problem, too, of determining in each case the invariant character of the sextic, corresponding to the vanishing of the skew invariant for the case of the quadratic substitution, is undoubtedly one of considerable difficulty; this I have made no attempt to solve.

On the Analytical Theory of the Congruency. *By* Professor CAYLEY.

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If the lines of a congruency are considered as issuing from the several points of a surface, or say as the quasi-normals of a surface, then the fundamental geometrical theory is established by an analysis closely similar to that for the theory of the curvature of a surface; viz., it is shown that each quasi-normal is intersected by two consecutive quasi-normals in two points respectively (corresponding to the centres of curvature), or say in two foci; we have on the surface two series of curves of quasi-curvature—only these do not in general intersect at right angles; the intersecting quasi-normals form two series of developable surfaces, each touching the surface of centres (or focal surface), along its cuspidal edge, &c.; and, in particular, each quasi-normal is a bitangent of the focal surface, touching it at the two foci respectively.

But the analysis assumes a very different form if we consider the congruency by itself, without thus connecting it with a surface.