On the Thirty Cubes that can be constructed with Six differently Coloured Squares. By Major P. A. MacMahon, R.A., F.R.S.

Read February 9th, 1893. Received March 21st, 1893.

1. It has been long known that the number of rotations which bring a regular solid into coincidence with itself is equal to twice the number of its edges. If, then, a polyhedron possess F faces, and E edges, it is seen that

$$\frac{F!}{2E}$$

different polyhedra may be made by numbering or colouring the faces differently. Thus

	Faces.	Edges.	Number.
Tetrahedron	4	6	2
Cube	6	12	30
Octahedron	8	12	1680
Dodecahedron	12	30	$\frac{12!}{60}$
Icosahedron	20	30	$\frac{20!}{60}$

2. Coming now to the case of the cube, observe that we have found that thirty different cubes may be obtained by numbering the faces with the numbers 1, 2, 3, 4, 5, 6.

Choosing from these a particular cube, say

$$\begin{matrix} & 3 & \\ 2 & 1 & 4 \\ & 5 & \end{matrix}$$

where it is to be understood that 6 is on the face opposite to the face 1 (the face 1 being uppermost), and the remaining numbers are on the remaining (vertical) faces in the circular order shown, observe that this cube remains unaltered for a group of

$$\frac{6!}{30} = 24 \text{ substitutions,}$$

and that *a priori* the order of the group is equal to the number of rotations of the cube. The group is

1;	(2543);	(24)(35)	(23)(45)(16)	(263)(154)
	(2345);	(24)(16)	(25)(34)(16)	(215)(463)
	(3156);	(35)(16)	(24)(36)(15)	(213)(546)
	(3651);		(24)(13)(56)	(265)(413)
	(2146);		(26)(35)(14)	(256)(143)
	(2641);		(21)(35)(46)	(236)(145)
				(231)(456)
				(251)(436)

and it is singly transitive and imprimitive. It is further holohedrically isomorphous with the group of twenty-four permutations of the four diagonals of the cube.

Exchanging the numbers upon any two opposite faces of the cube we obtain a different cube, which remains unaltered by the same substitutions, and which therefore belongs to the same group as the former cube. These two cubes, whose pairs of opposite faces are marked with the same numbers, which belong to the same group of substitutions, it is convenient to designate "associated cubes."

The thirty cubes are thus separated into fifteen pairs of associated cubes.

Denote the cubes as follows:—

A 3 2 1 4 5	*A'* 5 2 1 4 3	*B* 3 2 1 5 4	*B'* 4 2 1 5 3	*C* 4 2 1 3 5	*C'* 5 2 1 3 4
D 3 1 2 4 5	*D'* 5 1 2 4 3	*E* 3 1 2 5 4	*E'* 4 1 2 5 3	*F* 4 1 2 3 5	*F'* 5 1 2 3 4
G 1 2 3 4 5	*G'* 5 2 3 4 1	*H* 1 2 3 5 4	*H'* 4 2 3 5 1	*I* 4 2 3 1 5	*I'* 5 2 3 1 4
J 3 2 4 1 5	*J'* 5 2 4 1 3	*K* 3 2 4 5 1	*K'* 1 2 4 5 3	*L* 1 2 4 3 5	*L'* 5 2 4 3 1
M 3 2 5 4 1	*M'* 1 2 5 4 3	*N* 3 2 5 1 4	*N'* 4 2 5 1 3	*O* 4 2 5 3 1	*O'* 1 2 5 3 4

where *A*, *A'* are an associated pair, and so on.

The two cubes A, A' have the opposites

$$1-6$$
$$2-4$$
$$3-5;$$

rejecting all the cubes which have *any pair* of these opposites, we are left with the following sixteen, viz.:—

$$E,\ F,\ H,\ I,\ K,\ L,\ N,\ O,$$
$$E',\ F',\ H',\ I',\ K',\ L',\ N',\ O';$$

these sixteen may be further divided into two sets, of eight cubes each, which possess a very remarkable and elegant property.

We have: First set $K\ L\ F'\ E'\ H'\ O'\ I\ N$,

Second set $F\ E\ K'\ L'\ I'\ N'\ H\ O$.

In regard to the first set, I say that they are connected with the cube A in the following manner:—

It is possible to form the eight cubes of the set into a single cube in such wise that contiguous faces of the cubes are similarly numbered, and also so that the resulting large cube has four identical numbers exhibited on each face, and from its numbering is identifiable with the cube A.

There are two, and only two solutions, which I exhibit by writing first the lower layer of cubes and beneath it the top layer.

FIRST SOLUTION.

Lower layer.

$K\ F'$		3 2 4 5 1	3 5 2 4 1
	or		
$L\ E'$		1 2 4 3 5	1 3 2 4 5

Upper layer.

$H'\ I$		3 2 1 5 6	3 5 1 4 6
	or		
$O'\ N$		6 2 1 3 5	6 3 1 4 5

SECOND SOLUTION.

Lower Layer.

N O'	or	3 2 5 1 4	3 1 5 4 2
I H'		4 2 3 1 5	2 1 3 4 5

Upper layer.

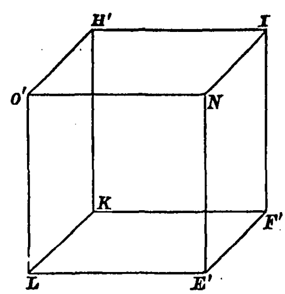
E' L	or	3 2 1 6 4	3 6 1 4 2
F' K		4 2 1 6 5	2 6 1 4 5

The second set of eight cubes is similarly connected with the cube A'.

For the examination of this property it is convenient to make a few simple definitions.

I speak of the cube A as *containing* each of the eight cubes

$$K, \; L, \; F', \; E', \; H', \; O', \; I, \; N.$$



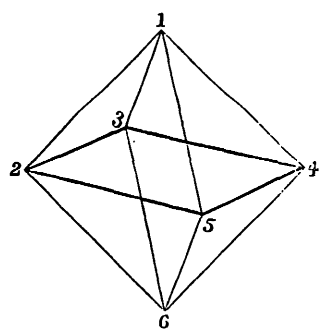
I call the cubes K and N (see figure) *diagonally opposite* with respect to the cube A. So also the pairs L, I; F', O'; E', H' are diagonally opposite with respect to the same cube.

The cubes L, F', H' I speak of as being *adjacent* to K with respect to A; and of the cubes E', I, O' as being *diametrically opposite* to K with respect to A.

It will be evident, as regards the location of the cubes, that in the example given the second solution is derivable from the first by interchanging the cubes in each diagonally opposite pair.

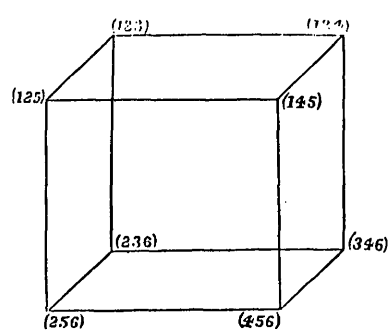
Associated with a cube having numbered faces is an octahedron having numbered summits, formed by joining the middle point of each face with the middle points of the four faces which have with it one edge in common.

Thus the cube *A* yields the octahedron



which may be supposed on a horizontal plane with the diagonal 16 vertical. We have eight octahedral faces having a one-to-one correspondence with the eight summits of the cube. The face 514 corresponds to that summit of the cube which is the point of intersection of the faces numbered 5, 1, 4; the opposite face 236 of the octahedron corresponds to the cube summit determined by the intersection of the faces 2, 3, 6, which is diagonally opposite to the former summit. The latter summit is the cube-summit opposite to the face 514 of the octahedron.

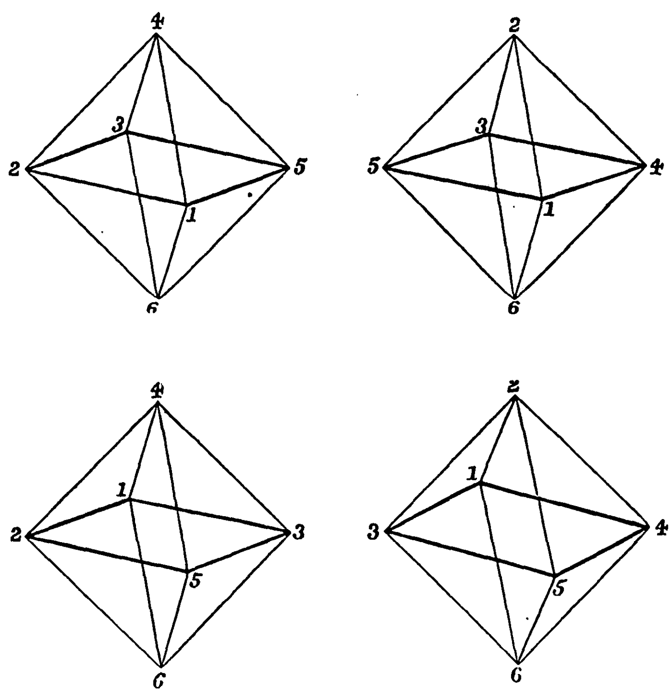
Denote the eight summits of the cube *A* as below.



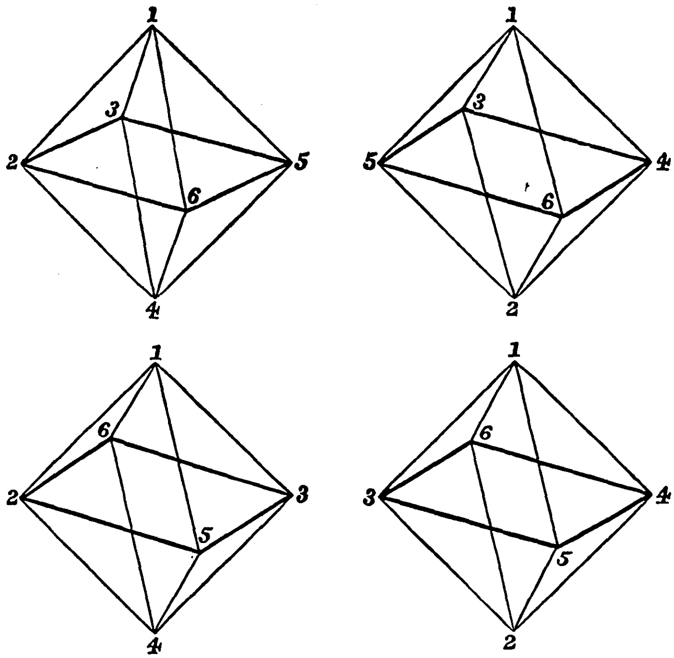
The problem is to properly place the eight octahedra contained by the octahedron A at the eight summits of this cube.

For the summit (236), take the octahedral face 145, which corresponds with the diagonally opposite summit 145, and, regarding the octahedron from an external point, perform the counter-clock-wise substitution (514). The resulting octahedron is to be placed without rotation at the cube summit 236. Similarly, for the summit 346, we perform the clock-wise substitution (512), and place the resulting octahedron without rotation at the summit 346. Proceeding in this way, employing the counter-clock-wise substitution in the cases of those summits which are diametrically opposite to that summit first considered, and clock-wise substitutions for the remaining summits, we obtain the following result:—

Lower layer.



Upper layer.



which constitutes a solution of the problem, and may be identified with the first solution above given. This solution not only selects the proper eight cubes, but places them in their right places and with their right rotations.

The second solution is obtained by merely employing counter-clockwise rotations of octahedral faces where clock-wise rotations are employed in the first solution, and *vice versâ*.

The corresponding substitutions are

Lower layer.	*Upper layer.*
(145), (125),	$(645) \equiv (123)^2$, $(625) \equiv (143)^2$,
(143), (123),	$(643) \equiv (125)^2$, $(623) \equiv (145)^2$.

If a particular cube of the eight be obtained from the containing cube by a circular substitution of the third order, the diagonally opposite cube is obtained by the square of the same substitution.

The condition that a cube Y may be contained by a cube X is clearly that, on replacing them by octahedra, the Y octahedron may be obtainable from the X octahedron by a circular substitution performed upon the summits which determine a triangular face. In other words, if the Y cube is obtainable from the X cube by a circular substitution performed upon the faces which determine a cube summit, the cube X contains the cube Y.

Hence follows the reciprocal relation between the cubes, and we may say that if a cube X contain the cube Y, then also the cube Y contains the cube X.*

Before proceeding further, it will be convenient to present the whole of the thirty sets of eight cubes.

A	contains	$KLF'E'H'O'IN$,	A'	contains	$FEK'L'I'N'HO$,
B	,,	$MO'FD'G'LI'J$,	B'	,,	$F'DM'OIJ'GL'$,
C	,,	$H'GD'EKM'JN'$,	C'	,,	$DE'HG'J'NK'M$,
D	,,	$LKC'B'GMI'N'$,	D'	,,	$CBL'K'ING'M'$,
E	,,	$O'MCA'HKIJ'$,	E'	,,	$C'AOM'I'JH'K'$,
F	,,	$GH'A'BLOJ'N$,	F'	,,	$AB'G'HJN'L'O'$,
G	,,	$CB'O'NFK'DJ$,	G'	,,	$ON'C'BD'J'F'K$,
H	,,	$C'A'LJF'M'EN$,	H'	,,	$L'J'CAE'N'FM$,
I	,,	$B'AED'K'MLO$,	I'	,,	$E'DBAL'O'KM'$,
J	,,	$BCE'F'HOGM$,	J'	,,	$EFB'C'G'M'H'O'$,
K	,,	$G'I'ACENDO$,	K'	,,	$A'C'GID'O'E'N'$,
L	,,	$N'M'BAIFHD$,	L'	,,	$B'A'NMH'D'I'F'$,
M	,,	$JL'DEIC'H'B$,	M'	,,	$D'E'J'LHB'I'C$,
N	,,	$AC'D'FGL'HK$,	N'	,,	$DF'A'CH'K'G'L$,
O	,,	$A'B'JKG'E'IF$,	O'	,,	$J'K'ABI'F'GE$.

In the cube O', J' and E, K' and G, A and F', B and I' are diagonally opposite, respectively, and so on; in every case letters symmetrically placed with regard to the extremities of the row of eight letters denote diagonally opposite cubes.

In any set of cubes, any cube contains the cube diagonally opposite to it, but no other cube of the set. For the cube in question can be seen, by inspection of the substitutions by which the cubes of the set are derived from the containing cube, to be the only cube of the

* Obviously, also, if a cube X contain a cube Y, the cube associated with X contains the cube associated with Y.

set derivable, by a rotation of an octahedral face, from the selected cube.

Suppose cubes X and Y a diagonally opposite pair with respect to Z, and the cube X to be derived by a circular substitution (abc) from Z. Then the substitutions from Z are respectively

$$\begin{array}{ccc} X & Y & Z \\ (abc) & (abc)^2 & 1, \end{array}$$

and, performing the substitution (abc), we get

$$\begin{array}{ccc} X & Y & Z \\ (abc)^2 & 1 & (abc), \end{array}$$

and, again performing the substitution (abc),

$$\begin{array}{ccc} X & Y & Z \\ 1 & (abc) & (abc)^2, \end{array}$$

results which establish that Z and X are a diagonally opposite pair with respect to Y, and further that Y and Z are a diagonally opposite pair with respect to X.

Ex. gr., from the above table, we see that

J' and E	are diagonally opposite	with regard to	O',
O' and J'	,,	,,	E,
E and O'	,,	,,	J'.

This law of reciprocity includes, of course, that previously established.

If, in any set of eight cubes, the cubes W, X, Y be diametrically opposite to Z, it can be shown that the cube which is associated with Z, viz., Z', contains the three cubes W, X, Y. This is obvious on examination of any set of eight cubes as represented by octahedra. Transforming Z to Z', it is found that Z can be transformed into W, X, or Y, by a circular substitution of the third order performed upon some three summits which determine a face of the octahedron.

Each set of eight cubes may be separated into two tetrads of cubes, the cubes in each tetrad being diametrically opposite.

The property of a tetrad is that the cube associated with any cube of the tetrad contains the three other cubes of the tetrad.

Altogether there are sixty tetrads.

Any tetrad of cubes, together with the cube which contains them, further constitute a pentad of cubes, which it is interesting to examine.

The cube A contains the tetrad K, E', O', I.
The pentad is therefore

$$A, \quad K, \quad E', \quad O', \quad I.$$

It can be shown that, from the five cubes

$$A, \quad K, \quad E, \quad O, \quad I,$$

and their associates $A', \quad K', \quad E', \quad O', \quad I'$,

there can be formed altogether ten pentads. Since

$$A \text{ contains } K, E', O', I,$$

$$A, K' \text{ both contain } E', O', I,$$

by the previous proposition.

Therefore K' contains A', E', O', I,

giving the pentad K', A', E', O', I; and so on.

The ten pentads are

$$A; \quad K, E', O', I,$$
$$K; \quad A, E, O, I',$$
$$E'; \quad A, K', O, I',$$
$$O'; \quad A, K', E, I',$$
$$I; \quad A, K', E, O,$$
$$A'; \quad K', E, O, I',$$
$$K'; \quad A', E', O', I,$$
$$E; \quad A', K, O', I,$$
$$O; \quad A', K, E', I,$$
$$I'; \quad A', K, E', O.$$

The pentad $A; \quad K, E', O', I$,

shows that the cubes

$$A, K' \text{ each contain the three cubes } E', O', I,$$

and that the cubes

$$E', O', I \text{ each contain the two cubes } A, K';$$

and, from the above ten pentads, we find that there are twenty pairs of cubes which contain three cubes in common, and twenty triads of cubes which contain two cubes in common.

There are fifty other pentads, viz., ten each derived from the pentads

$$
\begin{array}{l}
A;\ L,\ F',\ H',\ N, \\
B;\ M,\ D',\ L,\ I', \\
B;\ O',\ F,\ G',\ J, \\
C;\ H',\ E,\ M',\ J, \\
C;\ G,\ D',\ K,\ N'.
\end{array}
$$

Altogether there are 120 pairs of cubes which contain three cubes common to each pair, and 120 triads of cubes, the cubes of each triad containing two cubes in common.

Thursday, March 9th, 1893.

A. B. BASSET, Esq., F.R.S., Vice-President, in the Chair.

The following gentlemen were elected members:—F. W. Dyson, M.A., Fellow of Trinity College, Cambridge; J. P. Johnston, M.A. Dub., B.A. Cambridge; T. R. Lee, B.A., late Scholar of Pembroke College, Cambridge; and J. E. A. Steggall, M.A., Professor of Mathematics in University College, Dundee.

Mr. T. J. Dewar exhibited twenty stereographs of the Regular Solids, which were examined with the aid of a stereoscope. He was shown the diagrams furnished by the late Prof. Clerk-Maxwell to the second volume of the *Proceedings.*

The following communications were made:—

Note on the Stability of a Thin Rod loaded vertically: Mr. Love.

On Complex Primes formed with the Fifth Roots of Unity: Prof. Tanner.

On a Threefold Symmetry in the Elements of Heine's Series: Prof. L. J. Rogers.

The Dioptrics of Gratings: Dr. J. Larmor.

The following presents were received:—

A Cabinet Likeness of Mr. Rhodes, presented by Mr. Rhodes.

"Beiblätter zu den Annalen der Physik und Chemie." Band XVII., Stück 2.

"Journal of the Institute of Actuaries," Vol. xxx., Pt. 4, No. 168.

"Proceedings of the Royal Society," Vol. LII., No. 318.

"Mathematical Questions and Solutions," edited by W. J. C. Miller, Vol. LVIII.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," Band III., Heft 3; 1893.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVI., Livraisons 4, 5.

"Report of the Superintendent of the U. S. Naval Observatory" for the year ending 1892, June 30.

"Jahrbuch über die Fortschritte der Mathematik," Band XXII., Jahrgang 1890, Heft 1.

"Nieuw Archief voor Wiskunde," Deel xx., Stük 1; Amsterdam, 1893.

"Bulletin de la Société Mathématique de France," Tome xx., Nos. 7 and 8.

"Bulletin of the New York Mathematical Society," Vol. II., No. 5.

"Levensbericht van F. J. van den Berg en Lijst Zijner Geschriften," door D. B. de Haan; Amsterdam, 1893.

"Bulletin des Sciences Mathématiques," Tome XVII.; January, 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," 5ª Serie, Vol. II., Fasc. 1, 2, Sem. 1; Roma, 1893.

"Annales de la Faculté des Sciences de Toulouse," Tome VI., Pt. 4; 1892.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VI., Fasc. 1; Napoli, 1893.

"Educational Times," March, 1893.

"Journal für die reine und angewandte Mathematik," Bd. CXI., Heft 2.

"Transactions of the Royal Irish Academy," Vol. xxx., Parts 3 and 4.

"Indian Engineering," Vol. XIII., Nos. 3, 4, 5, and Index to Vol. XII., Pt. 2.

Note on the Stability of a Thin Elastic Rod. By A. E. H. Love.

Read March 9th, 1893.

1. The stability of a thin rod or column, vertical when unstrained, and loaded at its upper end, was first investigated by Euler in 1757. He showed that when the load exceeds a certain limit, the rod will be bent under its own weight, and he found the limiting load under which the central line of the rod can take up a form differing very little from the straight form, and crossing its initial position a given number of times. When the load is greater than that needed to produce flexure, and less than that needed to make the central line