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THEORY OF THE GENERAL ALTERNATING CURRENT TRANSFORMER.

BY CHARLES P. STEINMETZ.

The simplest alternating current apparatus is the alternating current transformer. It consists of a magnetic circuit, interlinked with two electric circuits or sets of electric circuits. The one, the primary circuit, is excited by an impressed E. M. F., while in the other, the secondary circuit, an E. M. F. is induced. Thus, in the primary circuit, power is consumed, in the secondary circuit a corresponding amount of power is produced, or in other words, power is transferred through space, from primary to secondary circuit. This transfer of power finds its mechanical equivalent in a repulsive thrust acting between primary and secondary. Thus, if the secondary coil is not held rigidly as in the stationary transformer, it will be repelled and move away from the primary. This mechanical effect is made use of in the induction motor, which represents a transformer whose secondary is mounted movably with regard to the primary in such a way that, while set in rotation, it still remains in the primary field of force.

We see thus that the stationary transformer and the induction motor are merely different applications of the same apparatus, comprising a magnetic circuit interlinked with two electric circuits. Such an apparatus can properly be called a "*general alternating current transformer*," and its equations in complex quantities are given in the following:—

The equations of the stationary transformer and those of the induction motor are merely specializations of the general alternating current transformer equations.

The general alternating current transformer transforms between electrical and mechanical power, and changes not only E. M. F.'s and currents, but frequencies also.

Besides the magnetic flux interlinked with both primary and secondary electric circuit, a magnetic cross flux passes in the transformer between primary and secondary, surrounding one coil only, without being interlinked with the other. This magnetic cross flux is proportional to the current flowing in the electric circuit, and constitutes what is called the self-induction of the transformer. As seen, as self-induction of a transformer circuit, not the total flux produced by and interlinked with this circuit is understood, but only that—usually small—part of the flux which surrounds the one circuit without interlinking with the other, and is thus produced by the M. M. F. of one circuit only.

The common magnetic flux of the transformer is produced by the resultant M. M. F. of both electric circuits. It is determined by the counter E. M. F., the number of turns, and the frequency of the electric circuit, by the equation:

$$E = \sqrt{2}\,\pi\, N\, n\, M\, 10^{-8}.$$

Where E = effective E. M. F.
N = frequency.
n = number of turns.
M = maximum magnetic flux.

The M. M. F. producing this flux, or the resultant M. M. F. of primary and secondary circuit, is determined by shape and magnetic characteristic of the material composing the magnetic circuit, and by the magnetic induction. At open secondary circuit, this M. M. F. is the M. M. F. of the primary current, which in this case is called the exciting current, and consists of an energy component, the magnetic energy current, and a reactive component, the magnetizing current.

In the general alternating current transformer, where the secondary is movable with regard to the primary, the rate of cutting of the secondary electric circuit with the mutual magnetic flux is different from that of the primary. Thus, the frequencies of both circuits are different, and the induced E. M. F.'s are not proportional to the number of turns as in the stationary transformer, but to the product of number of turns into frequency.

Let, in a general alternating current transformer:

$$k = \text{ratio } \frac{\text{secondary}}{\text{primary}} \text{ frequency, or "slip,"}$$

thus, if:

N = primary frequency, or frequency of impressed E. M. F.,

$$k\,N = \text{secondary frequency,}$$

and the E. M. F. induced per secondary turn by the mutual flux has to the E. M. F. induced per primary turn the ratio k,

$k = 0$ represents synchronous motion of the secondary,

$k < 0$ represents motion above synchronism—driven by external mechanical power, as will be seen,

$k = 1$ represents standstill,

$k > 1$ represents backward motion of the secondary, that is, motion against the mechanical force acting between primary and secondary (thus representing driving by external mechanical power).

Let:

n_0 = number of primary turns in series per circuit,

n_1 = number of secondary turns in series per circuit,

$a = \frac{n_0}{n_1}$ = ratio of turns,

$Y_0 = \rho_0 + j\,\sigma_0$ = primary admittance, per circuit, where

ρ_0 = effective conductance,

σ_0 = susceptance,

$U_0 = r_0 - j\,s_0$ = internal primary impedance per circuit, where

r_0 = effective resistance of primary circuit,

s_0 = reactance of primary circuit,

$U_{11} = r_1 - j\,s_1$ = internal secondary impedance per circuit at standstill, or for $k = 1$, where

r_1 = effective resistance of secondary coil,

s_1 = reactance of secondary coil at standstill, or full frequency: $k = 1$.

Since the reactance is proportional to the frequency, at the slip k, or the secondary frequency $k\,N$, the secondary impedance is:

$$U_1 = r_1 - j\,k\,s_1.$$

Let the secondary circuit be closed by an external resistance r, and an external reactance, and denote the latter by s at frequency N, then at frequency $k\,N$, or slip k, it will be $= k\,s$, and thus:

$U = r - j\,k\,s =$ external secondary impedance.[1]

Let:

E_0 = primary impressed E. M. F. per circuit.

E_0' = E. M. F. consumed by primary counter E. M. F.

E_1 = secondary terminal E. M. F.

E_1' = secondary induced E. M. F.

e = E. M. F. induced per turn by the mutual magnetic flux, at full frequency N.

C_0 = primary current.

C_{00} = primary exciting current.

C_1 = secondary current.

It is then:

Secondary induced E. M. F.:

$$E_1' = k\,n_1\,e.$$

Total secondary impedance:

$$U_1 + U = (r_1 + r) - j\,k\,(s_1 + s);$$

hence, secondary current:

$$C_1 = \frac{E_1'}{U_1 + U} = \frac{k\,n_1\,e}{(r_1 + r) - j\,k\,(s_1 + s)}.$$

Secondary terminal voltage:

$$E_1 = E_1' - C_1\,U_1 = C_1\,U$$

$$= k\,n_1\,e\,\left\{1 - \frac{r_1 - j\,k\,s_1}{(r_1 + r) - j\,k\,(s_1 + s)}\right\} = \frac{k\,n_1\,e\,(r - j\,k\,s)}{(r_1 + r) - j\,k\,s_1 + s)}.$$

E. M. F. consumed by primary counter E. M. F.:

$$E_0' = -\,n_0\,e;$$

hence, primary exciting current:

$$C_{00} = E_0'\,Y_0 = -\,n_0\,e\,(\rho_0 + j\,\sigma_0).$$

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1 This applies to the case, where the secondary contains inductive reactance only, or rather that kind of reactance, which is proportional to the frequency. In a condenser, the reactance is inversely proportional to the frequency, in a synchronous motor under circumstances independent of the frequency. Thus in general we have to set: $s = s' + s'' + s'''$, where s' is that part of the reactance, which is proportional to the frequency, s'' that part of the reactance independent of the frequency, and s''' that part of the reactance, which is inversely proportional to the frequency, and have thus, at slip k, or frequency $k\,N$, the external secondary reactance: $k\ s' + s'' + \frac{s'''}{k}$.

Component of primary current corresponding to secondary current C_1:

$$C_1' = -\frac{C_1}{a}$$

$$= -\frac{n_0\, k\, e}{a^2\,\{(r_1 + r) - j\, k\, (s_1 + s)\}};$$

hence, total primary current:

$$C_0 = C_{00} + C_1'$$

$$= -k\, n_0\, e\left\{\frac{1}{a_2}\,\frac{1}{(r_1 + r) - j\, k\, (s_1 + s)} + \frac{\rho_0 + j\, k\, \sigma_0}{k}\right\}.$$

Primary impressed E. M. F.:

$$E_0 = E_0' + C_0\, U_0$$

$$= -n_0\, e\left\{1 + \frac{k}{a^2}\,\frac{r_0 - j\, s_0}{(r_1 + r) - j\, k\, (s_1 + s)} + (r_0 - j\, s_0)\,(\rho_0 + j\, \sigma_0)\right\}$$

We get thus, as the

Equations of the General Alternating Current Transformer,

of ratio of turns: a, and ratio of frequencies: k, with the E. M. F. induced per turn at full frequency: e, as parameter, the values:

Primary impressed E. M. F.:

$$E_0 = -n_0\, e\left\{1 + \frac{k}{a^2}\,\frac{r_0 - j\, s_0}{(r_1 + r) - j\, k\, (s_1 + s)} + (r_0 - j\, s_0)\,(\rho_0 + j\, \sigma_0)\right\}.$$

Secondary terminal voltage:

$$E_1 = k\, n_1\, e\left\{1 - \frac{r_1 - j\, k\, s_1}{(r_1 + r) - j\, k\, (s_1 + s)}\right\} = k\, n_1\, e\,\frac{r - j\, k\, s}{(r_1 + r) - j\, k\, (s_1 + s)}.$$

Primary current:

$$C_0 = -k\, n_0\, e\left\{\frac{1}{a^2}\,\frac{1}{(r_1 + r) - j\, k\, (s_1 + s)} + \frac{\rho_0 + j\, k\, \sigma_0}{k}\right\}.$$

Secondary current:

$$C_1 = \frac{k\, n_1\, e}{(r_1 + r(- j\, k\, (s_1 + s)}.$$

Therefrom we get:

Ratio of currents:

$$\frac{C_0}{C_1} = -\frac{1}{a}\left\{1 + \frac{a^2}{k}\,(\rho_0 + j\, \sigma_0)\,[(r_1 + r) - j\, (s_1 + s)]\right\}.$$

Ratio of E. M. F.'s:

$$\frac{E_0}{E_1} = -\frac{a}{k}\left\{\frac{1+\frac{k}{a^2}\frac{r_0-j\,s_0}{(r_1+r)-j\,k(s_1+s)}+(r_0-j\,s_0)(\rho_0+j\,\sigma_0)}{1-\frac{r_1-j\,k\,s_1}{(r_1+r)-j\,k\,(s_1+s)}}\right\}$$

Total apparent primary impedance:

$$U_t = \frac{E_0}{C_0} = \frac{a^2}{k}\;\{(r_1+r)-j\,(s_1+s)\}$$

$$\left\{\frac{1+\frac{k}{a^2}\,\frac{r_0-j\,s_0}{(r_1+r)+j\,k\,(s_1+s)}+(r_0-j\,s_0)\;(\rho_0+j\,\sigma_0)}{1+\frac{a^2}{k}\,(\rho_0+j\,\sigma_0)\;[(r_1+r)-j\,k\,(s_1+s)]}\right\}.$$

Where:

$$s = s' + \frac{s''}{k} + \frac{s'''}{k^2}$$

in the general secondary circuit as discussed in foot note, page 4.

Substituting in these equations:

$$k = 1,$$

gives the

General Equations of the Stationary Alternating Current Transformer:

$$E_0 = -\,n_0\,e\,\{1+\frac{1}{a^2}\;\frac{U_0}{U_1+U}+U_0\;Y_0\}.$$

$$E_1 = n_1\,e\,\{1-\frac{U_1}{U_1+U}\} = n_1\,e\,\frac{U}{U_1+U}.$$

$$C_0 = -\,n_0\,e\left\{\frac{1}{a^2\,(U_1+U)}+Y_0\right\}.$$

$$C_1 = \frac{n_1\,e}{U_1+U}.$$

$$\frac{C_0}{C_1} = -\frac{1}{a}\left\{1+a^2\;Y_0\,(U_1+U)\right\}.$$

$$\frac{E_0}{E_1} = -\,a\left\{\frac{1+\frac{U_0}{a^2\,(U_1+U)}+U_0\;Y_0}{1-\frac{U_1}{U_1+U}}\right\}$$

$$U_t = \frac{E_0}{C_0} = a^2\,(U_1+U)\left\{\frac{1+\frac{U_0}{a^2\,(U_1+U)}+U_0\;Y_0}{1+a^2\;Y_0\,(U_1+U)}\right\}$$

Substituting in the equations of the general alternating current transformer:

$$U = 0,$$

gives the

General Equations of the Induction Motor:

$$E_0 = -n_0\, e \left\{1 + \frac{k}{a^2}\,\frac{r_0 - j\, s_0}{r_1 - j\, k\, s_1} + (r_0 + j\, s_0)\,(\rho_0 + j\, \sigma_0)\right\}.$$

$$E_1 = 0.$$

$$C_0 = -k\, n_0\, e \left\{\frac{1}{a^2\,(r_1 - j\, k\, s_1)} + \frac{\rho_0 + j\, \sigma_0}{k}\right\}$$

$$C_1 = \frac{k\, n_1\, e}{r_1 - j\, k\, \sigma_1}.$$

$$\frac{C_0}{C_1} = -\frac{1}{a}\left\{1 + \frac{a^2}{k}\,(\rho_0 + j\, \sigma_0)\,(r_1 - j\, k\, s_1)\right\}.$$

$$U_t = \frac{a^2}{k}\,(r_1 - j\, k\, s_1)\left\{\frac{1 + \dfrac{k}{a^2}\,\dfrac{r_0 - j\, s_0}{r_1 - j\, k\, s_1} + (r_0 - j\, s_0)\,(\rho_0 + j\, \sigma_0}{1 + \dfrac{a^2}{k}\,(r_1 - j\, k\, s_1)\,(\rho_0 + j\, \sigma_0)}\right\}$$

Returning now to the general alternating current transformer, we have by substituting:

$$(r_1 + r)^2 + k^2\,(s_1 + s)^2 = u_k^{\,2},$$

and separating the real and imaginary quantities:

$$E_0 = -n_0\, e \left\{\left[1 + \frac{k}{a^2\, u_k^{\,2}}\Big(r_0\,(r_1 + r) + k\, s_0\,(s_1 + s)\Big) + (r_0\,\rho_0 + s_0\,\sigma_0)\right] + j\left[\frac{k}{a^2\, u_k^{\,2}}\Big(k\, r_0\,(s_1 + s) - s_0\,(r_1 + r)\Big) + (r_0\,\sigma_0 - s_0\,\rho_0)\right]\right\}.$$

$$C_0 = -k\, n_0\, e \left\{\left[\frac{r_1 + r}{a^2\, u_k^{\,2}} + \frac{\rho_0}{k}\right] + j\left[\frac{k\,(s_1 + s)}{a^2\, u_k^{\,2}} + \frac{\sigma_0}{k}\right]\right\}.$$

$$C_1 = \frac{k\, n_1\, e}{u_k^{\,2}}\left\{(r_1 + r) + j\, k\,(s_1 + s)\right\}.$$

Neglecting the exciting current, or rather considering it as a separate and independent shunt circuit outside of the transformer, as can approximately be done, and assuming the primary impedance reduced to the secondary circuit as equal to the secondary impedance:

$$Y_0 = 0,$$

$$\frac{U_0}{a^2} = U_{11}.$$

Substituting this in the equations of the general transformer, we get:

$$E_0 = - n_0 e \left\{ 1 + \frac{k}{u_k^2} \left[r_1 (r_1 + r) + k s_1 (s_1 + s) \right] + \frac{j k}{u_k^2} \left[k r_1 (s_1 + s) - s_1 (r_1 + r) \right] \right\}.$$

$$E_1 = \frac{k n_1 e}{u_k^2} \left\{ \left[r (r_1 + r) + k^2 s (s_1 + s) \right] + j k \left[r s_1 - s r_1 \right] \right\}$$

$$C_0 = - \frac{k n_1 e}{a u_k^2} \left\{ (r_1 + r) j k (s_1 + s) \right\}.$$

$$C_1 = \frac{k n_1 e}{u_k^2} \left\{ (r_1 + r) j k (s_1 + s) \right\}.$$

If:

$E = a + j b =$ E. M. F., in complex quantities, and

$C = c + j d =$ current, in complex quantities,

the power is:

$$P = | E, C | = E C \cos (E, C) = a c + b d.$$

Making use of this, and denoting:

$$\frac{k n_1^2 e^2}{u_k^2} = w$$

gives:

Secondary output of the transformer:

$$W_1 = | E_1, C_1 | = \left(\frac{k n_1 e}{u_k} \right)^2 r = k r w.$$

Internal loss in secondary circuit:

$$W_1^1 = c_1^2 r_1 = \left(\frac{k n_1 e}{u_k} \right)^2 r_1 = k r_1 w.$$

Total secondary power:

$$W_1 + W_1^1 = \left(\frac{k n_1 e}{u_k} \right)^2 (r + r_1) = k w (r + r_1).$$

Internal loss in primary circuit:

$$W_0^1 = c_0^2 r_1 = \left(\frac{k n_1 e}{u_k} \right)^2 r_1 = k r_1 w.$$

Total electrical output plus loss:

$$W^1 = W_1 + W_1^1 + W_0^1 = \left(\frac{k\,n_1\,e}{u_k}\right)^2 (r + 2\,r_1) = k\,w\,(r + 2\,r_1).$$

Total electrical input of primary:

$$W_0 = |\,E_0,\ C_0\,| = k\left(\frac{n_1\,e}{u_k}\right)^2 (r + r_1 + k\,r_1) = w\,(r + r_1 + k\,r_1).$$

Hence, mechanical output of transformer:

$$W = W_0 - W^1 = w\,(1 - k)\,(r + r_1).$$

$$\text{Ratio: } \frac{\text{mechanical output}}{\text{total secondary power}} = \frac{W}{W_1 + W_1^1} = \frac{1 - k}{k} = \frac{\text{speed}}{\text{slip}}.$$

Thus:

In a general alternating transformer of ratio of turns a, and ratio of frequencies k, neglecting exciting current, it is:

Electrical input in primary:

$$W_0 = \frac{k\,n_1^2\,e^2\,(r + r_1 + k\,r_1)}{(r_1 + r)^2 + k^2\,(s_1 + s)^2}.$$

Mechanical output:

$$W = \frac{k\,(1 - k)\,n_1^2\,e^2\,(r + r_1)}{(r_1 + r)^2 + k^2\,(s_1 + s)^2}.$$

Electrical output of secondary:

$$W_1 = \frac{k^2\,n_1^2\,e^2\,r}{(r_1 + r)^2 + k^2\,(s_1 + s)^2}.$$

Losses in transformer:

$$W_0^1 + W_1^1 = W^1 = \frac{1\,k^2\,n_1^2\,e^2\,r_1}{(r_1 + r)^2 + k^2\,(s_1 + s)^2}.$$

Of these quantities, W^1 and W_1 are always positive; W_0 and W can be positive or negative, according to the value of k. Thus the apparatus can either produce mechanical power, acting as a motor, or consume mechanical power, and it can either consume electrical power or produce electrical power, as a generator.

At:

$k = 0$, synchronism, $W_0 = 0,\ W = 0,\ W_1 = 0.$

At:

$0 < k < 1$, between synchronism and standstill.

W_1, W and W_0 are positive, that is, the apparatus consumes electrical power W_0 in the primary, and produces mechanical power W and electrical power $W_1 + W_1^1$ in the secondary, which is partly— W_1^1—consumed by the internal secondary resistance, partly— W_1—available at the secondary terminals.

In this case it is : $$\frac{W_1 + W_1^1}{W} = \frac{k}{1 - k},$$

That is, of the electrical power consumed in the primary circuit — W_0—a part W_0^1 is consumed by the internal primary resistance, the remainder transmitted to the secondary, and divides between electrical power $W_1 + W_1^1$ and mechanical power W in the proportion of the slip, or drop below synchronism—k—, to the speed : $1 - k$.

In this range, the apparatus is a motor.

At :

$$k > 1, \text{ or backwards driving,}$$

$W < 0$, or negative, that is, the apparatus requires mechanical power for driving.

It is then :

$$W_0 - W_0^1 - W_1^1 < W_1,$$

that is : the secondary electrical power is produced partly by the primary electrical power, partly by the mechanical power, and the apparatus acts simultaneously as transformer and as alternating current generator, with the secondary as armature.

The ratio of mechanical input to electrical input is the ratio of speed to synchronism.

In this case, the secondary frequency is higher than the primary.

At : $K < 0$, beyond synchronism.

$W < 0$, that is, the apparatus has to be driven by mechanical power.

$W_0 < 0$, that is, the primary circuit produces electrical power, from the mechanical input.

At :

$$r + r_1 + k\, r_1 = 0, \qquad \text{or:} \qquad k < -\frac{r + r_1}{r},$$

the electrical power produced in the primary becomes less than required to cover the losses of power, and W_0 becomes positive again.

We have thus : $$k < -\frac{r + r_1}{r},$$

consumes mechanical and primary electric power ; produces secondary electric power.

$$-\frac{r + r_1}{r} < k < 0,$$

consumes mechanical, and produces electrical power in primary and in secondary circuit.

$$0 < k < 1,$$

consumes primary electric power, and produces mechanical and secondary electrical power.

$$1 < k,$$

consumes mechanical and primary electrical power; produces secondary electrical power.

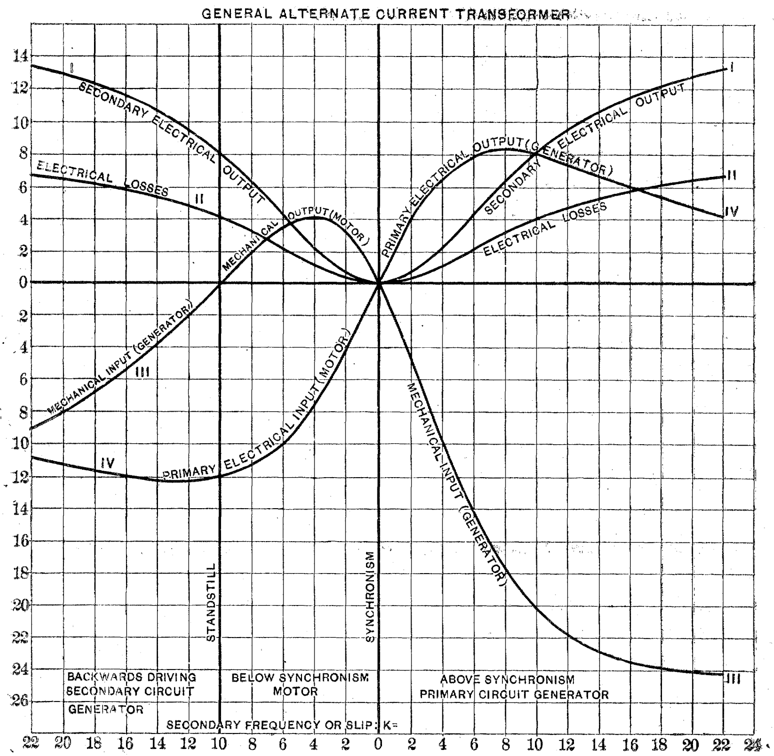

FIG. 1.

As an instance, in Fig. 1 are plotted, with the slip k as abscissae, the values of:

secondary electrical output	as curve I,
total internal loss	as curve II,
mechanical output	as curve III,
primary electrical input	as curve IV,

for the values:

$$n_1 e = 100,$$
$$r_1 = .1,$$
$$s_1 = .2,$$
$$r = .4,$$
$$s = .3;$$

hence:

$$W_1 = \frac{16,000\ k^2}{1 + k^2},$$

$$W_0^1 + W_1^1 = \frac{8,000\ k^2}{1 + k^2},$$

$$W_0 = \frac{4,000\ k + (5 + k)}{1 + k^2},$$

$$W = \frac{20,000\ k\ (1 - k)}{1 + k^2}.$$

Discussion.

Dr. M. I. Pupin:—I was given this very interesting paper a few days ago, but unfortunately I lost it and have not had a chance to look over it carefully again, and it is a paper that requires very careful study before any comment can be made upon it. It seems to me that from the viewpoint of the comprehensiveness of the subject with which the paper deals, the author accomplishes just what he intended. That is, to give in a very small compass the general theory of the alternating current machinery that is used to-day in commercial work, and he does it in a most elegant way. But it is elegant on account of its simplicity. This simplicity is made possible by the introduction of a peculiar method of accurate analysis which Mr. Steinmetz has worked out all by himself.

There is another point which recommends this general treatment of current transformers, namely: not only is the method a simple one, but it is easily translated into graphical methods. We can actually take these equations and by pencil and ruler transfer their physical meaning to paper by means of simple lines and curves.

I saw a machine at the Frankfort Exhibition, which, if I remember correctly, was constructed by Schückert. It would give you an alternating current, a two-phase direct current, a three-phase current, or anything else you wished, and for that reason it was called the "maid-of-all-work." This theory of alternating current machinery which Mr. Steinmetz has given us may be called the "maid-of-all-work" in alternating current machinery, because it gives information not only about the alternating current transformer, but also about the alternating current generator and alternating current induction