ON THE EXPANSION OF POLYNOMIALS IN SERIES OF FUNCTIONS

By L. N. G. FILON.

[Received and Read May 10th, 1906.—Received in revised form August 10th, 1906.]

1. *Introduction and Summary.*

The problem of expanding a given function $f(x)$ in a series of functions of given form—thus:

$$f(x) = a_1\phi(\kappa_1, x)+a_2\phi(\kappa_2, x)+\ldots+a_n\phi(\kappa_n, x)+\ldots, \tag{1}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_n, \ldots$ are the roots of a transcendental equation

$$\psi(z) = 0 \tag{2}$$

—is one which has been familiar to mathematicians since the days of Fourier. This problem, in most cases which occur in mathematical physics, is usually solved by the method of normal functions; that is, functions $\chi(\kappa, x)$ are determined such that

$$\int_a^\beta \chi(\kappa_r, x)\,\phi(\kappa_s, x)\,dx = 0 \tag{3}$$

when r, s are different, but has some definite value when $r = s$. Thus, multiplying (1) by $\chi(\kappa_n, x)$ and integrating from a to β, the coefficient a_n is readily determined.

The great disadvantages of this method are that it gives no clue for the discovery of the functions χ when the form of the latter is not obvious from other considerations, and that it gives no means of predicting, given the functions ϕ and the transcendental equation (2), whether the required expansion is possible or not.

Another method has been given by Cauchy, and is described in Picard's *Cours d'Analyse* (pp. 169 *et seq.*). This method depends on the calculus of residues. Cauchy (and Picard after him) restricted himself to the case of trigonometrical series (see Cauchy, *Œuvres Complètes*, t. VII, 2^e Série: "Sur les Résidus des Fonctions exprimées par des Intégrales définies," p. 393), but the process by which the result is arrived at seems artificial. The function [denoted below by $F(z)$] on which the whole expansion hinges is selected from an *a priori* knowledge of the coefficients in

Fourier's expansion, and no method is given for finding it in the general case.

Dini, in his book on Fourier series (*Serie di Fourier e altri rappresentazione analitiche delle funzioni di una variabile reale*, Pisa, 1880), has employed a mixed method, depending partly on normal functions, partly on Cauchy's residue theorem. He gives a determination of the function $F(z)$ of the present paper, but in order to do so seems to assume (*loc. cit.*, pp. 131, 132) that the conjugate functions are practically already known, and that $\chi(\kappa, x) = \phi(\kappa, x)\,\theta(x)$, $\theta(x)$ being a function of x independent of κ. This restricts very considerably the generality of his results.

Dini's analysis seems to be directed rather to giving exact proofs of expansions already known than to developing methods for obtaining new expansions.

The object of the present paper is to extend and generalize the application of Cauchy's method of residues to expansions, and to give a rule for finding the *form* of the expansion in certain large classes of cases.

In what follows the functions to be expanded are supposed finite polynomials. This enables us to dispense at present with troublesome considerations of convergence.

The paper begins by establishing a general theorem for expanding a polynomial in a series of functions of the form $\phi(\kappa x)$, κ being a root of $\psi(z) = 0$. The theorem is practically contained in equations (6) and (15). Exceptional cases, when $z = 0$ is a zero of $\psi(z)$, are next dealt with. An example of the method is then given, showing how to expand a function $f(x)$ in the form $\Sigma\,\{A_n \cos(\kappa_n x) + B_n \sin(\kappa_n x)\}$, the κ_n's being roots of the transcendental equation $J_0(\kappa a) = 0$.

It is also verified that the method will give the expansions of Fourier, Schlömilch's expansion, and expansions in Bessel functions of order zero which occur in physical examples. New forms are obtained for the coefficients in the expansions in Bessel functions of order zero.

Also, in each case, the method enables us to find the range of validity of the expansion and the values of the series at the extremities of the range of validity. Thus the results (34), (35), which give the values at the ends of the range for Fourier's second trigonometrical series, I have not been able to find anywhere.

The latter part of the paper, after a brief consideration of the possibility of extending the results to functions other than polynomials, is devoted to applying the method to series of functions $\phi(\kappa, x)$ where κ, x do not appear exclusively as a product κx. The results are applied to an expansion in functions occurring in the theory of elasticity, which expansion I believe to be new. Mr. John Dougall, in a paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate" (*Trans. Roy. Soc. Edin.*, Vol. XLI., Part I., 8, 1904), has used functions of this type and obtained the sum of a few series involving them by the method of residues, but he has not attempted the converse problem of determining the coefficients, given the sum.

2. *Determination of the Coefficients.*

Let $\phi(z, x)$ be a function of two variables, z being complex and x real. Consider the function $F(z)\phi(z, x)/\psi(z)$, and integrate it round any closed contour C in the z-plane, enclosing the origin. Then

$$\frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz = \text{sum of the residues of the function inside } C.$$

Suppose now that $\phi(z, x)$ is a function without poles, and that $F(z)$ has poles only at the origin. Then the poles of the function which contribute to the residues are the zeros of $\psi(z)$ inside C and the origin. To begin with, we shall assume that $\psi(z)$ has no zero at the origin.

Let $\kappa_1, \kappa_2, \ldots, \kappa_n$ be the zeros of $\psi(z)$ inside C, arranged in the order of magnitude of their moduli. Then

$$\frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz = \sum_{r=1}^{n} \frac{F(\kappa_r)\phi(\kappa_r, x)}{\psi'(\kappa_r)} + \text{residue at the origin.}$$

The residue at the origin will be some function of x. Denote it by $f(x)$. We have

$$f(x) = -\sum_{r=1}^{n} \frac{F(\kappa_r)\phi(\kappa_r, x)}{\psi'(\kappa_r)} + \frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz. \tag{4}$$

If, now, as the contour C becomes larger and larger,

$$\mathrm{L}\,\frac{1}{2\pi i}\int_C \frac{F(z)\phi(z, x)}{\psi(z)}\,dz = 0, \tag{5}$$

we obtain, on proceeding to the limit

$$f(x) = -\sum_{r=1}^{r=\infty} \frac{F(\kappa_r)\phi(\kappa_r, x)}{\psi'(\kappa_r)}, \tag{6}$$

the roots κ_r occurring in the order of their moduli. The problem is so to determine $F(z)$ that $f(x)$ shall be identical with a given polynomial and that (5) shall hold.

Consider
$$\frac{1}{2\pi i}\int \frac{F(z)\phi(z, x)}{\psi(z)}\,dz$$

taken round a small circle enclosing the origin. Since $\phi(z, x)$ is without poles, its expansion in powers of z is absolutely and uniformly convergent over this circle. Also, if the radius of this circle be $< |\kappa_1|$, $1/\psi(z)$ may be replaced by the equivalent Taylor series. The product of the two series is absolutely and uniformly convergent over the path of integration. Thus $\phi(z, x)/\psi(z)$ may be replaced by the two series and the result of multiplying these out integrated term by term. We shall further suppose that $\psi(z)$ itself has no pole infinitely close to $z = 0$, and take the radius of the circle of integration less than the modulus of its nearest pole (if any), so that $\psi(z)$ itself can be replaced by a power series.

Take, as a particular case,

$$F(z) = z^{-(n+1)}\psi_n(z) \tag{7}$$

where
$$\psi_n(z) = a_0+a_1 z+\ldots+a_n z^n, \tag{8}$$

i.e., $\psi_n(z)$ denotes the first $(n+1)$ terms of the denominator $\psi(z)$ of the above integral.

$$\psi(z) = a_0+a_1 z+\ldots+a_n z^n+a_{n+1}z^{n+1}+\ldots; \tag{9}$$

$$\frac{F(z)}{\psi(z)} = z^{-(n+1)}\frac{[\psi(z)-a_{n+1}z^{n+1}-a_{n+2}z^{n+2}-\ldots]}{\psi(z)}$$

$$= z^{-(n+1)}-\frac{(a_{n+1}+a_{n+2}z+\ldots)}{\psi(z)}$$

$$= z^{-(n+1)}+ \text{ power series in } z.$$

Thus
$$\frac{1}{2\pi i}\int\frac{\psi_n(z)}{z^{n+1}}\,\frac{\phi(z,x)}{\psi(z)}\,dz = \frac{1}{2\pi i}\int\frac{\phi(z,x)}{z^{n+1}}\,dz.$$

Let
$$\phi(z,x) = f_0(x)+zf_1(x)+\ldots+z^n f_n(x)+\ldots.$$

Then required residue $= f_n(x)$.

If we take
$$F(z) = \frac{p_0\psi_0(z)}{z}+\frac{p_1\psi_1(z)}{z^2}+\ldots+\frac{p_n\psi_n(z)}{z^{n+1}} \tag{10}$$

where $p_0, p_1, \ldots, p_n$ are constants, then

$$f(x) = p_0 f_0(x)+p_1 f_1(x)+\ldots+p_n f_n(x). \tag{11}$$

We will now consider more specially the important particular case where
$$f_n(x) = q_n x^n$$

or $\phi(z, x) =$ a function of zx only: thus

$$\phi(z,x) = \phi(zx) = q_0+q_1 zx+\ldots+q_n(zx)^n+\ldots. \tag{12}$$

Hence
$$q_n = \frac{\phi^n(0)}{n!}. \tag{13}$$

(11) gives
$$f(x) = p_0q_0+p_1q_1x+\ldots+p_nq_nx^n$$

or
$$p_nq_n = \frac{f^n(0)}{n!}, \tag{14}$$

that is
$$p_n = f^n(0)/\phi^n(0).$$

We shall suppose $\phi^n(0)$ is not zero; so that no term is missing in the expansion of $\phi(z)$.

The required form for $F(z)$ is therefore

$$F(z) = \frac{f(0)}{\phi(0)} \frac{\psi_0(z)}{z} + \frac{f'(0)}{\phi'(0)} \frac{\psi_1(z)}{z^2} + \ldots + \frac{f^n(0)}{\phi^n(0)} \frac{\psi_n(z)}{z^{n+1}}, \tag{15}$$

It will be convenient in what follows to conceive the series (15) as extending to infinity. This will prove in many cases to be really a simplification, and, if we remember that, after a certain term, all the terms of the series vanish, no difficulties of convergency will be introduced.

3. *Case where $\psi(z) = 0$ at the origin.*

Suppose that $z = 0$ is a zero of $\psi(z)$ of ρ-th order. Then

$$\psi(z) = a_\rho z^\rho + a_{\rho+1} z^{\rho+1} + \ldots + a_{\rho+s} z^{\rho+s} + \ldots.$$

In this case $\psi_0(z)$, $\psi_1(z)$, ..., $\psi_{\rho-1}(z)$ all vanish. Consider

$$\frac{\psi_{\rho+s-1}(z)}{z^{\rho+s}\psi(z)} = \frac{1}{z^{\rho+s}} \left\{ \frac{a_\rho z^\rho + \ldots + a_{\rho+s-1} z^{\rho+s-1}}{a_\rho z^\rho + \ldots + a_{\rho+s} z^{\rho+s} + \ldots} \right\}$$

$$= \frac{1}{z^{\rho+s}} \left\{ 1 - \frac{a_{\rho+s} z^{\rho+s} + a_{\rho+s+1} z^{\rho+s+1} + \ldots}{a_\rho z^\rho + \ldots + a_{\rho+s} z^{\rho+s} + \ldots} \right\}$$

$$= \frac{1}{z^{\rho+s}} \left\{ 1 - \frac{(a_{\rho+s} z^s + a_{\rho+s+1} z^{s+1} + \ldots)}{a_\rho + a_{\rho+1} z + \ldots + a_{\rho+s} z^s + \ldots} \right\}. \tag{16}$$

In the expansion of the above in ascending powers of z only the negative powers are required. Therefore it is sufficient to expand

$$\frac{1}{a_\rho + a_{\rho+1} z + \ldots + a_{\rho+s} z^s + \ldots}$$

as far as $z^{\rho-1}$.

Let

$$(a_\rho + a_{\rho+1} z + \ldots + a_{\rho+s} z^s + \ldots)^{-1} = b_0 + b_1 z + \ldots + b_{\rho-1} z^{\rho-1} + \ldots; \tag{17}$$

then we have the equations, to find the b's,

$$\left. \begin{aligned} 1 &= b_0 a_\rho \\ 0 &= b_0 a_{\rho+1} + b_1 a_\rho \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 &= b_0 a_{\rho+t} + b_1 a_{\rho+t-1} + \ldots + b_t a_\rho \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 &= b_0 a_{2\rho-1} + b_1 a_{2\rho-2} + \ldots + b_{\rho-1} a_\rho \end{aligned} \right\}. \tag{18}$$

Using (17), (16) becomes

$$\frac{\psi_{\rho+s-1}(z)}{z^{\rho+s}\psi(z)} = \frac{1}{z^{\rho+s}}\{1-z^s[b_0a_{\rho+s}+z(b_0a_{\rho+s+1}+b_1a_{\rho+s}) + z^2(b_0a_{\rho+s+2}+b_1a_{\rho+s+1}+b_2a_{\rho+s})+\dots + z^{\rho-1}(b_0a_{2\rho+s-1}+b_1a_{2\rho+s-2}+\dots+b_{\rho-1}a_{\rho+s})]\}$$
$$+ \text{ positive powers of } z. \qquad (19)$$

(19) has been established for positive values of s. But it is easy to see that it will still hold for $s = 0$ or $s =$ a negative integer numerically less than ρ.

If s be negative (or zero), then, using (18) and remembering $a_t = 0$ if $t < \rho$, we see that the only term in the square bracket in (19) which does not vanish is $z^{-s}b_0a_\rho$, *i.e.*, z^{-s}. Thus, if $s \leqslant 0$, (19) gives

$$\psi_{\rho+s-1}(z)/z^{\rho+s}\psi(z) = 0,$$

as it should.

We may therefore write (19)

$$\frac{\psi_s(z)}{z^{s+1}\psi(z)} = \frac{1}{z^{s+1}} - \frac{1}{z^\rho}[b_0a_{s+1}+z(b_0a_{s+2}+b_1a_{s+1})+z^2(b_0a_{s+3}+b_1a_{s+2}+b_2a_{s+1}) +\dots+z^{\rho-1}(b_0a_{s+\rho}+b_1a_{s+\rho-1}+\dots+b_{\rho-1}a_{s+1})]$$
$$+ \text{ positive powers of } z, \qquad (20)$$

and (20) holds for $s = 0$ or any positive integer. If we take, as before,

$$F(z) = \sum_{s=0}^{s=\infty} p_s z^{-(s+1)}\psi_s(z)$$

and

$$\phi(z, x) = \sum_{s=0}^{s=\infty} z^s f_s(x),$$

then the residue at the origin of $F(z)\phi(z, x)/\psi(z)$ is

$$\sum_{s=0}^{s=\infty} p_s f_s(x) - f_{\rho-1}(x)\sum_{s=0}^{s=\infty} b_0a_{s+1}p_s - f_{\rho-2}(x)\sum_{s=0}^{s=\infty}(b_0a_{s+2}+b_1a_{s+1})p_s - \dots$$
$$- f_0(x)\sum_{s=0}^{s=\infty}(b_0a_{s+\rho}+\dots+b_{\rho-1}a_{s+1})p_s.$$

If, now, we have, as before, been able to expand $f(x)$ in a series of functions $f_s(x)$ so that

$$f(x) = \sum_{s=0}^{s=\infty} p_s f_s(x),$$

it follows that $f(x)$ is not expansible *solely* in functions $\phi(z, x)$, but contains a finite number of terms of a different form. We have

$$f(x) = f_0(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho} + \ldots + b_{\rho-1} a_{s+1}) p_s$$
$$+ f_1(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho-1} + \ldots + b_{\rho-2} a_{s+1}) p_s + \ldots + f_{\rho-1}(x) \sum_{s=0}^{s=\infty} b_0 a_{s+1} p_s$$
$$- \sum_{r=1}^{r=\infty} \frac{F(\kappa_r)}{\psi'(\kappa_r)} \phi(\kappa_r, x). \tag{21}$$

In the important particular case where $\phi(z, x)$ is given by (12)

$$f_n(x) = \frac{\phi^n(0)}{n!} x^n, \qquad p_s = \frac{f^s(0)}{\phi^s(0)}.$$

Then

$$f(x) = \phi(0) \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho} + \ldots + b_{\rho-1} a_{s+1}) \frac{f^s(0)}{\phi^s(0)}$$
$$+ \frac{x\phi'(0)}{1!} \sum_{s=0}^{s=\infty} (b_0 a_{s+\rho-1} + \ldots + b_{\rho-2} a_{s+1}) \frac{f^s(0)}{\phi^s(0)} + \ldots$$
$$+ \frac{x^{\rho-1}\phi^{\rho-1}(0)}{(\rho-1)!} \sum_{s=0}^{s=\infty} (b_0 a_{s+1}) \frac{f^s(0)}{\phi^s(0)} - \sum_{r=1}^{r=\infty} \frac{F(\kappa_r)}{\psi'(\kappa_r)} \phi(\kappa_r, x) \tag{22}$$

where $F(z)$ is given by (15).

4. *Application to a New Trigonometrical Expansion.*

As an example of the application of this method to trigonometrical expansions in general, let it be proposed to expand $f(x)$ in the form

$$f(x) = \Sigma (a_{n_s} \cos n_s x + b_{n_s} \sin n_s x)$$

where n_s is a root of the transcendental equation $J_0(z) = 0$.

We take $\quad \phi(xz) = e^{xz}, \qquad \psi(z) = I_0(z) = J_0(iz).$

We proceed to calculate $F(z)$. We have here

$$\phi(0) = \phi'(0) = \ldots = \phi^n(0) = \ldots = 1.$$

Then (15) gives

$$F(z) = \left[\left(\frac{\psi_0(z)}{z} + \frac{D\psi_1(z)}{z^2} + \ldots + \frac{D^n \psi_n(z)}{z^{n+1}} \right) f(u) \right]_{u=0} \tag{23}$$

where $D \equiv d/du$. Or, writing out $\psi_n(z)$,

$$F(z) = \left[\left\{ a_0 \frac{1}{z} + a_0 \frac{D}{z^2} + a_1 \frac{D}{z} + a_0 \frac{D^2}{z^3} + a_1 \frac{D^2}{z^2} + a_2 \frac{D^2}{z} + \ldots \right\} f(u) \right]_{u=0}. \tag{24}$$

The series (24) contains only a finite number of terms. It is, therefore, permissible to change the order of the terms. Collecting terms in z^{-1}, z^{-2}, ..., we find

$$F(z) = \left[\left\{\frac{1}{z}(a_0+a_1D+a_2D^2+\ldots)+\frac{D}{z^2}(a_0+a_1D+a_2D^2+\ldots)\right.\right.$$
$$\left.\left.+\frac{D^2}{z^3}(a_0+a_1D+a_2D^2+\ldots)\right\} f(u)\right]_{u=0}$$
$$= \left[\frac{\psi(D)}{z-D} f(u)\right]_{u=0}. \tag{25}$$

We may note here that (25) is the symbolic value of $F(z)$ for *all* trigonometric series, since it has been obtained without reference to the form of $\psi(z)$.

In the present case $\psi(z)$ has no zero $z=0$. But in those cases [*e.g.*, that of a Fourier series, where $\psi(z) = \sinh(bz)$] where $\psi(z)$ has a single zero $z=0$ the additional terms in (22) take a simple form. For they then reduce to the first term, namely,

$$b_0 \sum_{s=0}^{s=\infty} a_{s+1} f^s(0) = \left[\frac{b_0 \psi(D)}{D} f(u)\right]_{u=0}. \tag{25A}$$

We proceed to evaluate $$\frac{\psi(D)}{z-D} f(u).$$

By a known transformation in the theory of differential operators, this is

$$\psi(D)\, e^{zu}\left(-\frac{1}{D}\right) e^{-zu} f(u) = e^{zu}\psi(D+z)\int_u^a e^{-zu} f(u)\,du,$$

a being some upper limit.

It is not necessary to evaluate a. In fact, a may be given any convenient value.

For, if we change a to β, the difference between the two values of

$$\frac{\psi(D)}{z-D} f(u)$$

is
$$e^{zu}\psi(D+z)\int_a^\beta e^{-zu} f(u)\,du,$$

that is, since the limits of the integral are now constants with regard to u,

$$e^{zu}\psi(z)\int_a^\beta e^{-zu} f(u)\,du.$$

But we are going to compute the value of $F(z)$ only for such values of z as are roots of $\psi(z)=0$. The above difference, which contains $\psi(z)$ as a factor, is therefore irrelevant.

We have then

$$F(z) = \left[\psi(D+z)\int_u^a e^{-zu} f(u)\,du\right]_{u=0} + \psi(z)\,\{\text{some factor}\}. \tag{26}$$

So far our results are independent of the form of $\psi(z)$; that is, they hold for all trigonometric series.

Take now the value for $\psi(z)$ assumed at the beginning of the present section. We have (see Gray and Mathews' *Bessel Functions*, p. 89)

$$J_0(q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iq\sin\theta}\,d\theta.$$

Thus

$$\psi(z) = J_0(iz) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\,d\theta,$$

$$\psi(D+z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\,e^{D\sin\theta}\,d\theta.$$

Therefore, using the symbolic form of Taylor's theorem,

$$\left[\psi(D+z)\int_u^a e^{-uz} f(u)\,du\right]_{u=0} = \left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\left\{\int_a^{u+\sin\theta} e^{-uz} f(u)\,du\right\}d\theta\right]_{u=0}$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\sin\theta}\left\{\int_a^{\sin\theta} e^{-uz} f(u)\,du\right\}d\theta.$$

Since a is arbitrary, we may take it equal to 0. Hence

$$F(z) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\sin\theta} e^{z(\sin\theta-u)} f(u)\,d\theta\,du.$$

The terms in the expansion corresponding to the roots $z = \pm in_s$ are

$$+\frac{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^{\sin\theta} e^{in_s(\sin\theta-u)} f(u)\,d\theta\,du}{iJ'(-n_s)}\,e^{in_s}$$

$$+\frac{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\int_0^{\sin\theta} e^{-in\,(\sin\theta-u)}\;(u)\,d\theta\,du}{iJ'(n_s)}\,e^{-in_s x}$$

$$= \frac{1}{\pi J'(n_s)}\int_{-\pi}^{\pi}\int_0^{\sin\theta} \sin n_s(u-x-\sin\theta)\,f(u)\,d\theta\,du,$$

whence we get

$$f(x) = \sum_{s=1}^{s=\infty}\left[\quad\frac{1}{\pi J'(n\,)}\cos n_s x\int_{-\pi}^{\pi}\int_0^{\sin\theta}\sin\{n_s(u-\sin\theta)\}\,f(u)\,d\theta\,du\right.$$

$$\left.-\frac{1}{\pi J_0'(n_s)}\sin n_s x\int_{-\pi}^{\pi}\int_0^{\sin\theta}\cos\{n_s(u-\sin\theta)\}\,f(u)\,d\theta\,du\right]. \tag{27}$$

5. *Validity and Limits of the last Expansion.*

To obtain the conditions under which the expansion (27) is valid we have to consider the integral

$$\int_C \frac{F(z)\, e^{zx}}{I_0(z)}\, dz,$$

taken round a very large contour C in the z-plane.

The form (25) for $F(z)$ shows that when z becomes infinite in any manner

$$\underset{z=\infty}{\mathrm{L}}\; zF(z) = [\psi(D)\, f(u)]_{u=0} = [J_0(iD) f(u)]_{u=0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\sin\theta)\, d\theta,$$

which is finite.

With regard to the values of $\dfrac{\phi(xz)}{\psi(z)}$, that is of $\dfrac{e^{xz}}{I_0(z)}$, we have, when z is large,

$$J_0(z) = \sqrt{\left(\frac{2}{\pi z}\right)} \cos\left(z - \frac{\pi}{4}\right) \qquad \text{(real part of } z > 0),$$

$$J_0(z) = \sqrt{\left(\frac{2}{-\pi z}\right)} \cos\left(z + \frac{\pi}{4}\right) \qquad \text{(real part of } z < 0),$$

$$J_0(iy) = J_0(-iy) = (2\pi y)^{-\frac{1}{2}} e^y,$$

where the square roots are so taken that their real part is positive (see Hankel, *Math. Annalen*, Vol. I., pp. 500, 501).

From these we deduce

$$I_0(z) = (2\pi z)^{-\frac{1}{2}} (e^z + e^{-z+\frac{1}{2}i\pi}) \qquad \text{(imaginary part of } z > 0),$$

$$I_0(z) = (2\pi z)^{-\frac{1}{2}} (e^z + e^{-z-\frac{1}{2}i\pi}) \qquad \text{(imaginary part of } z < 0),$$

$$I_0(z) = (2\pi z)^{-\frac{1}{2}} e^z \qquad (z \text{ real and positive}),$$

$$I_0(z) = (-2\pi z)^{-\frac{1}{2}} e^{-z} \qquad (z \text{ real and negative}),$$

the values of the roots above being determined by taking $\sqrt{z}$ to be the positive real root of z when z is real and positive.

It is clear that, if $x = \pm 1$, $e^{xz}/I_0(z)$ is in general comparable with $(2\pi z)^{\frac{1}{2}}$, and

$$\int_C \frac{F(z)\, e^{xz}}{I_0(z)}\, dz$$

need not be finite. Accordingly the expansion is not valid for the end values $x = \pm 1$. Still less is it valid if $|x| > 1$.

Consider $|x| < 1$. Let the path of integration be a circle of radius R passing through $z = i(n\pi + \frac{1}{4}\pi)$;

$$\frac{e^{zx}}{I_0(z)} = \frac{(2\pi z)^{\frac{1}{2}}}{e^{z(1-x)} + e^{-z(1+x) \pm \frac{1}{2}\pi i}},$$

when $|z|$ is large.

Let $z = p + iq$ be any point on the path of integration.

Consider first those parts of the contour which lie inside an angle 2ϵ enclosing the imaginary axis: let us restrict for the present our attention to that arc which is bisected by the positive half of the above axis.

Here $$I_0(z) = (2\pi z)^{-\frac{1}{2}}(e^z + e^{-z+\frac{1}{2}i\pi}),$$

$$|I_0(z)| = (2\pi R)^{-\frac{1}{2}}|e^z|\,|1+e^{-2z+\frac{1}{2}i\pi}| = (2\pi R)^{-\frac{1}{2}} e^p\,|1+e^{-2p}e^{i(-2q+\frac{1}{2}\pi)}|$$
$$= (2\pi R)^{-\frac{1}{2}} e^{-p}\,|1+e^{2p}e^{i(2q-\frac{1}{2}\pi)}|.$$

Thus $$|I_0(z)| > (2\pi R)^{-\frac{1}{2}} e^{|p|}[1+e^{-2|p|}\cos(2q-\tfrac{1}{2}\pi)],$$

since $$|a+ib| > |a|,$$

and $$\left|\frac{e^{zx}}{I_0(z)}\right| < \frac{(2\pi R)^{\frac{1}{2}} e^{-|p|(1-|x|)}}{1+e^{-2|p|}\cos(2q-\frac{1}{2}\pi)}.$$

Let p_0 be given by the equation

$$e^{p_0(1-|x|)} = \lambda(2\pi R)^{\frac{1}{2}},$$

λ being any constant.

Then, if $p > p_0$,

$$\left|\frac{e^{zx}}{I_0(z)}\right| < \frac{1}{\lambda[1+e^{-2|p|}\cos(2q-\frac{1}{2}\pi)]} < \frac{1}{\lambda(1-e^{-2p_0})}.$$

$<$ a fixed finite quantity Q if R exceeds a fixed value, since p_0 becomes infinite with R.

Let the arc bounded by $\pm p_0 + iq_0$ subtend an angle $2\epsilon'$ at the origin.

Then $$\sin\epsilon' = \frac{p_0}{R} = \frac{1}{2}\frac{\log R}{R[1-|x|]} + \frac{\log\{\lambda\sqrt{(2\pi)}\}}{R[1-|x|]}.$$

Thus ϵ' tends to zero as R increases, and the arc $2\epsilon'$ ultimately lies inside the arc 2ϵ.

The integral over the arc $2\epsilon - 2\epsilon'$

$$< Q(2\epsilon - 2\epsilon') \underset{z=\infty}{\mathrm{L}} zF(z)$$

in the limit.

Over the arc $2\epsilon'$ $\qquad R > q > R - p_0^2/R.$

Now p_0^2/R is of order $\dfrac{1}{(1-|x|)^2}\left(\dfrac{\log\sqrt{R}}{\sqrt{R}}\right)^2$ and tends to zero as R increases. Hence, if R is large enough,

$$n\pi+\tfrac{1}{4}\pi > q > n\pi+\tfrac{1}{4}\pi-\tfrac{1}{2}\theta,$$

θ being any assigned positive quantity which may be taken as small as we please, and

$$\cos(2q-\tfrac{1}{2}\pi) > \cos\theta.$$

Hence over the arc $2\epsilon'$

$$\left|\frac{e^{zx}}{I_0(z)}\right| \leqslant \frac{(2\pi R)^{\frac{1}{2}}\, e^{-|p|(1-|x|)}}{1+e^{-2|p|}\cos\theta} < \frac{(2\pi R)^{\frac{1}{2}}}{1+e^{-2|p|}\cos\theta} < (2\pi R)^{\frac{1}{2}}.$$

The integral over the arc $2\epsilon'$ is therefore less than

$$2\epsilon'\,(2\pi R)^{\frac{1}{2}} \mathop{\mathrm{L}}_{z=\infty} zF(z)$$

in the limit.

But
$$\mathop{\mathrm{L}}_{R=\infty} 2\epsilon'\,(2\pi R)^{\frac{1}{2}} = \mathop{\mathrm{L}}_{R=\infty} \text{const.}\ \frac{\log R}{\sqrt{R}} = 0.$$

Thus the whole integral over the arc 2ϵ tends to a quantity less than

$$2Q\epsilon \mathop{\mathrm{L}}_{z=\infty} zF(z),$$

when R becomes indefinitely great.

Thus, making ϵ small, we see that the arc bisected by the positive half of the imaginary axis ultimately contributes nothing to the contour integral. By symmetry the arc bisected by the negative half of the imaginary axis also contributes nothing.

Now, over the parts of the contour lying outside the angle 2ϵ, it is obvious that $\mathrm{L}\left(\frac{e^{zx}}{I_0(z)}\right) = 0$. Hence these parts also contribute nothing. Therefore, if $|x| < 1$,

$$\int_C \frac{F(z)\,\phi(zx)}{\psi(z)}\,dx = 0.$$

When the radius of C becomes infinitely great the expansion then holds.

6. *The Expansions of Fourier.*

The same method may be applied to deduce from the general theorem of Art. 2 the well known series of Fourier

$$f(x) = a_0+a_1\cos\frac{\pi x}{b}+a_2\cos\frac{2\pi x}{b}+\ldots+b_1\sin\frac{\pi x}{b}+b_2\sin\frac{2\pi x}{b}+\ldots, \quad (28)$$

and another series, also given by Fourier,

$$f(x) = a_1\cos n_1x+a_2\cos n_2x+\ldots+b_1\sin n_1x+b_2\sin n_2x+\ldots, \quad (29)$$

where $n_1, n_2, \ldots$ are the roots of the equation

$$\cos nr - \frac{\lambda \sin nr}{nr} = 0, \tag{30}$$

λ being < 1.

In the case of Fourier's series, we find, in a manner similar to that employed on p. 403, using the symbolic form of Taylor's theorem,

$$F(z) = \tfrac{1}{2}\int_b^a e^{-zu} e^{zb} f(u)\,du - \tfrac{1}{2}\int_{-b}^a e^{-zu} e^{-zb} f(u)\,du + \psi(z)\{\text{some factor}\},$$

where a, as before, is arbitrary and $\psi(z) = \sinh zb$. Taking $a = -b$ and putting $z =$ any root κ of $\psi(z) = 0$, we have

$$F(\kappa) = -\tfrac{1}{2}\int_{-b}^{+b} e^{\kappa(b-u)} f(u)\,du. \tag{31}$$

The constant term is obtained from the additional term due to the single root $z = 0$ of $\psi(z)$. It is

$$\left[\frac{\psi(D)}{D} f(u)\right]_{u=0} = \frac{1}{2b}\int_{-b}^{b} f(u)\,du. \tag{32}$$

It is easily verified that (31) and (32) lead to the well known expansion. An investigation similar to that of Art. 5 will then show that the expansion is valid if $-b < x < b$, $\underset{z=\infty}{\mathrm{L}}\, zF(z)$ being here equal to

$$\tfrac{1}{2}[f(b) - f(-b)].$$

In the case where $x = \pm b$ it is easily shown (see also Picard, *Cours d'Analyse*, pp. 167–177) that

$$\int_C \frac{F(z)e^{zb}}{\sinh zb}\,dz = \pi i[f(b) - f(-b)]$$

and

$$\int_C \frac{F(z)e^{-zb}}{\sinh zb}\,dz = -\pi i[f(b) - f(-b)],$$

whence we get the well known result that the value of the series at the ends of the range is

$$\tfrac{1}{2}[f(b) + f(-b)].$$

With regard to the series (29), Fourier showed (*Théorie de la Chaleur*, p. 348 *et seq.*) how to expand an *odd* function in terms of sines. The coefficients a were therefore absent in his expansion. Picard in his *Cours d'Analyse* (pp. 179–183, first edition) has generalized Fourier's result, so as to include the even terms. But he has proceeded in what appears to be a rather arbitrary manner, with the result that he has introduced into his expansion a constant term which is unnecessary.

If we treat this example by the method of the present paper, putting

$$\psi(z) = \cosh zr - \frac{\lambda \sinh zr}{zr},$$

we find, by a process very similar to that used before,

$$F(z) = \frac{e^{rz}}{2}\left(1-\frac{\lambda}{rz}\right)\int_r^a e^{-zu} f(u)\,du + \frac{e^{-rz}}{2}\left(1+\frac{\lambda}{rz}\right)\int_{-r}^a e^{-zu} f(u)\,du$$
$$-\frac{\lambda}{2rz}\int_{-r}^{+r} f(u)\,du + \psi(z)\,G,$$

where G is some factor which we do not require to determine.

Whence, after some reductions,

$$f(x) = \sum_{s=1}^{s=\infty}\left\{\cos n_s x \frac{\int_{-r}^{r}\{\cos n_s u - \cos n_s r\} f(u)\,du}{r-(\sin 2n_s r)/2n_s} + \sin n_s x \frac{\int_{-r}^{r} \sin n_s u\, f(u)\,du}{r-(\sin 2n_s r)/2n_s}\right\}, \tag{33}$$

which gives the expansion required.

The sine terms in this expansion agree with those given by Fourier (*loc. cit.*) for the expansion of an odd function.

Picard's result differs from (36) in that the coefficient of $\cos n_s x$ is

$$\frac{1}{\{r-(\sin 2n_s r)/2n_s\}}\int_{-r}^{r} \cos n_s u\, f(u)\,du,$$

instead of the coefficient given in (33), and there is an absolute term introduced.

That such an absolute term is not really required is obvious from the present work, since $z = 0$ is not a zero of $\psi(z)$. In fact (33) allows us to expand a constant in a series $\Sigma A_s \cos n_s x$, and when we replace the absolute term in Picard's result by its expansion in a series of cosines, the new expansion is found to agree with (33).

That (33) is the natural expansion may also be seen from the fact that

$$\int_{-r}^{r} (\cos n_s u - \cos n_s r) \cos n_t u\, du = 0,$$

if s and t are different—a result which is easily verified directly and which would allow us to obtain the expansion by a method analogous to that of normal functions.

Also, if we investigate as before the validity of this expansion by considering $\int_C F(z)\frac{\phi(zx)}{\psi(z)}\,dz$, we find that

$$\underset{|z|=\infty}{\mathrm{L}}\; zF(z) = \tfrac{1}{2}[f(r)+f(r)] - \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du.$$

Then it may be shown, as in Art. 5, that, if $|x|>r$, $|\phi(xz)/\psi(z)| = \infty$ over one half of the contour of integration, so that the expansion is then not legitimate, but that, if $|x|<r$, $|\phi(xz)/\psi(z)| = 0$ when $|z| = \infty$, except within an angle 2ϵ enclosing the imaginary axis; and the parts of the path of integration within this angle can easily be shown to contribute nothing ultimately to the integral, so that, if $|x|<r$, the expansion is valid.

When $x = r$, $\underset{z=\infty}{\mathrm{L}}\;\frac{\phi(xz)}{\psi(z)} = 2$ when the real part of z is positive, and $\underset{z=\infty}{\mathrm{L}}\;\frac{\phi(xz)}{\psi(z)} = 0$ when the real part of z is negative.

Hence, if $x = r$,

$$f(r) = \text{series} + \tfrac{1}{2}[f(r)+f(-r)] - \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du,$$

or

$$\text{series} = \tfrac{1}{2}[f(r)-f(-r)] + \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du. \tag{34}$$

Similarly, if $x = -r$,

$$\text{series} = \tfrac{1}{2}[f(-r)-f(r)] + \frac{\lambda}{2r}\int_{-r}^{r} f(u)\,du. \tag{35}$$

These two end values for the series are not given by Picard, and I have been unable to find them anywhere else.

It follows from (34) and (35) that when $f(x)$ is an odd function the expansion in sines holds right up to the limit $x = \pm r$. But, if $f(x)$ be an even function, there is a discontinuity at the ends of the range. This is precisely the reverse of what happens with the ordinary Fourier's series.

7. *Schlömilch's Expansion.*

Here we require to expand $f(x)$ in a series of Bessel functions of zero order in the form

$$f(x) = A_0 + A_1 J_0(x) + A_2 J_0(2x) + \dots.$$

Clearly, if this expansion is to hold for negative as well as for positive values of x, $f(x)$ must be an even function.

It will be more convenient to consider a more general expansion, where $f(x)$ is not restricted to be even, namely,

$$f(x) = A_0+A_1 J_0(x)+A_2 J_0(2x)+\ldots+B_1 L_0(x)+B_2 L_0(2x)+\ldots,$$

where $$L_0(x) = \frac{2}{\pi}\left\{\frac{x}{1^2}-\frac{x^3}{1^2.3^2}+\frac{x^5}{1^2.3^2.5^2}-\ldots\right\}. \qquad (36)$$

This function occurs in the theory of the vibrations of a circular plate, and its properties have been discussed by Lord Rayleigh (*Theory of Sound*, Vol. II., § 302). Rayleigh denotes this function by $K(x)$.

We write

$$Q(z) = 1+\frac{2}{\pi}\frac{z}{1^2}+\frac{z^2}{2^2}+\frac{2}{\pi}\frac{z^3}{1^2.3^2}+\frac{z^4}{2^2.4^2}+\frac{2}{\pi}\frac{z^5}{1^2.3^2.5^2}+\ldots, \qquad (37)$$

and take $\qquad \phi(zx) = Q(zx), \qquad \psi(z) = \sinh \pi z.$

We then find easily

$$\frac{1}{\phi^s(0)} = \int_0^1 \frac{st^{s-1}}{(1-t^2)^{\frac{1}{2}}}\,dt = \int_0^1 (1-t^2)^{-\frac{1}{2}}\frac{d}{dt}(t^s)\,dt \qquad (38)$$

whether s be odd or even, with the exception of $1/\phi(0) = 1$.

Substituting into the expression (15) for $F(z)$, we find

$$F(z) = \psi_0(z)\,\frac{f(0)}{z}$$

$$+\left[\int_0^1 (1-t^2)^{-\frac{1}{2}}\frac{d}{dt}\left\{\frac{\psi_0(z)}{z}+\frac{tD\psi_1(z)}{z^2}+\ldots+\frac{t^rD^r\psi_r(z)}{z^{r+1}}+\ldots\right\}f(u)\,dt\right]_{u=0},$$

and, treating the series in curled brackets as was done in Art. 4, we find

$$F(z) = \psi_0(z)\,\frac{f(0)}{z}+\left[\int_0^1 (1-t^2)^{-\frac{1}{2}}\frac{d}{dt}\left\{\frac{\psi(tD)}{z-tD}\right\}f(u)\,dt\right]_{u=0}.$$

Hence, remembering that $[(tD)^r f(u)]_{u=0} = [D^r f(ut)]_{u=0}$,

$$F(z) = \psi_0(z)\,\frac{f(0)}{z}+\left[\frac{\psi(D)}{z-D}\int_0^1 (1-t^2)^{-\frac{1}{2}}uf'(ut)\,dt\right]_{u=0}. \qquad (39)$$

(39) is the general expression for $F(z)$ whatever the form of $\psi(z)$. It is therefore applicable to all expansions of the type

$$\Sigma\{A_s J_0(n_s x)+B_s L_0(n_s x)\}.$$

Taking now $\psi(z) = \sinh \pi z$, we find that

$$\frac{\sinh \pi D}{z-D}\,\chi(u) = -\tfrac{1}{2}\int_{-\pi}^{\pi} e^{z(\pi-u)}\chi(u)\,du+\psi(z)\,G.$$

Applying this result to (39),

$$F(z) = -\tfrac{1}{2}\int_{-\pi}^{\pi} du \int_0^1 dt\,(1-t^2)^{-\frac{1}{2}}\, e^{z(\pi-u)}\, u f'(ut) + \psi(z)\, G, \qquad (40)$$

since $\psi_0(z) = 0$ in the present case.

There will be an absolute term, since $\psi(z)$ has a simple zero at the origin.

This absolute term is given by the first term in (22), namely,

$$\phi(0)\sum_{s=0}^{s=\infty} b_0 a_{s+1}\frac{f^s(0)}{\phi^s(0)} = \phi(0)\, b_0\left[a_1 f(0) + \left(\frac{\psi(D)}{D}\int_0^1 \frac{u f'(ut)}{(1-t^2)^{\frac{1}{2}}}\,dt\right)_{u=0}\right], \qquad (41)$$

which, when we put $\phi(0) = 1$, $b_0 = 1/\pi$, $a_1 = \pi$, $\psi(D) = \sinh \pi D$, becomes

$$f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{u f'(ut)}{(1-t^2)^{\frac{1}{2}}}\,dt. \qquad (42)$$

Thus, from (22), (40), (42),

$$f(x) = f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{u f'(ut)}{(1-t^2)^{\frac{1}{2}}}\,dt + \sum_{\kappa} \frac{Q(\kappa x)}{2\pi\cosh \pi\kappa}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt\, e^{\kappa(\pi-u)}\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}},$$

$z = \kappa$ being any zero of $\sinh \pi z$ other than $z = 0$. Whence, grouping terms in pairs,

$$f(x) = f(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}} + J_0(nx)\,\frac{1}{\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt \cos nu\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}} + L_0(nx)\,\frac{1}{\pi}\int_{-\pi}^{\pi} du \int_0^1 \frac{dt \sin nu\, u f'(ut)}{(1-t^2)^{\frac{1}{2}}}. \qquad (43)$$

The even terms give Schlömilch's well known expansion. The odd terms complete this expansion, the function $L_0(x)$ having here to $J_0(x)$ the same relation that the sine has to the cosine.

In order to investigate the validity of this expansion, it is necessary to know the order of magnitude of $Q(z)$ when $|z|$ is large.

We have (see Rayleigh, *loc. cit.*)

$$Q(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{z \sin x}\,dx = \frac{2}{\pi}\int_0^1 \frac{e^{zt}}{(1-t^2)^{\frac{1}{2}}}\,dt.$$

Writing $(1-t) = u$,

$$Q(z) = \frac{2}{\pi}\, e^z \int_0^1 \frac{e^{-zu}\,du}{\sqrt{\{u(2-u)\}}}$$

$$= \frac{2}{\pi}\, e^z \int_0^1 \frac{e^{-zu}}{\sqrt{(2u)}}\left[1 + \frac{u}{4} + \left(\frac{u}{2}\right)^2 \frac{1.3}{2.4} + \ldots\right] du.$$

It may then be shown that, when z is large and its *real part is positive*, the most important term in $Q(z)$ is the first. This first term becomes, on writing $zu = v$,

$$\frac{2}{\pi}\frac{e^z}{\sqrt{(2z)}}\int_0^z \frac{e^{-v}\,dv}{\sqrt{v}},$$

the integrations with regard to v being taken along the straight line joining the origin to the point z.

By considering a contour $ABCD$ in the v-plane, AD being the arc of a very small circle, centre the origin, CB a concentric arc through the point $v = z$, CD a portion of the line joining $v = 0$ to $v = z$, and AB a portion of the real axis in the v-plane, we can readily show that the first term in $Q(z)$ is approximately equal to

$$\frac{2}{\pi}\frac{e^z}{\sqrt{(2z)}}\int_0^{|z|} \frac{e^{-v}\,dv}{\sqrt{v}}$$

when $|z|$ is large, the path of integration being now real.

Therefore the most important term in $Q(z)$ is

$$\frac{2}{\pi}\frac{e^z}{\sqrt{(2z)}}\int_0^{\infty} \frac{e^{-v}\,dv}{\sqrt{v}} = e^z\sqrt{\left(\frac{2}{\pi z}\right)}.$$

Take now the real part of z negative. Write $z = -\zeta$; then the real part of ζ is positive.

$$Q(z) = \frac{2}{\pi}\int_0^1 (1-t^2)^{-\frac{1}{2}} e^{-\zeta t}\,dt.$$

Now expand $(1-t^2)^{-\frac{1}{2}}$ by the binomial theorem

$$Q(z) = \frac{2}{\pi}\int_0^1 e^{-\zeta t}\left(1+\frac{1}{2}t^2+\frac{1.3}{2.4}t^4+\ldots\right)dt$$

$$= \frac{2}{\pi}\int_0^{\zeta} e^{-v}\left(\frac{1}{\zeta}+\frac{1}{2}\frac{v^2}{\zeta^3}+\frac{1.3}{2.4}\frac{v^4}{\zeta^5}+\ldots\right)dv.$$

By reasoning similar to that employed above, the most important term in $Q(x)$ is

$$\frac{2}{\pi\zeta}\int_0^{\infty} e^{-v}\,dv = \frac{2}{\pi\zeta} = -\frac{2}{\pi z}.$$

The case where z is a pure imaginary has been worked out by Lord Rayleigh. Taking the value given for $L_0(z)$ [his $K(z)$] in ascending powers of $1/z$ in his *Theory of Sound*, § 302, we have

$$Q(i\zeta) = \frac{2i}{\pi\zeta}+e^{i(\zeta-\frac{1}{4}\pi)}\sqrt{\left(\frac{2}{\pi z}\right)} = -\frac{2}{\pi z}+e^z\sqrt{\left(\frac{2}{\pi z}\right)},$$

the same convention being adopted with regard to $\sqrt{z}$ as on p. 405. Similarly when $z = -i\zeta$.

Thus
$$Q(z) = -\frac{2}{\pi z} + e^{z}\sqrt{\left(\frac{2}{\pi z}\right)} \tag{44}$$
will give approximately the order of $Q(z)$ for all large values of z.

It may then be readily shown

(*a*) That

$$\underset{z=\infty}{\mathrm{L}}\, zF(z) = \left[\sinh \pi D \int_0^1 (1-t^2)^{-\frac{1}{2}} u f'(ut)\, dt\right]_{u=0}$$
$$= \int_0^1 \tfrac{1}{2}(1-t^2)^{-\frac{1}{2}}\{\pi f'(\pi t) + \pi f'(-\pi t)\}\, dt,$$
which is finite.

(*b*) That, if $x > 0$,

$\dfrac{Q(zx)}{\sinh \pi z}$ tends to 0 if the real part of z is negative and approximates to $2\sqrt{2}(\pi z x)^{-\frac{1}{2}} e^{z(x-\pi)}$ if the real part of z be positive, that is, it tends to 0 or ∞ according as $x \not> \pi$ or $x > \pi$.

Similarly, if $x < 0$,

$\dfrac{Q(zx)}{\sinh \pi z}$ tends to 0 everywhere if $x \not< -\pi$, but tends to ∞ when the real part of z is negative and $x < -\pi$.

(*c*) That the parts of the contour integral in the neighbourhood of the imaginary axis are evanescent in the limit.

It follows that Schlömilch's expansion is valid if $-\pi \leqslant x \leqslant \pi$. No exception is to be made for the extremities of the range.

8. *Other Expansions in Bessel Functions of Zero Order.*

The method can also be applied to obtain an expansion of the type
$$f(x) = \Sigma\{A_s J_0(n_s x) + B_s L_0(n_s x)\}, \tag{45}$$
n_s being any root of the transcendental equation
$$J_0(na) = 0. \tag{46}$$
Such expansions occur frequently in mathematical physics in problems relating to vibrations where the boundaries are circular.

We have here
$$\phi(zx) = Q(zx), \quad \psi(z) = J_0(iaz).$$
Thus
$$\psi_0(z) = 1$$
and
$$F(z) = \frac{f(0)}{z} + \left[\frac{J_0(iaD)}{z-D}\int_0^1 (1-t^2)^{-\frac{1}{2}} u f'(ut)\, dt\right]_{u=0}.$$

Proceeding as in Art. 4, this leads to

$$F(z) = \frac{f(0)}{z} - \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} e^{z(a\sin\theta - u)}\, du \int_0^1 (1-t^2)^{-\frac{1}{2}} u f'(ut)\, dt + \psi(z)\, G$$

or, writing $t = \sin\phi$,

$$F(z) = \frac{f(0)}{z} - \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} du \int_0^{\frac{1}{2}\pi} d\phi\, u f'(u\sin\phi)\, e^{z(a\sin\theta-u)} + \psi(z)\, G,$$

whence

$$f(x) = \sum_{s=1}^{s=\infty} \frac{J_0(n_s x)}{a J_0'(n_s a)}$$

$$\times\left[-\frac{2f(0)}{n_s} - \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} du \int_0^{\frac{1}{2}\pi} d\phi\, u f'(u\sin\phi)\sin n_s(a\sin\theta - u)\right]$$

$$+\sum_{s=1}^{s=\infty} \frac{L_0(n_s x)}{a J_0'(n_s a)}\left[-\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{a\sin\theta} du \int_0^{\frac{1}{2}\pi} d\phi\, u f'(u\sin\phi)\cos n_s(a\sin\theta - u)\right]. \tag{47}$$

The form (47) is very different from the one usually employed, which gives

$$f(x) = \sum_{s=1}^{s=\infty} \frac{2J_0(n_s x)}{a^2\{J_0'(n_s a)\}^2}\int_0^a J_0(n_s x) f(x)\, x\, dx \tag{48}$$

when $f(x)$ is an even function.

The forms (47) and (48) are not easily comparable directly, but, if we go back to the form (15) for $F(z)$, it is found that, if $i\kappa = n_s$, so that κ is any zero of $\psi(z)$,

$$\psi_{2r+1}(\kappa) = \psi_{2r}(\kappa) = -2i(\tfrac{1}{2}\kappa)^{2r+1}\frac{a^{-1}}{(r!)^2}\int_0^a \frac{x^{2r+1}J_0(n_s x)}{J_0'(n_s a)}\, dx,$$

whence

$$F(\kappa) = -ia^{-1}\int_0^a \frac{xJ_0(n_s x)}{J_0'(n_s a)}\, dx\left[\Sigma\frac{x^{2r}D^{2r}}{(2r)!} f(u)\right]_{u=0}$$

$$-\frac{i\pi}{2\kappa}a^{-1}\int_0^a \frac{J_0(n_s x)}{J_0'(n_s a)}\left[\Sigma\frac{x^{2r+1}D^{2r+1}}{(2r+1)!}\left(\frac{1.3\ldots(2r-1)(2r+1)}{2.4\ldots(2r)}\right)^2 f(u)\right]_{u=0} dx$$

$$= -ia^{-1}\int_0^a \frac{xJ_0(n_s x)}{J_0'(n_s a)}\,\frac{f(x)+f(-x)}{2}\, dx$$

$$-\frac{2i}{\pi\kappa}a^{-1}\int_0^a \frac{J_0(n_s x)}{J_0'(n_s a)}\, dx\int_0^1 dt\int_0^1 dv$$

$$\times\tfrac{1}{2}(1-t^2)^{-\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}\frac{d^2}{dt\,dv}\{f(xtv) - f(-xtv)\},$$

using the identity

$$\left(\frac{1.3\ldots(2r+1)}{2.4\ldots(2r)}\right)^2 = \frac{4}{\pi^2}\int_0^1 dt\int_0^1 dv\,(1-t^2)^{-\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}\frac{d^2}{dt\,dv}(tv)^{2r+1}.$$

This leads to the form

$$f(x) = \sum_{s=1}^{s=\infty} \frac{J_0(n_s x)}{a^2\{J_0'(n_s a)\}^2}\int_0^a J_0(n_s x)\{f(x)+f(-x)\}\,x\,dx$$

$$+\sum_{s=1}^{s=\infty} \frac{L_0(n_s x)}{\pi n_s a^2\{J_0'(n_s a)\}^2}\int_0^a J_0(n_s x)\,dx\int_0^1 dt\int_0^1 dv$$

$$\times(1-t^2)^{-\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}\frac{d^2}{dt\,dv}\{f(xtv)-f(-xtv)\}. \quad (49)$$

The even terms have the coefficients found by the usual methods. The odd terms have their coefficients in a new form.

The comparison of the coefficients in (47) and (49) will be found to yield several interesting theorems connecting definite integrals involving the function J_0, which it would be difficult to establish otherwise.

Returning to the expansion in the form (47) which presents itself more naturally in this connection, we find that

$$\underset{z=\infty}{\mathrm{L}}\, zF(z) = f(0)+\left[J_0(iaD)\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\,d\phi\right]_{u=0}$$

$$= f(0)+\frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{aD\sin\theta}\,d\theta\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\,d\phi\right]_{u=0}$$

$$= f(0)+\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\int_0^{\frac{1}{2}\pi} a\sin\theta\, f'(a\sin\theta\sin\phi)\,d\phi.$$

By considering this integral as taken over the surface of a sphere of radius a, θ being the colatitude and ϕ the longitude, and y being $a\sin\theta\sin\phi$, we find

$$\int_0^{\pi} d\theta\int_0^{\frac{1}{2}\pi} a\sin\theta\, f'(a\sin\theta\sin\phi)\,d\phi = \int f'(y)\,a^{-1}\,dS,$$

taken over the area of the lune bounded by $\phi = 0$, $\phi = \frac{1}{2}\pi$,

$$= \pi\int_0^a f'(y)\,dy = \pi[f(a)-f(0)].$$

In like manner

$$\int_{-\pi}^{0} d\theta\int_0^{\frac{1}{2}\pi} a\sin\theta\, f'(a\sin\theta\sin\phi)\,d\phi = \pi[f(-a)-f(0)].$$

Thus $$\underset{z=\infty}{\mathrm{L}}\, zF(z) = \tfrac{1}{2}[f(a)+f(-a)].$$

Also, when $|z|$ is large,

$$\frac{Q(zx)}{J_0(iaz)} = \frac{e^{zx}\left(\dfrac{2}{\pi zx}\right)^{\frac{1}{2}}-\dfrac{2}{\pi zx}}{(2\pi az)^{-\frac{1}{2}}(e^{az}+e^{-az+\frac{1}{2}\pi i})}$$

if the imaginary part of z is positive; and

$$\frac{Q(xz)}{J_0(iaz)} = \frac{e^{zx}\left(\dfrac{2}{\pi zx}\right)^{\frac{1}{2}} - \dfrac{2}{\pi zx}}{(-2\pi az)^{-\frac{1}{2}}(e^{-az}+e^{az+\frac{1}{2}\pi i})}$$

if the imaginary part of z is negative.

Reasoning strictly analogous to that used in previous cases shows that, if $|x| < a$, $|Q(zx)/J_0(iaz)|$ tends to zero when z tends to infinity in any direction save that of the imaginary axis.

If $x = a$, $Q(zx)/J_0(iaz)$ tends to 2 if the real part of z be positive, and to 0 if the real part of z be negative.

If $x = -a$, $Q(xz)/J_0(iaz)$ tends to 0 if the real part of z be positive, and to 2 if the real part of z be negative.

If $|x| > a$, $|Q(zx)/J_0(iaz)|$ tends to ∞ over one half of the contour C, and to 0 over the other half.

Finally, it may be proved that the neighbourhood of the imaginary axis contributes nothing in the limit to the contour integral.

Thus, if $|x| < a$, $\quad \mathrm{L}\, \dfrac{1}{2\pi i}\displaystyle\int_C F(z)\frac{\phi(zx)}{\psi(z)}\,dz = 0,$

and the series converges to $f(x)$.

If $|x| > a$, $\mathrm{L}\, \dfrac{1}{2\pi i}\displaystyle\int_C F(z)\frac{\phi(zx)}{\psi(z)}\,dz$ need not be finite.

If $x = \pm a$, $\mathrm{L}\, \dfrac{1}{2\pi i}\displaystyle\int_C \frac{F(z)\,\phi(zx)}{\psi(z)}\,dz = \tfrac{1}{2}[f(a)+f(-a)]$.

$$\left.\begin{array}{l}\text{Therefore value of series when } x = a \text{ is } \tfrac{1}{2}[f(a)-f(-a)] \\ \text{value of series when } x = -a \text{ is } \tfrac{1}{2}[f(-a)-f(a)]\end{array}\right\}. \qquad (50)$$

This result shows that, whereas the even part of the series is, in general, discontinuous for the ends of the range of validity, being zero for $x = \pm a$ [which is, indeed, immediately obvious from the equation $J_0(n_s a) = 0$], the odd part remains continuous up to the ends of the range inclusive.

9. *Possibility of Extension of the above Results to Functions other than Polynomials.*

It is well known that every function $f(x)$ which can be represented by a Fourier's series between a and b can also be represented throughout the same range as the limit of polynomials.

Thus, let $f(x) = \underset{n=\infty}{\mathrm{L}}\, P_n(x)$, where $P_n(x)$ is a polynomial.

$P_n(x)$, by the preceding work, can be expanded in a series of suitable functions

$$P_n(x) = \sum_{r=1}^{r=\infty} A_{r,n}\phi(\kappa_r, x).$$

Thus,

$$f(x) = \mathop{\mathrm{L}}_{n=\infty} \sum_{r=1}^{r=\infty} A_{r,n}\phi(\kappa_r, x).$$

It seems difficult to prove generally that in the cases where the expansion of $P_n(x)$ is possible the limiting sign can be taken through the sign of summation.

If we *assume* that this can be done, then

$$f(x) = \sum_{r=1}^{r=\infty} A_r\phi(\kappa_r, x)$$

where

$$A_r = \mathop{\mathrm{L}}_{n=\infty} A_{r,n}.$$

In one fairly simple case, where $f(x)$ is expansible in an infinite power series, so that $P_n(x)$ = sum of the first n terms of the Taylor series for $f(x)$, we can prove that, under certain restrictions, the present method allows us to calculate the coefficients A_r—in other words, that $A_{r,n}$ tends to a limit when n increases.

We now proceed to prove this.

Consider the expression (15) for $F(z)$ and write it

$$\begin{aligned}F(z) = \psi(z)\left[\frac{f(0)}{\phi(0)}\frac{1}{z}+\frac{f'(0)}{\phi'(0)}\frac{1}{z^2}+\dots+\frac{f^n(0)}{\phi^n(0)}\frac{1}{z^{n+1}}\right]\\ -(a_1+a_2z+a_3z^2+\dots)\frac{f(0)}{\phi(0)}\\ -(a_2+a_3z+a_4z^2+\dots)\frac{f'(0)}{\phi'(0)}\\ -\quad\dots\quad\dots\quad\dots\quad\dots\\ -(a_{n+1}+a_{n+2}z+a_{n+3}z^2+\dots)\frac{f^n(0)}{\phi^n(0)}.\end{aligned} \quad (51)$$

Consider first the part of $F(z)$ in square brackets. $\phi(z)$ is an integral function; hence by a well known result

$$|\phi^n(0)| < \frac{Mn!}{\zeta^n},$$

ζ being any positive constant, however large.

Also, if the power series for $f(x)$ have a finite radius of convergence ρ, then it is clear that we *cannot* have for *all* values of n, however large,

$$|f^n(0)| < \frac{\mu n!}{\rho'^n},$$

ρ' being any quantity greater than ρ.

Therefore terms must exist in the sum in square brackets which are numerically greater than

$$\frac{\mu}{M}\,\frac{\zeta^n}{|z|^{n+1}\rho'^n},$$

and this must occur for values of n as large as we please.

Thus, if we make n infinite, terms exceeding any given magnitude will appear in the series in square brackets, which is therefore divergent. If $f(x)$ were an integral function whose coefficients decreased at a sufficiently rapid rate, this part of $F(z)$ might be convergent; but this will be a comparatively rare case.

The divergence of this part of the expression for $F(z)$ is, however, immaterial, since in calculating the coefficients we put $z=\kappa$ in $F(z)$, where $\psi(\kappa)=0$. The part in question therefore disappears.

To deal with the other part we notice that in all the examples considered $\psi(z)$ has been an integral function.

We shall suppose that $\psi(z)$ is such a function, and further that from a certain value of r, $|a_r| < q^r/r!$, q being some positive quantity, a condition that will always be satisfied by integral functions of order zero. (See Poincaré, "Mémoire sur les fonctions entières," *Bulletin de la Société Mathématique de France*, 1883.)

Then, from this value of r,

$$\begin{aligned}|a_{r+1}+a_{r+2}z+\dots| &\leqslant |a_{r+1}|+|a_{r+2}|\,|z|+\dots\\ &\leqslant \frac{q^{r+1}}{(r+1)!}\left\{1+\frac{q\,|z|}{r+2}+\frac{(q\,|z|)^2}{(r+2)(r+3)}+\dots\right\}\\ &< \frac{q^{r+1}}{(r+1)!}\left(1-\frac{q\,|z|}{r+2}\right)^{-1} < \frac{q^{r+1}}{(r+1)!}\,\frac{\lambda}{\lambda-1},\end{aligned}$$

r being taken so large that $r+2>\lambda q\,|z|$, where λ is any fixed number greater than 1. Hence, if the series

$$\Sigma\,\frac{q^{r+1}}{(r+1)!}\,\frac{f^r(0)}{\phi^r(0)} \tag{52}$$

be absolutely convergent, the second part of $F(z)$ is also absolutely convergent. We may therefore increase n without limit, and use this series to calculate the limiting values of the coefficients. So far I have not been able to complete the demonstration and to show that these are the actual coefficients in the expansion of $f(x)$ itself.

10. *Application of the Theorem to Cases where $\phi(z, x)$ is not a Function of zx only.*

Return now to the more general case, where $\phi(z, x)$ is not a function of the product zx only. In this case the expansion in functions $\phi(\kappa_r, x)$ is known by (10), when the expansion (11) of $f(x)$ in terms of the functions $f_0(x) \dots f_n(x), \dots$, which are the coefficients in the expansion of $\phi(z, x)$ in powers of z, is known.

Now, if $f(x)$ be a polynomial, the expansion of $f(x)$ in functions $f_n(x)$ is easily obtained in the following case, namely, when $f_0(x), f_1(x), \dots, f_n(x)$ are themselves polynomials of increasing degrees 0, 1, 2, ..., n.

In this case, if $f(x)$ be a polynomial of degree n, all the quantities $p_{n+1}, p_{n+2}, \dots$ in the expansion (11) may be taken zero. Let

$$\left.\begin{aligned} f_0(x) &= q_{00} \\ f_1(x) &= q_{10}+q_{11}x \\ f_2(x) &= q_{20}+q_{21}x+q_{22}x^2 \\ &\dots \quad \dots \quad \dots \\ f_n(x) &= q_{n0}+q_{n1}x+q_{n2}x^2+\dots+q_{nn}x^2 \end{aligned}\right\}. \tag{53}$$

Let
$$f(x) = a_0+a_1x+\dots+a_nx^n. \tag{54}$$

Then, to determine $p_0, p_1, \dots, p_n$, we have the $(n+1)$ equations

$$\left.\begin{aligned} p_0q_{00}+p_1q_{10}+p_2q_{20}+\dots+p_nq_{n0} &= a_0 \\ p_1q_{11}+p_2q_{21}+\dots+p_nq_{n1} &= a_1 \\ p_2q_{22}+\dots+p_nq_{n2} &= a_2 \\ \dots \quad \dots \quad \dots & \\ p_nq_{nn} &= a_n \end{aligned}\right\}, \tag{55}$$

of which the solution is

$$p_n = \frac{a_n}{q_{nn}},$$

$$p_{n-1} = \frac{a_{n-1}}{q_{n-1,\,n-1}} - \frac{a_n q_{n,\,n-1}}{q_{nn}q_{n-1,\,n-1}},$$

$$p_{n-2} = \frac{a_{n-2}}{q_{n-2,\,n-2}} - \frac{a_{n-1}q_{n-1,\,n-2}}{q_{n-1,\,n-1}\,q_{n-2,\,n-2}} + \frac{a_n \begin{vmatrix} q_{n-1,\,n-2} & q_{n,\,n-2} \\ q_{n-1,\,n-1} & q_{n,\,n-1} \end{vmatrix}}{q_{n,\,n}\,q_{n-1,\,n-1}\,q_{n-2,\,n-2}}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

... ...

$$
\begin{aligned}
p_{n-r} &= \frac{a_{n-r}}{q_{n-r,\,n-r}} \\
&- \frac{a_{n-r+1}\, q_{n-r+1,\,n-r}}{q_{n-r+1,\,n-r+1}\, q_{n-r,\,n-r}} \\
&+ \frac{a_{n-r+2} \begin{vmatrix} q_{n-r+1,\,n-r} & q_{n-r+2,\,n-r} \\ q_{n-r+1,\,n-r+1} & q_{n-r+2,\,n-r+1} \end{vmatrix}}{q_{n-r+2,\,n-r+2}\, q_{n-r+1,\,n-r+1}\, q_{n-r,\,n-r}} \\
&+ \frac{(-1)^s a_{n-r+s} \begin{vmatrix} q_{n-r+1,\,n-r} & q_{n-r+2,\,n-r} & \cdots & q_{n-r+s-1,\,n-r} & q_{n-r+s,\,n-r} \\ q_{n-r+1,\,n-r+1} & q_{n-r+2,\,n-r+1} & \cdots & q_{n-r+s-1,\,n-r+1} & q_{n-r+s,\,n-r+1} \\ 0 & q_{n-r+2,\,n-r+2} & \cdots & q_{n-r+s-1,\,n-r+2} & q_{n-r+s,\,n-r+2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & q_{n-r+s-1,\,n-r+s-1} & q_{n-r+s,\,n-r+s-1} \end{vmatrix}}{q_{n-r+s,\,n-r+s} \cdots q_{n-r,\,n-r}} \\
&+ \ldots \\
&+ \frac{(-1)^r a_n \begin{vmatrix} q_{n-r+1,\,n-r} & q_{n-r+2,\,n-r} & \cdots & q_{n-1,\,n-r} & q_{n,\,n-r} \\ q_{n-r+1,\,n-r+1} & q_{n-r+2,\,n-r+1} & \cdots & q_{n-1,\,n-r+1} & q_{n,\,n-r+1} \\ 0 & q_{n-r+2,\,n-r+2} & \cdots & q_{n-1,\,n-r+2} & q_{n,\,n-r+2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & q_{n-1,\,n-1} & q_{n,\,n-1} \end{vmatrix}}{q_{n,\,n} \cdots q_{n-r,\,n-r}}; \qquad (56)
\end{aligned}
$$

so that the coefficients p are obtained in finite form. $F(z)$ is then known by (10) and the expansion (6) follows.

11. *Application to Functions occurring in the Theory of Elasticity.*

The equations for the mean stresses P, Q, S in the plane of an elastic plate are

$$\frac{dP}{dx} + \frac{dS}{dy} = 0, \qquad \frac{dS}{dx} + \frac{dQ}{dy} = 0,$$

and these are satisfied by

$$P = -\frac{d^2E}{dy^2}, \qquad Q = -\frac{d^2E}{dx^2}, \qquad S = \frac{d^2E}{dx\,dy},$$

where $\nabla^4 E = 0$. (See a paper by the author on "An Approximate Solution for the Bending of a Beam of Rectangular Cross-Section," *Phil. Trans.*, A, Vol. 201, pp. 63–155; and also a paper by John Dougall, M.A., "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," *Trans. Roy. Soc. Edin.*, Vol. XLI., Pt. 1, No. 8, 1904. See also for an analysis of the above Professor Love's *Theory of Elasticity*, second edition, chapters v. and ix.)

If we take $E = (C \sinh \kappa x + Dx \cosh \kappa x) \cos \kappa y$,

then
$$P = \kappa^2 [C \sinh \kappa x + Dx \cosh \kappa x] \cos \kappa y,$$
$$Q = -\kappa [(C\kappa + 2D) \sinh \kappa x + D\kappa x \cosh \kappa x] \cos \kappa y,$$
$$S = -\kappa [(C\kappa + D) \cosh \kappa x + D\kappa x \sinh \kappa x] \sin \kappa y.$$

If $P = 0$, $S = 0$, when $x = \pm b$, then
$$C \sinh \kappa b + Db \cosh \kappa b = 0,$$
$$(C\kappa + D) \cosh \kappa b + D\kappa b \sinh \kappa b = 0.$$

Hence
$$\sinh 2\kappa b - 2\kappa b = 0. \tag{57}$$

We have
$$Q = \text{const.} [\kappa x \cosh \kappa x \sinh \kappa b - \kappa b \cosh \kappa b \sinh \kappa x + 2 \sinh \kappa b \sinh \kappa x] \cos \kappa y.$$

It is easy to verify that when κ is a root of (57)
$$\int_{-b}^{b} Qx\, dx = 0. \tag{58}$$

Now for various purposes it is desirable to be able to expand a given function of x in terms of functions $\phi(\kappa_r, x)$ where
$$\phi(z, x) = zx \cosh zx \sinh zb - zb \cosh zb \sinh zx + 2 \sinh zb \sinh zx, \tag{59}$$
where $z = \kappa_r$ is any root of
$$\psi(z) = \sinh 2zb - 2zb = 0. \tag{60}$$

For example, if Q be given over $y = 0$, between $x = -b$ and $x = +b$, such an expansion will give us the coefficients of the typical terms which build up the complete solution.

We notice, however, that, if $f(x)$ be the function to be expanded, then, owing to (58),
$$\int_{-b}^{b} f(x)\, x\, dx = 0\,; \tag{61}$$
and therefore $f(x)$ is not entirely arbitrary.

The expansion of $\phi(z, x)$ in powers of z is as follows :—
$$\phi(z, x) = \sum_{r=1}^{r=\infty} \frac{z^{2r}}{(2r)!} [(b+x)^{2r} - (b-x)^{2r} - r(b^2 - x^2)\{(b+x)^{2r-2} - (b-x)^{2r-2}\}]. \tag{62}$$

Thus
$$f_{2r+1}(x) = 0,$$
$$f_0(x) = 0,$$
$$f_{2r}(x) = \frac{1}{(2r)!} [(b+x)^{2r} - (b-x)^{2r} - r(b^2 - x^2)\{(b+x)^{2r-2} - (b-x)^{2r-2}\}]$$
$$= 2 \sum_{s=1}^{s=r} \frac{x^{2s-1} b^{2r-2s+1} (2s-r)}{(2s-1)!\,(2r-2s+1)!}. \tag{63}$$

It follows that only an *odd* function will be suitable for $f(x)$.

If $$f(x) = a_1 x + a_3 x^3 + \dots + a_{2r-1} x^{2r-1},$$

then, since from (76) it is seen that the highest power of x in $f_{2r}(x)$ is x^{2r-1}, we have
$$f(x) = p_2 f_2(x) + p_4 f_4(x) + \dots + p_{2r} f_{2r}(x), \tag{64}$$

where $p_2, p_4, \dots, p_{2r}$ can be found in the manner indicated in the preceding section.

The equations for $p_2, p_4, \dots, p_{2r}$ may be here quoted: they are

$$\left.\begin{aligned}
\frac{(2-1)\,p_2 b^2}{1!} + \frac{(2-2)\,p_4 b^4}{3!} + \frac{(2-3)\,p_6 b^6}{5!} + \dots + \frac{(2-r)\,p_{2r} b^{2r}}{(2r-1)!} &= \frac{1!\,a_1 b}{2} \\
\frac{(4-2)\,p_4 b^4}{1!} + \frac{(4-3)\,p_6 b^6}{3!} + \dots + \frac{(4-r)\,p_{2r} b^{2r}}{(2r-3)!} &= \frac{3!\,a_3 b^3}{2} \\
\frac{(6-3)\,p_6 b^6}{1!} + \dots + \frac{(6-r)\,p_{2r} b^{2r}}{(2r-5)!} &= \frac{5!\,a_5 b^5}{2} \\
\dots \qquad \dots \qquad \dots \qquad & \dots \\
\frac{(2r-r)\,p_{2r} b^{2r}}{1!} &= \frac{(2r-1)!\,a_{2r-1} b^{2r-1}}{2}
\end{aligned}\right\}. \tag{65}$$

12. *Zeroes of* sinh $2zb - 2zb = 0$.

We have now to consider the distribution of the zeroes of $\psi(z)$.

In the first place the expansion of $\sinh 2zb - 2zb$ begins with a term in z^3, so that the origin is a triple zero.

To find the other zeroes write
$$2zb = \xi + i\eta.$$

Then $$\sinh(\xi + i\eta) = \xi + i\eta,$$

and, equating real and imaginary parts,
$$\sinh \xi \cos \eta = \xi, \tag{66}$$
$$\sin \eta \cosh \xi = \eta. \tag{67}$$

If we put $\eta = 0$, we have $\sinh \xi = \xi$,

which is impossible, unless $\xi = 0$; and, if we put $\xi = 0$, we have
$$\sin \eta = \eta,$$

which is also impossible, unless $\eta = 0$.

Thus no root lies on the axes, except the triple root $z = 0$.

Consider now the position of the roots of very large modulus.

Squaring (66) and (67) and adding, we find

$$\cosh^2 \xi = \cos^2 \eta + \xi^2 + \eta^2.$$

If, therefore, $|\xi + i\eta| = \sqrt{(\xi^2+\eta^2)}$ is large, ξ must be large.

Hence, by (66),
$$\cos \eta = \frac{\xi}{\sinh \xi},$$
and is small; therefore
$$\eta = n\pi + \tfrac{1}{2}\pi$$
approximately. Putting this value into (67), then, since $\cosh \xi$ must be positive, n must be *even*.

Thus
$$\eta = 2r\pi + \tfrac{1}{2}\pi, \qquad \cosh \xi = 2r\pi + \tfrac{1}{2}\pi$$
very nearly, or
$$\xi = \log_e (4r\pi + \pi)$$
approximately. The roots fall into groups of four, symmetrically placed with regard to the axes, the four members of each group being given by $\pm \log_e (4r\pi + \pi) \pm i\,(2r\pi + \tfrac{1}{2}\pi)$ approximately.

13. *Additional Terms due to the Triple Zero at $z = 0$.*

Referring to the expression (21), we see that the additional terms due to the triple root $z = 0$ are

$$f_0(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+3} + b_1 a_{s+2} + b_2 a_{s+1})\, p_s + f_1(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+2} + b_1 a_{s+1})\, p_s$$
$$+ f_2(x) \sum_{s=0}^{s=\infty} (b_0 a_{s+1})\, p_s,$$

and, since $f_0(x) = 0$, $f_1(x) = 0$, and $f_2(x) = 2xb$, this reduces to

$$2xb \sum_{s=0}^{s=\infty} (b_0 a_{s+1})\, p_s.$$

Now
$$\psi(z) = \frac{(2zb)^3}{3!} + \frac{(2zb)^5}{5!} + \dots + \frac{(2zb)^{2r+1}}{(2r+1)!} + \dots.$$

Thus
$$a_0 = a_1 = a_2 = 0, \qquad a_{2r} = 0,$$
$$a_{2r+1} = \frac{(2b)^{2r+1}}{(2r+1)!}, \qquad b_0 = \frac{1}{a_3} = \frac{3!}{(2b)^3}.$$

Thus the additional term is

$$\frac{3x}{2b^2} \sum_{s=1}^{s=r} \frac{(2b)^{2s+1}}{(2s+1)!}\, p_{2s},$$

since, if $s > 2r$, $p_s = 0$. The quantities p_{2r} are here given by (65).

But this term may be evaluated as follows:—Rewriting the first expression (63) for $f_{2r}(x)$,

$$f_{2r}(x) = \frac{1}{(2r)!}\left[(r+1)\left\{(b+x)^{2r}-(b-x)^{2r}\right\}-2br\left\{(b+x)^{2r-1}-(b-x)^{2r-1}\right\}\right],$$

whence

$$\begin{aligned} xf_{2r}(x) = \frac{1}{(2r)!}\big[(r+1)&\left\{(b+x)^{2r+1}+(b-x)^{2r+1}\right\} \\ &-b(3r+1)\left\{(b+x)^{2r}+(b-x)^{2r}\right\} \\ &+2rb^2\left\{(b+x)^{2r-1}+(b-x)^{2r-1}\right\}\big] \end{aligned}$$

and

$$\begin{aligned} \int_{-b}^{b} xf_{2r}(x)\,dx &= \frac{1}{(2r)!}\left[\frac{r+1}{2r+2}\,2(2b)^{2r+2}-\frac{b(3r+1)}{2r+1}\,2(2b)^{2r+1}+\frac{2rb^2}{2r}\,2(2b)^{2r}\right] \\ &= \frac{2^{2r+1}\,b^{2r+2}}{(2r+1)!}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{s=1}^{s=r}\frac{(2b)^{2s+1}p_{2s}}{(2s+1)!} &= \frac{1}{b}\int_{-b}^{b} x\sum_{s=1}^{s=r} p_{2s}f_{2s}(x)\,dx \\ &= \frac{1}{b}\int_{-b}^{b} xf(x)\,dx \quad \text{by (64).} \end{aligned}$$

The additional term in the expansion is therefore

$$\frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx. \tag{68}$$

Therefore

$$\begin{aligned} f(x) &= \frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx-\Sigma\,\frac{F(\kappa_r)}{\psi'(\kappa_r)}\,\phi(\kappa_r,\,x) \\ &= \frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx-\Sigma\,\frac{F(\kappa_r)\left[\kappa_r x\cosh\kappa_r x\sinh\kappa_r b-\kappa_r b\cosh\kappa_r b\sinh\kappa_r x+2\sinh\kappa_r b\sinh\kappa_r x\right]}{2b\left\{\cosh 2\kappa_r b-1\right\}} \end{aligned}$$

where κ_r is any root of (60), and

$$\begin{aligned} F(z) = \sum_{s=1}^{s=r}\frac{p_{2s}\psi_{2s}(z)}{z^{2s+1}} = \;& \frac{1}{z^2}\left\{p_4\frac{(2b)^3}{3!}+p_6\frac{(2b)^5}{5!}+\ldots+p_{2r}\frac{(2b)^{2r-1}}{(2r-1)!}\right\} \\ &+\frac{1}{z^4}\left\{p_6\frac{(2b)^3}{3!}+p_8\frac{(2b)^5}{5!}+\ldots+p_{2r}\frac{(2b)^{2r-3}}{(2r-3)!}\right\} \\ &+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\ &+\frac{1}{z^{2r-2}}\left\{p_{2r}\frac{(2b)^3}{3!}\right\}. \end{aligned} \tag{69}$$

Consider the roots in groups of four. Connecting the two in each group which have opposite signs, then, since $F(z)$ contains only even powers of z, we may write

$$f(x) = \frac{3x}{2b^3}\int_{-b}^{b} xf(x)\,dx - \Sigma \frac{F(\kappa_r)\left[\kappa_r x\cosh\kappa_r x\sinh\kappa_r b - \kappa_r b\cosh\kappa_r b\sinh\kappa_r x + 2\sinh\kappa_r b\sinh\kappa_r x\right]}{b\left\{\cosh 2\kappa_r b - 1\right\}} \tag{70}$$

where the Σ now extends only to those roots of (60) of which the real part is positive.

As an example, if we wish to expand x^3 in such a series, we have from (65), putting $a_1 = 0$, $a_3 = 1$, and all the other a's zero,

$$p_4 = \frac{3}{2b}.$$

Hence
$$F(z) = \frac{2b^2}{z^2}$$

and
$$\int_{-b}^{b} xf(x)\,dx = \frac{2b^5}{5}.$$

Therefore

$$x^3 = \tfrac{3}{5}xb^2 - 2b\,\Sigma\frac{1}{\kappa_r^2}\,\frac{\left[\kappa_r x\cosh\kappa_r x\sinh\kappa_r b - \kappa_r b\cosh\kappa_r b\sinh\kappa_r x + 2\sinh\kappa_r b\sinh\kappa_r x\right]}{(\cosh 2\kappa_r b - 1)}.$$

We notice that the terms introduced by the zeroes at the origin ensure that the function which is represented by the sum of the series in (70) always satisfies the condition (61).

14. *Limits and Validity of this Expansion.*

Looking at the expression (69) for $F(z)$, we see that the most important terms when $|z|$ is large involve $1/z^2$, the terms in $1/z$ being absent from the expansion.

$$\underset{z=\infty}{\mathrm{L}}\, z^2F(z) = p_4\frac{(2b)^3}{3!} + p_6\frac{(2b)^5}{5!} + \ldots + p_{2r}\frac{(2b)^{2r-1}}{(2r-1)!}.$$

Now

$$f'_{2s}(x) = \frac{1}{(2s-1)!}\left[(s+1)\left\{(b+x)^{2s-1} + (b-x)^{2s-1}\right\} - (2s-1)\,b\left\{(b+x)^{2s-2} + (b-x)^{2s-2}\right\}\right].$$

Hence, when $s > 1$, $$f'_{2s}(b) = \tfrac{3}{2}\frac{(2b)^{2s-1}}{(2s-1)!}.$$

Thus $p_4 \frac{(2b)^3}{3!} + p_6 \frac{(2b)^5}{5!} + \ldots + p_{2r} \frac{(2b)^{2r-1}}{(2r-1)!} = \frac{2}{3}\left[p_4 f_4'(b) + \ldots + p_{2r} f_{2r}'(b)\right]$

$$= \tfrac{2}{3}\left[f'(b) - p_2 f_2'(b)\right].$$

But $$f_2(x) = 2bx.$$

Therefore $$f_2'(b) = 2b.$$

Hence $$\mathop{\mathrm{L}}_{z=\infty} z^2 F(z) = \tfrac{2}{3} f'(b) - \tfrac{4}{3} b p_2. \qquad (71)$$

It follows that the remainder after n terms in (70)—the terms involving conjugate imaginaries being counted as one term—is given by

$$\frac{1}{2\pi i}\int_C \{z^2 F(z)\} \left\{\frac{x \cosh zx \sinh zb - b \cosh zb \sinh zx + (2/z) \sinh zb \sinh zx}{\sinh 2zb - 2zb}\right\} \frac{dz}{z}, \qquad (72)$$

the contour C being a circle passing through the points $\pm i(4r+3)\pi/4b$.

Referring to the results of Art. 12, we see that when r is large

$$\frac{\xi}{\eta} = \frac{\log_e (4r+1)\pi}{\frac{1}{2}(4r+1)\pi},$$

and this tends to zero with r. The roots, after a certain value of r, are contained within an angle 2ϵ enclosing the imaginary axis, where 2ϵ may be taken as small as we please.

Again, if r be large enough, all the roots for which $\eta \leqslant (\pi/4b)(4r+1)$ lie inside the circle C.

For, if (ξ', η) be *on* the circle and

$$\eta = (\pi/4b)(4r+1), \qquad \xi'^2 = (\pi/2b)(2R - \pi/2b),$$

R being the radius of the circle, that is $R = (\pi/4b)(4r+3)$. Thus

$$\xi' = \pi/2b\sqrt{(4r+2)}$$

and $$\xi/\xi' = \log\{4r+1)\pi\}/\pi\sqrt{(4r+2)} = 0$$

when $r = \infty$.

Therefore, if r be large enough, (ξ, η) lies inside the circle C, and the roots for which $\eta < (\pi/4b)(4r+1)$ can easily be shown to lie inside the circle C.

We will consider first those parts of the contour integral which lie inside the angle 2ϵ on the arc which is bisected by the positive half of the imaginary axis; the work for the arc which is bisected by the negative half of the imaginary axis is precisely similar.

Let $z = p+iq$, as before, be any point on the circle C. Then

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| = \left|\frac{(x\cosh zx \sinh zb - b\cosh zb \sinh zx + (2/z)\sinh zb \sinh zx)}{\sinh 2zb - 2zb}\right|$$

$$\leqslant \frac{|x|\,|\cosh zx|\,|\sinh zb| + b\,|\cosh zb|\,|\sinh zx| + (2/R)\,|\sinh zb|\,|\sinh zx|}{|\sinh 2zb - 2zb|}.$$

Now $$|\cosh(px+iqx)| < \tfrac{1}{2}|e^{px+iqx}| + \tfrac{1}{2}|e^{-px-iqx}|$$
$$< \tfrac{1}{2}(e^{px}+e^{-px}) < e^{p|x|}.$$

Similarly $|\sinh zb| < e^{pb}$, $|\cosh zb| < e^{pb}$, $|\sinh zx| < e^{p|x|}$.

Thus $$\left|\frac{\phi(z, x)}{z\psi(z)}\right| < \frac{\{|x|+b+2/R\}\, e^{p(b+|x|)}}{|\sinh 2zb - 2zb|}.$$

Take p so large that $$\sinh 2pb - 2Rb > \tfrac{1}{2}\lambda\, e^{2pb}, \tag{73}$$

λ being some positive quantity < 1. Then

$$|\sinh 2zb - 2zb| > |\sinh 2zb| - |2zb|,$$

and $$|\sinh 2zb| > \tfrac{1}{2}|e^{2zb}| - \tfrac{1}{2}|e^{-2zb}| > \tfrac{1}{2}e^{2pb} - \tfrac{1}{2}e^{-2pb} > \sinh 2pb.$$

Thus $$|\sinh 2zb - 2zb| > \sinh 2pb - 2Rb > \tfrac{1}{2}\lambda\, e^{2pb},$$

and for values of p which satisfy the inequality (73)

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| < \frac{2}{\lambda}\left\{|x| + b + \frac{2}{R}\right\} e^{p[|x|-b]};$$

and is therefore finite if $-b \leqslant x \leqslant +b$, however large R may be.

The first value of p which satisfies the inequality (73) is given by

$$\tfrac{1}{2}(1-\lambda)\, e^{2pb} = \tfrac{1}{2}e^{-2pb} + 2Rb.$$

This leads to a large value of p when R is large, and thus, to a first approximation, this limiting value of p is given by

$$\tfrac{1}{2}(1-\lambda)\, e^{2pb} = 2Rb$$

or $$2pb = \log\{4bR/(1-\lambda)\}.$$

Thus $$\frac{p}{\xi'} = \frac{\log\{4bR/(1-\lambda)\}}{\pi\sqrt{(4r+2)}} = \frac{\log\left\{\dfrac{\pi(4r+3)}{1-\lambda}\right\}}{\pi\sqrt{(4r+2)}}.$$

This ratio becomes very small as r (and therefore R) increases. Hence the parts of the contour for which p does not satisfy the inequality (73) are on a small arc $2\epsilon'$ bisected by the imaginary axis and ultimately very small compared with the arc 2ϵ.

Consider now the values of $\left|\dfrac{\phi(z, x)}{z\psi(z)}\right|$ over this arc $2\epsilon'$. We have

$$R-q < p^2/R, \qquad q > R-p^2/R.$$

Hence, *a fortiori*,

$$q > (4r+3)\frac{\pi}{4b} - \frac{1}{4b^2R}\left[\log\left(\frac{4Rb}{1-\lambda}\right)\right]^2,$$

by increasing R the second term can be made numerically less than any assigned quantity $\theta/2b$, and we have

$$\tfrac{1}{2}(4r+3)\pi > 2qb > \tfrac{1}{2}(4r+3)\pi - \theta$$

all over the arc $2\epsilon'$.

Now

$$|\sinh 2zb - 2zb| = |\sinh 2pb \cos 2qb - 2pb + i(\cosh 2pb \sin 2qb - 2qb)|$$
$$\geqslant |\cosh 2pb \sin 2qb - 2qb|.$$

Now $\sin 2qb$ lies between -1 and $-\cos\theta$. Therefore

$$\cosh 2pb \sin 2qb - 2qb$$

lies between $-\cosh 2pb - 2qb$ and $-\cosh 2pb \cos\theta - 2qb$.

Thus $|\cosh 2pb \sin 2qb - 2qb| > \cosh 2pb \cos\theta + 2qb > \tfrac{1}{2}e^{2pb}\cos\theta$.

Accordingly, inside the arc $2\epsilon'$

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| < 2\left\{|x| + b + \frac{2}{R}\right\} e^{p[|x|-b]} \sec\theta,$$

and this is finite for $-b \leqslant x \leqslant b$, since in this case $e^{p[|x|-b]} \leqslant 1$.

It follows that the parts of the integral due to the whole arc 2ϵ are less than

$$2\epsilon' 2\left\{|x| + b + \frac{2}{R}\right\}\sec\theta + (2\epsilon - 2\epsilon')\frac{2}{\lambda}\left\{|x| + b + \frac{2}{R}\right\},$$

and. since we can take $\cos\theta > \lambda$, this is less than

$$\frac{4\epsilon}{\lambda}\left\{|x| + b + \frac{2}{R}\right\},$$

which tends to zero with ϵ. These parts then contribute nothing in the limit to the integral.

Now consider the parts outside the angle 2ϵ. It is easy to show that, if $|x| \lessdot b$, then over those parts

$$\underset{R=\infty}{\mathrm{L}}\left|\frac{\phi(z, x)}{z\psi(z)}\right| = 0,$$

and they ultimately contribute nothing to the contour integral; the latter then vanishes when $R = \infty$, and the expansion holds.

If $x = \pm b$,

$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| = \left|\frac{2/z \sinh^2 zb}{\sinh 2zb - 2zb}\right| = \frac{1}{R}$$

when R is large. In this case also the contour integral vanishes when $R = \infty$. The expansion holds therefore for the ends of the range of validity.

If $|x| > b$,
$$\left|\frac{\phi(z, x)}{z\psi(z)}\right| = \infty$$

when $R = \infty$, and the expansion is not valid.