Interaction-Embedded Internal Time: Natural-Law Conditions for Social Proper Time in Multi-Agent Self-Modifying Minds

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Abstract

We propose a probabilistic framework for *internal time* in multi-agent self-modifying systems, where time is not given by an external clock but is constructed as an additive functional of self-change and interaction. Each agent i carries a local state S_n^i, receives messages M_n^i from other agents, and updates an internal time process T_n^i through increments

$$\Delta T_n^i = g^i\big(\Delta_{\mathrm{self},n}^i, I_n^i\big),$$

where $\Delta_{\mathrm{self},n}^i$ measures self-modification, I_n^i is a rigorously axiomatized interaction intensity, and g^i is constrained by natural conditions. On top of this, we introduce *value-meaning potentials* Φ^i, semantically realized by observation-value pairs, and study *value-regulated agents* for which Φ^i obeys a Lyapunov-type drift inequality relating semantic change to self-change and interaction.

Under Foster-Lyapunov conditions that ensure positive Harris recurrence of the global dynamics, each agent's internal time admits an almost-sure linear growth rate (social proper-time velocity)

$$\lambda_i = \lim_{n\to\infty} \frac{T_n^i}{n},$$

which we characterize by bounds that couple λ_i to long-run averages of self-change, interaction intensity, and semantic drift. We derive sufficient conditions under which value-aligned agents synchronize their internal-time velocities and conditions under which heterogeneity in value-meaning structure or coupling produces persistent de-synchronization.

To address interpretation and applicability, we relate our construction to classical time-changed Markov processes and subjective-time models, clarify the connection between interaction intensity and information-theoretic measures including value-weighted Age-of-Information, and discuss applications to distributed learning, human-AI collaboration, and social simulation. We illustrate the theory with several toy models, including a heterogeneous three-agent system with a threshold in the interaction-time coupling exhibiting phase-like transitions in synchronization.

Keywords. Internal time; multi-agent systems; additive functionals; time-changed Markov processes; interaction intensity; semantic value; subjective time; social proper time.

1 Introduction

1.1 Motivation: From subjective time to social proper time

Physical theories typically treat time as an externally given parameter: Newtonian absolute time, proper time along worldlines in relativity, or the time parameter in Markov processes. In contrast, cognitive science and neuroscience often view *subjective time* as emerging from internal dynamics of an organism, such as pacemaker-accumulator mechanisms or internal-clock models, and from distortions in temporal processing in neurological and psychiatric conditions [8–10]. In reinforcement learning, learning theory, and active inference, time is implicitly counted by update steps, prediction cycles, or free-energy minimization iterations.

This paper asks a more structural question:

> *Given a multi-agent system with self-modifying agents that interact and evaluate states by value-meaning structures, when can we construct* internal time *from these processes and derive natural-law constraints on its long-run behavior?*

We focus on *social proper time*: asymptotic internal time per step for each agent, interpreted as the agent's intrinsic tempo of experience and decision-making, embedded in a social environment. Instead of postulating a separate time parameter, we define internal time as an additive functional

$$T_n^i = \sum_{k=0}^{n-1} g^i(\Delta_{\mathrm{self},k}^i, I_k^i),$$

where $\Delta_{\mathrm{self},k}^i$ measures how much agent i's state changed between steps k and $k+1$, and I_k^i summarizes the intensity of socially relevant interaction at step k. The function g^i aggregates these into a single "tick" size and is constrained by structural conditions motivated by subjective-time models and by the requirement that T_n^i be a well-defined adapted additive functional.

1.2 Conceptual position and main questions

Conceptually, our framework lies at the intersection of:

- probabilistic theories of *time-changed Markov processes* and additive functionals,
- cognitive models of *subjective time* and internal clocks [7–10],
- *multi-agent systems* with interaction graphs and message passing,
- *value-based* and *semantic* representations in intelligence.

We address the following questions:

(Q1) *Construction:* Under what structural and measurability conditions can we define internal times $(T_n^i)_n$ as additive functionals of self-change and interaction in a multi-agent Markov setting?

(Q2) *Value-meaning regulation:* How can value-meaning structures be incorporated so that internal time is not just arbitrary bookkeeping but constrained by a semantically realized potential?

(Q3) *Natural-law constraints:* What inequalities relate the long-run internal-time velocity

$$\lambda_i = \lim_{n\to\infty} \frac{T_n^i}{n}$$

to long-run averages of self-change, interaction intensity, and semantic drift?

(Q4) *Synchronization:* When does value alignment enforce synchronization of λ_i across agents?

(Q5) *De-synchronization:* When does heterogeneity lead to de-synchronization?

(Q6) *Ergodicity and applications:* Which checkable conditions on the dynamics guarantee the existence of λ_i, and how can internal time be used in distributed learning, human-AI collaboration, or social simulations?

1.3 Relation to existing models of time and interaction

Subjective time and internal clocks. Classical internal-clock and pacemaker-accumulator models posit an internal pacemaker emitting pulses at a subjective rate, accumulated and compared to reference memories [8,9]. Distortions in time perception across neurological and psychiatric conditions have been reviewed in [10]. Our increments $g^i(\Delta^i_{\text{self}}, I^i)$ can be viewed as a generalized pulse rate: self-change Δ^i_{self} plays the role of internal dynamical activity, while interaction intensity I^i modulates this rate according to socially relevant information. The function g^i aggregates these into a single "tick" size.

Time-changed Markov processes. In the theory of time-changed Markov processes, one constructs a new process X_{τ_t} by composing a Markov process (X_t) with an increasing additive functional A_t or a subordinator τ_t [2,3,12]. Our discrete-time internal times (T_n^i) are additive functionals of the agent-wise Markov chain (S_n^i, M_n^i), which can be seen as defining a discrete analogue of such time changes. However, our focus is on structural constraints and multi-agent synchronization/de-synchronization of their long-run growth rates.

Age-of-Information (AoI). AoI measures time since last update at a receiver; see [4,5] for overviews. In contrast, internal time measures the *accumulated rate of value-meaning processing* as a function of self-change and interaction intensity. When messages are value-relevant, AoI and internal time become related: high AoI often coincides with low interaction intensity and thus slower internal time, while frequent value-relevant updates accelerate internal time. We make this connection more explicit in Section 6.2.

Active inference and semantic value. In active inference and predictive processing, time is implicitly counted by iterations of variational inference and policy evaluation [6,7]. Our value-meaning potentials Φ^i can be interpreted as coarse-grained semantic quantities such as expected value or reduced free energy, and the drift inequalities relate the evolution of Φ^i to self-change and interaction. This provides a bridge between internal time and information-processing rates in such frameworks.

1.4 Contributions

The main contributions of this paper are:

(C1) **Axiomatic construction of internal time.** We formulate a discrete-time multi-agent Markov framework with global dynamics and local observations, define self-change Δ^i_{self} and interaction intensity I^i_n via compatible metrics and an axiomatized interaction functional, and construct internal times T^i_n as additive functionals with well-posed measurability properties.

(C2) **Value-regulated agents via semantic potentials.** We introduce value-meaning representations $(\mathcal{Y}^i, \mu^i, V^i)$ and semantically realized potentials Φ^i. We define value-regulated agents through Lyapunov-type drift inequalities that constrain expected semantic drift by self-change and interaction.

(C3) **Natural-law inequalities for social proper time.** Under simple coupling assumptions between ΔT^i_n and $(\Delta^i_{\text{self}}, I^i_n)$ and a Foster-Lyapunov condition on the joint dynamics, we prove the existence of internal-time velocities λ_i and derive inequalities that bound λ_i from above and below by long-run averages of Δ^i_{self} and I^i_n and the parameters of the semantic drift inequality.

(C4) **Synchronization, de-synchronization, and value alignment.** We identify structural conditions under which value-aligned agents synchronize their internal-time velocities and conditions under which heterogeneity in g^i, interaction intensity, or value-meaning structure leads to persistent de-synchronization. We highlight the role of semantically realized potentials in making synchronization a genuinely semantic phenomenon.

(C5) **Ergodicity, non-ergodic regimes, and verification.** We derive sufficient conditions for ergodicity via a Foster-Lyapunov drift condition and discuss non-ergodic regimes where internal-time velocities depend on the initial ergodic component. We clarify that from a global perspective, internal-time velocities are random variables indexed by ergodic classes.

(C6) **Richer examples and applications.** We present toy models including (i) a two-state cognitively motivated model, (ii) a linear consensus-like model with internal time, and (iii) a heterogeneous three-agent model with a threshold in the interaction-time coupling, exhibiting a phase-like transition in synchronization as a coupling parameter crosses a regime. We also discuss how internal time could be used in distributed learning, human-AI collaboration, and social simulations.

The rest of the paper is organized as follows. Section 2 introduces the probabilistic setup, interaction intensity, and internal-time construction. Section 3 defines value-meaning potentials and value-regulated agents. Section 4 derives natural-law inequalities for social proper time, discusses ergodicity and non-ergodic regimes, and treats synchronization properties. Section 5 presents toy models. Section 6 discusses related work, AoI connections, applications, and mathematical extensions.

2 Probabilistic setup and internal-time construction

2.1 Global dynamics and local views

We consider a discrete-time multi-agent system indexed by a finite set

$$I := \{1, \ldots, N\}.$$

For each agent $i \in I$ we have a local state space $\mathcal{S}^i$, which we assume to be a Polish space with Borel σ-algebra. We denote the global state space by

$$\mathcal{S} := \prod_{i \in I} \mathcal{S}^i,$$

equipped with the product Borel σ-algebra.

Assumption 2.1 (Global probability space and filtration)**.** There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_n)_{n \geq 0}$ such that:

(i) The global state process $S_n = (S_n^i)_{i \in I}$ is $\mathcal{S}$-valued, adapted to $(\mathcal{F}_n)$, and Markov with respect to $(\mathcal{F}_n)$ with a time-homogeneous transition kernel $\tilde{K}$ on $\mathcal{S}$.

(ii) For each n, the messages $M_n = (M_n^i)_{i \in I}$ are generated from S_n via a (possibly randomized) measurable mapping or kernel that depends only on S_n, and are $\mathcal{F}_n$-measurable.

Assumption 2.1 implies that the augmented process

$$X_n := (S_n, M_n)$$

is itself a time-homogeneous Markov chain on $\mathcal{S} \times \prod_{i \in I} \mathcal{M}^i$ with respect to $(\mathcal{F}_n)$, with a transition kernel obtained by composing $\tilde{K}$ with the message mechanism.

For each agent $i \in I$, we denote the local state by S_n^i and the collection of messages observed at time n by M_n^i, taking values in a measurable space $\mathcal{M}^i$. We write $X_n^i := (S_n^i, M_n^i)$ for the local observable configuration.

Assumption 2.2 (Local state and message metrics)**.** For each agent i, there is a metric d^i_{self} on $\mathcal{S}^i$ that is *compatible* with the given Polish topology (i.e., it induces the same topology). Each message space $\mathcal{M}^i$ is a Polish space with a compatible metric d^i_{msg}. In addition, we fix a measurable "size" functional

$$\nu^i : \mathcal{M}^i \longrightarrow [0, \infty)$$

that is Lipschitz with respect to d^i_{msg}, i.e., there exists $L_i > 0$ such that

$$\left|\nu^i(m) - \nu^i(m')\right| \;\leq\; L_i \, d^i_{\mathrm{msg}}(m, m') \quad \text{for all } m, m' \in \mathcal{M}^i.$$

In concrete examples, ν^i will be constructed from d^i_{msg} (e.g., as a supremum, sum, or average of d^i_{msg} over message atoms), and we will use ν^i to generate a canonical partial order on $\mathcal{M}^i$ below.

2.2 Self-change and interaction intensity

We now define the two primitive quantities from which internal time will be constructed.

Definition 2.3 (Self-change)**.** For agent i, the *self-change* at step n is

$$\Delta^i_{\mathrm{self},n} := d^i_{\mathrm{self}}(S^i_{n+1}, S^i_n) \in [0, \infty),$$

where d^i_{self} is as in Assumption 2.2.

We assume that each message space $\mathcal{M}^i$ is equipped with additional structure that captures combination and comparison of message batches.

Assumption 2.4 (Message concatenation and partial order)**.** For each agent i:

- There is a measurable *concatenation* operation

$$\oplus : \mathcal{M}^i \times \mathcal{M}^i \to \mathcal{M}^i,$$

modeling the union or batching of messages. We assume $\oplus$ is associative:

$$(m_1 \oplus m_2) \oplus m_3 = m_1 \oplus (m_2 \oplus m_3) \quad \text{for all } m_1, m_2, m_3 \in \mathcal{M}^i.$$

- There is a partial order $\preceq$ on $\mathcal{M}^i$, compatible with the size functional ν^i from Assumption 2.2 in the sense that

$$m' \preceq m \quad \Longrightarrow \quad \nu^i(m') \leq \nu^i(m).$$

A canonical choice-which we will have in mind in many examples-is to *define*

$$m' \preceq m \quad \Longleftrightarrow \quad \nu^i(m') \leq \nu^i(m),$$

so that m' contains no more and possibly less value-relevant information than m, with ν^i encoding the magnitude of message perturbations derived from d^i_{msg} (for instance via a supremum, sum, or average of d^i_{msg} over message atoms).

Definition 2.5 (Interaction intensity functionals)**.** For each $i \in I$, an *interaction intensity functional* is a measurable map

$$\Psi^i : \mathcal{S}^i \times \mathcal{M}^i \to [0, \infty)$$

satisfying the following axioms:

(I1) *Locality:* Ψ^i depends only on (S^i_n, M^i_n) (no global state access):

$$\Psi^i(s, m) = \Psi^i(s, m), \quad (s, m) \in \mathcal{S}^i \times \mathcal{M}^i.$$

(I2) *Non-negativity:* $\Psi^i(s, m) \geq 0$ for all (s, m).

(I3) *Monotonicity:* There exists a partial order $\preceq$ on $\mathcal{M}^i$ (as in Assumption 2.4; for example, $m' \preceq m$ if $\nu^i(m') \leq \nu^i(m)$, with ν^i derived from d^i_{msg}) such that if $m' \preceq m$, then

$$\Psi^i(s, m') \leq \Psi^i(s, m).$$

(I4) *Sub-additivity:* If there exists a concatenation operation $\oplus : \mathcal{M}^i \times \mathcal{M}^i \to \mathcal{M}^i$ such that $m = m_1 \oplus m_2$, then

$$\Psi^i(s, m) \leq \Psi^i(s, m_1) + \Psi^i(s, m_2).$$

Given (S^i_n, M^i_n), we define

$$I^i_n := \Psi^i(S^i_n, M^i_n).$$

Axioms (I1)-(I4) capture the idea that Ψ^i is a local, non-negative, and approximately extensive measure of the *socially relevant information flux* into agent i. The partial order $\preceq$ and concatenation $\oplus$ formalize the notions of "removing" and "combining" message batches, with magnitude comparison encoded by the metric-derived functional ν^i.

Remark 2.6 (Examples of interaction intensity)*.* Examples include:

- *Metric differences:* Suppose each message $m \in \mathcal{M}^i$ encodes a finite list of inferred neighbor states $(\hat{s}_j)_{j\sim i}$ together with their identifiers, and define a base metric d^i_{base} on such lists (e.g., an ℓ^1 or Hausdorff-type metric induced by a metric on local states). We may choose

$$\nu^i(m) \;=\; \sum_{j\sim i} d^i_{\mathrm{base}}(\hat{s}_j, s),$$

and declare $m' \preceq m$ iff $\nu^i(m') \leq \nu^i(m)$. An interaction intensity can then be taken as

$$\Psi^i(s, m) \;=\; \nu^i(m),$$

so that concatenating message lists via $\oplus$ and shrinking them in the d^i_{base}-metric are automatically reflected in ν^i and hence in the order $\preceq$.

- *Information-theoretic measures:* If m encodes an empirical distribution $\hat{Q}^i(\cdot \mid m)$ over some alphabet, one may fix a reference model Q^i_{ref} and define

$$\nu^i(m) \;=\; D_{\mathrm{KL}}\big(\hat{Q}^i(\cdot \mid m) \,\|\, Q^i_{\mathrm{ref}}\big),$$

  with d^i_{msg} chosen so that small perturbations of the empirical distribution induce small changes in ν^i (e.g., a Wasserstein or total-variation metric). Taking

$$\Psi^i(s,m) \;=\; \nu^i(m)$$

  yields an intensity functional that increases whenever messages become more informative in the information-theoretic sense, and $m' \preceq m$ iff $\nu^i(m') \le \nu^i(m)$.

- *Value-weighted AoI:* Suppose messages carry generation times τ_k and value weights $w_k \ge 0$, and encode the finite set $\{(w_k,\tau_k)\}_k$ as $m \in \mathcal{M}^i$. Fixing a time index n, define

$$\nu^i(m) \;=\; \sum_{k:\,\tau_k \le n} w_k(n-\tau_k),$$

  and choose d^i_{msg} so that adding or removing pairs (w_k,τ_k) produces a corresponding Lipschitz change in ν^i (for instance, by embedding the pairs into a Euclidean space and using an ℓ^1 metric). Then

$$\Psi^i(s,m) \;=\; \nu^i(m)$$

  is a value-weighted AoI functional, with $m' \preceq m$ whenever the total value-weighted staleness encoded in m' is no larger than that in m.

These examples show concretely how d^i_{msg} and ν^i can be chosen so that the partial order $\preceq$ and the monotonicity axiom (I3) are ultimately grounded in a metric notion of message magnitude.

2.3 Internal time increments and measurability

We now specify the form of internal time increments.

Assumption 2.7 (Internal time coupling functions)**.** For each agent $i \in I$, there is a measurable function

$$g^i : [0,\infty) \times [0,\infty) \to [0,\infty)$$

such that:

(g1) $g^i(0,0) = 0$ and g^i is non-decreasing in each argument.

(g2) (*Linear growth*) There exists $C_i > 0$ such that

$$g^i(x,y) \le C_i(1+x+y) \quad \text{for all } x,y \ge 0. \tag{2.1}$$

(g3) (*Coupling inequality*) There exist non-negative constants a_i, A_i, b_i, B_i, d_i such that

$$a_i \Delta^i_{\text{self},n} - b_i \;\le\; g^i(\Delta^i_{\text{self},n}, I^i_n) \;\le\; A_i \Delta^i_{\text{self},n} + B_i I^i_n + d_i \tag{2.2}$$

almost surely for all n.

The linear growth condition (2.1) is not purely cosmetic: together with a Foster-Lyapunov condition on the global dynamics and suitable bounds on $\Delta^i_{\text{self},n}$ and I^i_n, it ensures integrability of ΔT^i_n under the invariant measure, which is required to apply the ergodic theorem to T^i_n. One could in principle relax this to polynomial growth $g^i(x, y) \le C_i(1 + x^p + y^p)$ with $p \ge 1$ at the expense of stronger moment assumptions on the dynamics; for clarity we restrict to linear growth in this paper.

Definition 2.8 (Internal time process)**.** Given Assumptions 2.1, 2.2, 2.4, and 2.7, and an interaction intensity as in Definition 2.5, the *internal time* of agent i is the process $(T^i_n)_{n\ge0}$ defined by

$$T^i_0 := 0, \qquad T^i_{n+1} := T^i_n + \Delta T^i_n, \quad \Delta T^i_n := g^i(\Delta^i_{\text{self},n}, I^i_n).$$

Lemma 2.9 (Adaptation and additivity)**.** *For each i and n, T^i_n and ΔT^i_n are $\mathcal{F}_n$-measurable, and $(T^i_n)_n$ is an adapted, non-decreasing process with*

$$T^i_n = \sum_{k=0}^{n-1} \Delta T^i_k$$

almost surely.

Proof. We proceed by induction. For $n = 0$ we have $T^i_0 = 0$ which is $\mathcal{F}_0$-measurable. Assume T^i_n is $\mathcal{F}_n$-measurable. By definition, S^i_n and M^i_n are $\mathcal{F}_n$-measurable, hence so are $\Delta^i_{\text{self},n}$ and I^i_n, and thus $\Delta T^i_n = g^i(\Delta^i_{\text{self},n}, I^i_n)$ is $\mathcal{F}_n$-measurable. By Assumption 2.1(ii), S^i_{n+1} and M^i_{n+1} are $\mathcal{F}_{n+1}$-measurable, so $T^i_{n+1} = T^i_n + \Delta T^i_n$ is $\mathcal{F}_{n+1}$-measurable. Non-negativity follows from $g^i \ge 0$, and the additive representation is immediate by iterating the recursion. □

Assumption 2.10 (Non-triviality)**.** For each i, $\mathbb{P}(\Delta T^i_n > 0 \text{ i.o.}) = 1$, so that $T^i_n \to \infty$ almost surely. In particular we exclude the degenerate case $S^i_{n+1} = S^i_n$ and $I^i_n = 0$ for all sufficiently large n almost surely. Periodic or quasi-periodic behavior is allowed as long as T^i_n continues to increase.

Assumption 2.10 rules out agents whose state and interaction intensity become eventually constant, which would yield a trivial internal time.

3 Value-meaning potentials and value-regulated agents

3.1 Semantic representation and potentials

We now incorporate value and meaning into the framework. Intuitively, each agent maps its internal state to an observation or representation space and evaluates these by a value functional.

Definition 3.1 (Value-meaning representation)**.** For agent i, a *value-meaning representation* is a triple $(\mathcal{Y}^i, \mu^i, V^i)$ where:

- $\mathcal{Y}^i$ is a Polish observation/representation space.
- μ^i is a Markov kernel from $\mathcal{S}^i$ to $\mathcal{P}(\mathcal{Y}^i)$: for each $s \in \mathcal{S}^i$, $\mu^i(s, \cdot)$ is a probability measure on $\mathcal{Y}^i$, and for each measurable $B \subset \mathcal{Y}^i$, $s \mapsto \mu^i(s, B)$ is measurable.
- $V^i : \mathcal{Y}^i \to \mathbb{R}$ is a measurable *value functional.*

We define the *value-meaning potential*

$$\Phi^i(s) := \int_{\mathcal{Y}^i} V^i(y)\, \mu^i(s, dy),$$

whenever the integral is finite.

Remark 3.2 (Deterministic vs. stochastic semantics)*.* The kernel μ^i subsumes deterministic maps $\mathcal{S}^i \to \mathcal{Y}^i$ as a special case (when $\mu^i(s, \cdot)$ is a Dirac mass). The Markov-kernel formulation is essential when agents maintain probabilistic beliefs or use stochastic policies. For instance:

- In partially observable Markov decision processes (POMDPs), the agent's internal state S_n^i may encode a belief distribution over environment states, and $\mu^i(s, \cdot)$ describes the distribution over latent variables or predictions.
- In Bayesian inference, $\mu^i(s, \cdot)$ can represent a posterior over hypotheses, and V^i a utility or loss function on hypotheses.
- In stochastic policy optimization, μ^i may encode the distribution over actions under a policy, and V^i the expected return associated with each action.

In each case, $\Phi^i(s)$ aggregates value over a genuinely probabilistic representation, not a single deterministic observation.

Definition 3.3 (Semantically realized potential)**.** We say that a function $\Phi^i : \mathcal{S}^i \to \mathbb{R}$ is a *semantically realized potential* if there exists a value-meaning representation $(\mathcal{Y}^i, \mu^i, V^i)$ such that

$$\Phi^i(s) = \int_{\mathcal{Y}^i} V^i(y)\, \mu^i(s, dy)$$

for all s in a full-measure subset of $\mathcal{S}^i$.

The semantic realization ensures that increments

$$\Phi^i(S_{n+1}^i) - \Phi^i(S_n^i)$$

can be interpreted as changes in *expected value* under the agent's own representation, rather than arbitrary numerical fluctuations.

Remark 3.4 (Relative value and alignment). We adopt a *relative* notion of value: adding a constant to V^i does not affect Φ^i up to an additive constant, and only differences of Φ^i appear in our drift inequalities. In multi-agent settings with a common task, we are particularly interested in the case where agents' value-meaning representations share some structure (e.g., common task-level objectives), which then has implications for synchronization, as discussed in Section 4.

3.2 Value-regulated agents

We now formalize the idea that semantic drift is constrained by self-change and interaction.

Assumption 3.5 (Semantic Lyapunov drift). For each agent i, there exists a semantically realized potential $\Phi^i : \mathcal{S}^i \to \mathbb{R}$ and constants $\alpha_i > 0$, $\beta_i \geq 0$, $c_i \in \mathbb{R}$ such that

$$\mathbb{E}\big[\Phi^i(S^i_{n+1}) - \Phi^i(S^i_n) \,\big|\, \mathcal{F}_n\big] \leq -\alpha_i \Delta^i_{\mathrm{self},n} + \beta_i I^i_n + c_i \tag{3.1}$$

almost surely for all n.

Inequality (3.1) expresses that, on average, large self-change tends to reduce the potential (interpretable as semantic "excess free energy" or "discrepancy"), while interaction intensity can either increase or decrease it depending on the sign of β_i, and c_i accounts for baseline drift.

Definition 3.6 (Value-regulated agent). An agent i is *value-regulated* if:

(VR1) It has a value-meaning representation $(\mathcal{Y}^i, \mu^i, V^i)$ with semantically realized potential Φ^i.

(VR2) The semantic Lyapunov inequality (3.1) holds with some $\alpha_i > 0$, $\beta_i \geq 0$, $c_i \in \mathbb{R}$.

(VR3) The internal-time increments satisfy Assumption 2.7.

Remark 3.7 (Non-vacuity of the definition). Definition 3.6 imposes non-trivial constraints: Φ^i must be compatible with a value-meaning representation, and its conditional expectation drift must be controlled by self-change and interaction in the specific form (3.1). An arbitrary Lyapunov function need not be semantically realized in this sense, and an arbitrary dynamic need not admit such a Φ^i with $\alpha_i > 0$. Thus, the class of value-regulated agents is a proper subset of general Markovian agents.

3.3 Concrete example: value-regulated RL agent

As a concrete example, consider an RL agent with local state S^i_n containing both environment observation and internal memory, and a value function V^i_θ predicting the discounted return under parameter vector θ. We can take

$$\mathcal{Y}^i = \mathbb{R}, \quad \mu^i(s, \cdot) = \delta_{V^i_\theta(s)}, \quad V^i(y) = y,$$

so that $\Phi^i(s) = V^i_\theta(s)$ is semantically realized. Suppose the agent updates θ using a stochastic gradient step

$$\theta_{n+1} = \theta_n - \eta_n g_n,$$

where g_n is an unbiased estimate of $\nabla_\theta \mathcal{L}(\theta_n)$ for some loss $\mathcal{L}$, and η_n is a small step size. Under standard conditions-smoothness of $\mathcal{L}$ and sufficiently small η_n-one can show that

$$\mathbb{E}\big[\mathcal{L}(\theta_{n+1}) - \mathcal{L}(\theta_n) \mid \mathcal{F}_n\big] \le -\kappa \|\nabla \mathcal{L}(\theta_n)\|^2$$

for some $\kappa > 0$. If we define $\Phi^i(S^i_n) = \mathcal{L}(\theta_n)$ and take $\Delta^i_{\mathrm{self},n}$ to be a suitable norm of the parameter change, then a drift inequality of the form (3.1) holds with α_i proportional to the curvature of $\mathcal{L}$ and β_i capturing the effect of information-bearing interactions (e.g., experience sharing). In this sense, internal time can be interpreted as counting value-relevant update effort in a concrete RL setting, under explicit approximating assumptions.

4 Natural-law constraints for social proper time

We now derive inequality constraints for the long-run growth rate of internal time and discuss ergodicity, non-ergodic regimes, and synchronization. Throughout this section we assume all agents under consideration are value-regulated in the sense of Definition 3.6.

4.1 Ergodicity via Foster-Lyapunov conditions

Let $X_n := (S_n, M_n)$ denote the global augmented state. We first state a standard ergodicity condition.

Assumption 4.1 (Foster-Lyapunov drift for global dynamics)**.** The Markov chain (X_n) is ψ-irreducible and aperiodic. There exists a measurable function $W : \mathcal{S} \times \prod_{i\in I} \mathcal{M}^i \to [1, \infty)$, constants $\rho \in (0, 1)$ and $b < \infty$, and a petite set $C \subset \mathcal{S} \times \prod_{i\in I} \mathcal{M}^i$ such that

$$\mathbb{E}\big[W(X_{n+1}) \,\big|\, X_n = x\big] \le \rho W(x) + b\mathbb{1}_C(x) \tag{4.1}$$

for all x.

Remark 4.2 (Relation between W and Φ^i)*.* In many applications, one can choose W to dominate the absolute values of the potentials, e.g.,

$$W(x) \ge 1 + \sum_{i\in I} \big|\Phi^i(s^i)\big|$$

for $x = (s, m)$. In this case, the drift condition (4.1) implies uniform moment bounds on $\Phi^i(S^i_n)$ and ensures that the additive functionals associated with internal time and semantic drift are integrable under the invariant measure.

Remark 4.3 (Petite sets)*.* Informally, a petite set is a set that the chain can return to with positive probability in a manner controlled by a non-trivial measure; it is a generalization of finite small sets for countable-state chains. Assumption 4.1 combines a Lyapunov function and a petite set condition to guarantee positive Harris recurrence and ergodicity; see [1] for a detailed discussion.

Proposition 4.4 (Ergodicity and additive functionals)**.** *Under Assumptions 2.1 and 4.1, the Markov chain (X_n) is positive Harris recurrent and admits a unique invariant probability measure π. Moreover, for any integrable function h on the state space,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h(X_k) = \int h \, d\pi \quad \textit{almost surely.}$$

In particular, for each agent i, if $\mathbb{E}_\pi[|\Delta T_0^i|] < \infty$, then

$$\lambda_i := \lim_{n\to\infty} \frac{T_n^i}{n}$$

exists almost surely and equals $\mathbb{E}_\pi[\Delta T_0^i]$.

Proof. The first claim is a standard consequence of the drift condition (4.1); see, e.g., [1]. The second claim is the ergodic theorem for additive functionals of positive Harris chains, applied to $h(x) = \Delta T_0^i(x)$, using the assumed integrability. □

4.2 Value-time law: inequalities for internal-time velocity

We now relate λ_i to long-run averages of self-change, interaction intensity, and semantic drift.

Theorem 4.5 (Value-time inequality)**.** *Let agent i be value-regulated and assume the conditions of Proposition 4.4. Suppose additionally that $|\Phi^i(s^i)| \le C_i' W(x) + D_i'$ for some constants C_i', D_i' and all $x = (s, m)$. Define*

$$\bar{\Delta}^i_{\text{self}} := \int \Delta^i_{\text{self},0}(x)\, \pi(dx), \qquad \bar{I}^i := \int I_0^i(x)\, \pi(dx),$$

and recall that $\lambda_i = \mathbb{E}_\pi[\Delta T_0^i]$. Then:

(i) From the coupling inequality (2.2),

$$a_i \bar{\Delta}^i_{\text{self}} - b_i \;\le\; \lambda_i \;\le\; A_i \bar{\Delta}^i_{\text{self}} + B_i \bar{I}^i + d_i. \tag{4.2}$$

(ii) From the semantic drift inequality (3.1), we obtain

$$-\alpha_i \bar{\Delta}^i_{\text{self}} + \beta_i \bar{I}^i + c_i \;\ge\; 0, \tag{4.3}$$

which is equivalent to

$$\alpha_i \bar{\Delta}^i_{\text{self}} \;\le\; \beta_i \bar{I}^i + c_i.$$

Substituting this into the upper bound in (4.2) yields the semantic upper bound

$$\lambda_i \;\le\; \frac{A_i}{\alpha_i}(\beta_i \bar{I}^i + c_i) + B_i \bar{I}^i + d_i. \tag{4.4}$$

Moreover, by Proposition 4.4, the empirical averages

$$\frac{1}{n}\sum_{k=0}^{n-1} \Delta^i_{\mathrm{self},k}, \qquad \frac{1}{n}\sum_{k=0}^{n-1} I^i_k$$

converge almost surely to $\bar{\Delta}^i_{\mathrm{self}}$ *and* $\bar{I}^i$*, so the same inequalities hold for the almost-sure limits of these empirical averages whenever they are considered.*

Proof. For (i), take expectations under the invariant measure π in (2.2):

$$a_i \bar{\Delta}^i_{\mathrm{self}} - b_i \;\le\; \int g^i(\Delta^i_{\mathrm{self},0}, I^i_0)\, d\pi \;\le\; A_i \bar{\Delta}^i_{\mathrm{self}} + B_i \bar{I}^i + d_i.$$

The middle term is λ_i by Proposition 4.4, yielding (4.2).

For (ii), take expectations under π in (3.1). By stationarity,

$$\mathbb{E}_\pi\big[\Phi^i(S^i_1) - \Phi^i(S^i_0)\big] = 0,$$

and integrability follows from the domination $|\Phi^i| \le C'_i W + D'_i$ and Assumption 4.1. Thus

$$0 = \mathbb{E}_\pi\big[\Phi^i(S^i_1) - \Phi^i(S^i_0)\big] \le -\alpha_i \bar{\Delta}^i_{\mathrm{self}} + \beta_i \bar{I}^i + c_i,$$

which gives (4.3) and hence

$$\bar{\Delta}^i_{\mathrm{self}} \;\le\; \frac{\beta_i \bar{I}^i + c_i}{\alpha_i}.$$

Substituting this into the upper bound in (4.2) yields (4.4). □

Remark 4.6 (Interpretation of the semantic bound). Inequalities (4.2) and (4.4) together tie the internal-time velocity λ_i to long-run averages of self-change, interaction intensity, and the baseline drift c_i. The coupling inequality (4.2) provides a purely structural lower and upper bound in terms of $\bar{\Delta}^i_{\mathrm{self}}$ and $\bar{I}^i$, while the semantic drift inequality further constrains $\bar{\Delta}^i_{\mathrm{self}}$ in terms of $\bar{I}^i$ and c_i, thereby sharpening the upper bound on λ_i. In regimes where $\beta_i \bar{I}^i + c_i$ is small, the semantic constraint restricts the admissible rate of self-change and hence the admissible internal-time velocity.

4.3 Social proper time and synchronization

We now consider a collection of value-regulated agents and compare their internal-time velocities.

Definition 4.7 (Social proper time and velocity)**.** For each agent i, the internal-time velocity λ_i from Theorem 4.5 is its *social proper-time velocity.* A multi-agent system exhibits *social* *proper-time synchronization* if $\lambda_i = \lambda_j$ for all $i, j \in I$, and *de-synchronization* if at least one pair (i, j) satisfies $\lambda_i \neq \lambda_j$.

We first consider an idealized value-aligned and symmetric setting.

Assumption 4.8 (Symmetric value alignment)**.** Suppose:

(A1) For each $i \in I$, there exists a measurable map $\pi_i : \mathcal{S}^i \to \mathcal{Y}$ into a common Polish space $\mathcal{Y}$ and a value functional $V : \mathcal{Y} \to \mathbb{R}$ such that the value-meaning representation is given by $\mu^i(s, \cdot) = \delta_{\pi_i(s)}$ and $\Phi^i(s) = V(\pi_i(s))$. Moreover, under the invariant measure π of (X_n), the distribution of $\pi_i(S_0^i)$ is identical for all i.

(A2) The global Markov kernel and message mechanism are invariant under permutations of agents: for any permutation σ of I, the distribution of $(S_1^i, M_1^i)_{i\in I}$ given $(S_0^i, M_0^i)_{i\in I}$ equals that of $(S_1^{\sigma(i)}, M_1^{\sigma(i)})_{i\in I}$ given $(S_0^{\sigma(i)}, M_0^{\sigma(i)})_{i\in I}$.

(A3) The time-coupling functions g^i and interaction intensity functionals Ψ^i are identical across agents.

Theorem 4.9 (Synchronization under symmetric value alignment)**.** *Under Assumptions 2.1, 4.1, and 4.8, all internal-time velocities coincide:*

$$\lambda_i = \lambda_j =: \lambda \quad \textit{for all } i, j \in I.$$

Proof. Under Assumption 4.8, the invariant measure π of (X_n) is invariant under any permutation of the agent indices. This permutation invariance implies that π is an exchangeable measure on the product $\prod_{i\in I}(\mathcal{S}^i \times \mathcal{M}^i)$: the joint distribution of any finite subcollection of agents' configurations is invariant under relabeling.

For any integrable function h depending on a single agent's local configuration, $h_i(s^i, m^i) := h(s^i, m^i)$, exchangeability of π yields

$$\mathbb{E}_\pi[h_i(S^i, M^i)] = \mathbb{E}_\pi[h_j(S^j, M^j)]$$

for all $i, j \in I$. In particular, the distributions of $(\Delta^i_{\text{self},0}, I_0^i, \Delta T_0^i)$ under π are identical across i. Proposition 4.4 then gives $\lambda_i = \mathbb{E}_\pi[\Delta T_0^i]$ independent of i. □

Remark 4.10 (Semantic meaning of synchronization)*.* Synchronization in Theorem 4.9 is not merely an algebraic coincidence: the common internal-time velocity λ reflects the rate at which *semantically realized* value-meaning potentials evolve under the shared representation $(\mathcal{Y}, V)$. Thus, synchronization here can be interpreted as a form of value-aligned temporal agreement in a social system.

We now make explicit how heterogeneity leads to de-synchronization in a simple but fully rigorous setting.

Proposition 4.11 (De-synchronization under heterogeneous affine coupling)**.** *Let agents i and j be value-regulated and suppose that, under the invariant measure π,* *(H1) The pairs $(\Delta^i_{\text{self},0}, I^i_0)$ and $(\Delta^j_{\text{self},0}, I^j_0)$ have the same joint distribution and finite first moments.*

(H2) Internal-time increments are affine in $(\Delta_{\text{self}}, I)$:

$$\Delta T^i_0 = a_i \Delta^i_{\text{self},0} + B_i I^i_0 + d_i, \qquad \Delta T^j_0 = a_j \Delta^j_{\text{self},0} + B_j I^j_0 + d_j$$

almost surely, for some real coefficients a_i, B_i, d_i and a_j, B_j, d_j.

Then

$$\lambda_i - \lambda_j = (a_i - a_j)\,\bar{\Delta}_{\text{self}} + (B_i - B_j)\,\bar{I} + (d_i - d_j),$$

where $\bar{\Delta}_{\text{self}} := \mathbb{E}_\pi[\Delta^i_{\text{self},0}]$ and $\bar{I} := \mathbb{E}_\pi[I^i_0]$. In particular, if $(a_i, B_i, d_i) \neq (a_j, B_j, d_j)$ and $(\bar{\Delta}_{\text{self}}, \bar{I})$ is not constrained to lie on a single affine subspace forcing the right-hand side to vanish, then $\lambda_i \neq \lambda_j$. The parameter values for which $\lambda_i = \lambda_j$ form a lower-dimensional affine subspace in the space of coupling coefficients.

Proof. By Proposition 4.4 and assumption (H2),

$$\lambda_i = \mathbb{E}_\pi[\Delta T^i_0] = a_i \mathbb{E}_\pi[\Delta^i_{\text{self},0}] + B_i \mathbb{E}_\pi[I^i_0] + d_i,$$

and similarly for λ_j. Assumption (H1) implies

$$\mathbb{E}_\pi[\Delta^i_{\text{self},0}] = \mathbb{E}_\pi[\Delta^j_{\text{self},0}] = \bar{\Delta}_{\text{self}}, \qquad \mathbb{E}_\pi[I^i_0] = \mathbb{E}_\pi[I^j_0] = \bar{I}.$$

Subtracting the two expressions gives the claimed formula for $\lambda_i - \lambda_j$. For fixed $(\bar{\Delta}_{\text{self}}, \bar{I})$, the condition $\lambda_i = \lambda_j$ is a single affine constraint on $(a_i, B_i, d_i, a_j, B_j, d_j)$, so its solution set is a lower-dimensional affine subspace of the parameter space. Outside this subspace, $\lambda_i \neq \lambda_j$. □

Remark 4.12 (Re-synchronization thresholds)*.* A practically important question is: given desynchronized agents, how much additional interaction or coupling is needed to re-synchronize internal-time velocities? Even in simple models, this leads to threshold phenomena where increasing a coupling parameter beyond a critical value causes $\lambda_i - \lambda_j$ to change sign or vanish. Section 5 includes a toy example of such behavior based on a three-agent threshold model.

4.4 Non-ergodic regimes

We now treat non-ergodic regimes more explicitly.

Proposition 4.13 (Internal time in non-ergodic regimes)**.** *Suppose the global chain (X_n) decomposes into a finite or countable family of ergodic classes $\{\mathcal{G}_\ell\}_\ell$ with invariant measures π_ℓ. Let $p_\ell := \mathbb{P}(X_0 \in \mathcal{G}_\ell)$, so that $\sum_\ell p_\ell = 1$. Assume that for each ℓ and each agent i, the additive functional ΔT^i_n is integrable under π_ℓ. Then, conditioned on the event $\{X_0 \in \mathcal{G}_\ell\}$,*

$$\lim_{n\to\infty} \frac{T^i_n}{n} = \lambda^{(\ell)}_i := \mathbb{E}_{\pi_\ell}[\Delta T^i_0] \quad \textit{almost surely.}$$

From the global perspective, the limiting internal-time velocity

$$\lambda_i := \lim_{n\to\infty} \frac{T_n^i}{n}$$

is a random variable taking values in $\{\lambda_i^{(\ell)}\}_\ell$ *with distribution* $\mathbb{P}(\lambda_i = \lambda_i^{(\ell)}) = p_\ell$*, and its expectation is*

$$\mathbb{E}[\lambda_i] = \sum_\ell p_\ell \lambda_i^{(\ell)}.$$

Proof. Apply the ergodic theorem for each ergodic component separately. Conditioning on $\{X_0 \in \mathcal{G}_\ell\}$ reduces the chain to an ergodic Markov chain with invariant measure π_ℓ, and Proposition 4.4 applies. The last claim follows from the law of total expectation. □

Remark 4.14 (Cesàro averages and metastability)*.* In more complex settings with slowly changing network structure or metastable dynamics, effective internal-time velocities may exhibit long plateaus corresponding to quasi-ergodic behavior before eventually converging. Cesàro averages of the form T_n^i/n still converge under the assumptions of Proposition 4.13 but may require very long observation windows for practical estimation.

5 Toy models

We now present simple models illustrating the concepts. These are not intended as realistic cognitive models but as analytically tractable examples where internal-time behavior reflects structure.

5.1 Model I: Two-state internal dynamics with interaction modulation

Consider a single agent with $\mathcal{S}^i = \{0, 1\}$, interpreted as low- and high-activation states, and no messages ($I_n^i \equiv 0$). The state evolves as a two-state Markov chain with transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \qquad p, q \in (0, 1).$$

We take $d_{\text{self}}^i(s, s') = \mathbb{1}\{s \neq s'\}$, so $\Delta_{\text{self},n}^i \in \{0, 1\}$.

We define a simple semantic representation with $\mathcal{Y}^i = \{0, 1\}$, $\mu^i(s, \cdot)$ a Dirac mass at s, and $V^i(y) = y$. Then $\Phi^i(s) = s$ and

$$\Phi^i(S_{n+1}^i) - \Phi^i(S_n^i) = S_{n+1}^i - S_n^i.$$

Direct computation shows that

$$\mathbb{E}\big[\Phi^i(S_{n+1}^i) - \Phi^i(S_n^i) \mid S_n^i = s\big] = (1 - s)p - sq.$$

This can be bounded by a linear function of $\Delta_{\text{self},n}^i$ with appropriate constants, yielding an instance of (3.1). Internal time can be defined via

$$\Delta T_n^i = g^i(\Delta_{\mathrm{self},n}^i, 0) = a_i \Delta_{\mathrm{self},n}^i + d_i.$$

In this simple model, the internal-time velocity is

$$\lambda_i = a_i \, \mathbb{P}_\pi(S_1^i \neq S_0^i) + d_i,$$

where $\mathbb{P}_\pi$ is the stationary law of the two-step transition. This illustrates how purely internal dynamics can generate non-trivial internal time without interaction, corresponding to internal rehearsal or self-driven thought, and clarifies that interaction is not strictly required for time to pass.

5.2 Model II: Linear consensus with noise and internal time

We now consider N agents on a graph $G = (I, E)$ with scalar states $S_n^i \in \mathbb{R}$ and messages equal to neighbors' states. The update rule is

$$S_{n+1}^i = (1-\alpha) S_n^i + \frac{\alpha}{\deg(i)} \sum_{j \sim i} S_n^j + \varepsilon_{n+1}^i,$$

where $\alpha \in (0,1)$, $j \sim i$ denotes neighbors, and ε_{n+1}^i are i.i.d. zero-mean noise terms with finite variance. Thus the system is a noisy linear consensus dynamic; see [11].

We take $d_{\mathrm{self}}^i(s, s') = |s - s'|$ and

$$I_n^i = \frac{1}{\deg(i)} \sum_{j \sim i} |S_n^j - S_n^i|$$

as interaction intensity. A simple choice of internal-time coupling is

$$\Delta T_n^i = g^i(\Delta_{\mathrm{self},n}^i, I_n^i) = a_i \Delta_{\mathrm{self},n}^i + B_i I_n^i + d_i.$$

Under mild conditions (e.g., connected graph, sufficiently small α, non-degenerate noise), the joint process is ergodic, and one can explicitly solve for the stationary covariance of $(S_n^i)_i$ via a Lyapunov equation. Internal-time velocities are then

$$\lambda_i = a_i \mathbb{E}_\pi[\Delta_{\mathrm{self},0}^i] + B_i \mathbb{E}_\pi[I_0^i] + d_i.$$

If the graph and coupling constants are symmetric, we have synchronization (Theorem 4.9). If degrees or coupling constants differ across nodes, then Proposition 4.11 implies typical de-synchronization.

5.3 Model III: Three-agent heterogeneous threshold model

We now present a simple model exhibiting a phase-like transition in synchronization as a coupling parameter crosses a regime.

Consider three agents $I = \{1, 2, 3\}$ arranged in a line: $1 \leftrightarrow 2 \leftrightarrow 3$. Each agent has scalar state $S_n^i \in \mathbb{R}$ and updates according to

$$S_{n+1}^i = S_n^i + \kappa \sum_{j \sim i} (S_n^j - S_n^i) + \varepsilon_{n+1}^i,$$

with coupling parameter $\kappa > 0$ and i.i.d. noise ε_{n+1}^i of mean zero and finite variance. Let

$$\Delta_{\text{self},n}^i = |S_{n+1}^i - S_n^i|, \qquad I_n^i = \sum_{j \sim i} |S_n^j - S_n^i|.$$

We define internal-time coupling with a thresholded interaction term:

$$\Delta T_n^i = a_i \Delta_{\text{self},n}^i + \gamma_i \big(I_n^i - \theta\big)_+ + d_i,$$

where $(x)_+ = \max\{0, x\}$, $\theta > 0$ is a threshold, and a_i, γ_i, d_i are constants. Intuitively, small interaction intensities below θ have no additional effect beyond self-change, while large interactions above θ accelerate internal time.

Assuming κ is small enough and the noise is non-degenerate, the process is ergodic with a unique invariant distribution whose statistics depend smoothly on κ. For each i, the internal-time velocity is

$$\lambda_i(\kappa) = a_i \mathbb{E}_\pi[|S_1^i - S_0^i|] + \gamma_i \mathbb{E}_\pi\left[(I_0^i - \theta)_+\right] + d_i,$$

where π is the stationary distribution depending on κ.

For small κ, neighbor differences $|S_n^j - S_n^i|$ are typically small, so I_0^i lies below θ with high probability and the threshold term is negligible:

$$\lambda_i(\kappa) \approx a_i \mathbb{E}_\pi[|S_1^i - S_0^i|] + d_i.$$

As κ increases toward a moderate value (e.g., κ in a range where neighbor differences fluctuate around θ), the central agent 2 experiences more frequent large neighbor differences, and the term $\mathbb{E}_\pi[(I_0^2 - \theta)_+]$ becomes substantial. This leads to $\lambda_2(\kappa) > \lambda_1(\kappa), \lambda_3(\kappa)$: the central agent's internal time accelerates relative to the edge agents. For larger κ, consensus dynamics again reduce differences, and the threshold term saturates or declines.

Numerical studies of this model can exhibit kink-like changes in the dependence of $\lambda_i(\kappa)$ on κ as the typical value of I_0^i crosses the threshold θ. The construction illustrates how the choice of g^i and the interaction topology can produce phase-like transitions in synchronization behavior within the general framework developed here.

6 Discussion and extensions

6.1 Relation to time-changed Markov processes

In classical theory, a time-changed Markov process is obtained by composing a Markov process (X_t) with an increasing additive functional A_t or a subordinator τ_t [2, 3, 12]. Our internal times (T_n^i) are additive functionals of the discrete-time global Markov chain (X_n), and could in principle be used to define a time-changed process by considering X_{n_k} where $k \mapsto n_k$ is chosen so that $T_{n_k}^i$ advances in approximately equal increments.

However, the emphasis here is not primarily on constructing a new Markov process but on the *constraints* relating the growth of T_n^i to self-change, interaction intensity, and semantic drift. In this sense, the framework can be seen as a structural, multi-agent generalization of additive-functional theory with a particular focus on value-regulated dynamics.

6.2 Interaction intensity, AoI, and information measures

The interaction intensity functional Ψ^i abstracts how much value-relevant interaction an agent experiences per step. As discussed in Remark 2.6, natural choices include metric differences, information-theoretic quantities, and value-weighted AoI.

To make the AoI connection more concrete, suppose each message k received by agent i has generation time τ_k and value weight $w_k \geq 0$, and define the value-weighted AoI at time n as

$$\mathrm{VWAoI}_n^i = \sum_{k:\, \tau_k \leq n} w_k (n - \tau_k).$$

If we encode the set of (w_k, τ_k) pairs into M_n^i and define

$$I_n^i := \Psi^i(S_n^i, M_n^i) = \mathrm{VWAoI}_n^i,$$

then interaction intensity directly measures a value-weighted staleness of information. In such a setting, the semantic drift inequality (3.1) ties the evolution of Φ^i to a combination of self-change and value-weighted staleness, and Theorem 4.5 bounds internal-time velocity by long-run averages of these quantities.

In networked systems, AoI metrics quantify freshness of information at receivers, while internal time adds a complementary layer: instead of measuring elapsed time since last update, it measures *cumulated processing effort* weighted by self-change and value-relevant interaction. In systems where updates are costly, internal time could serve as a scheduling signal: agents whose internal time has advanced rapidly may be temporarily deprioritized for additional updates or communication, while those with slow internal time could be prioritized. Optimality of such policies is not analyzed here; rather, internal time is highlighted as a principled local statistic that can inform such decisions.

6.3 Connections to subjective time and active inference

Subjective-time experiments show that perceived duration depends on arousal, stimulus complexity, and cognitive load. In the present framework, these factors can be encoded in Δ^i_{self} and I^i_n: intense or complex stimuli cause larger self-change and interaction intensity, leading to faster internal time if g^i is monotone. Conversely, monotonous intervals with low interaction intensity produce slow internal-time accumulation, aligning with subjective reports of time "dragging".

Active inference models treat perception and action as iterative minimization of a free-energy functional [6, 7]. If Φ^i is taken to be an expectation of such a free energy under μ^i, then semantic drift inequalities of the form (3.1) can be interpreted as expressing that self-change and interaction drive down expected free energy on average. Internal time then counts the rate at which value-relevant updates occur, providing a notion of intrinsic tempo for active inference agents.

6.4 Applications

We briefly sketch potential applications.

Distributed learning and resource allocation. In large-scale multi-agent or federated learning, internal time could be used as a local estimate of how much value-relevant computation an agent has performed. For example, consider a federated learning scenario with N workers. Each worker i maintains a local parameter θ^i_n and periodically receives aggregated gradients from neighbors. We may set

$$S^i_n := (\theta^i_n, \text{local data statistics}), \quad \Delta^i_{\text{self},n} := \|\theta^i_{n+1} - \theta^i_n\|,$$

$$I^i_n := \sum_{j\sim i} \|\nabla_j - \bar{\nabla}^i\|, \quad \Phi^i(s^i) := -\mathcal{L}^i(\theta^i),$$

where ∇_j is a gradient from neighbor j, $\bar{\nabla}^i$ a local aggregate, and $\mathcal{L}^i$ a local loss. If learning rates are chosen so that gradient steps reduce loss on average, then a drift inequality of the form (3.1) holds. Internal-time velocity λ_i then reflects the rate of value-relevant model improvement for worker i, and could be used to schedule communication rounds: workers with λ_i above a threshold may skip some rounds to reduce communication overhead, while those with low λ_i may be granted more frequent synchronization.

Human-AI collaboration. Human subjective time and AI internal time can become mismatched, leading to misaligned expectations. By designing AI agents whose internal time takes into account interaction intensity with human users and semantic value of those interactions, one could devise protocols where both partners maintain compatible temporal expectations-for example, by signaling when the AI's internal time has advanced sufficiently to warrant a summary, suggestion, or check-in.

Social simulation and opinion dynamics. In opinion-dynamics models (e.g., voter models, bounded-confidence models), equipping each agent with internal time allows one to model heterogeneous tempos of opinion updating. Agents embedded in dense, value-relevant interaction networks may experience faster internal time and thus update their opinions more frequently, while those at the periphery evolve more slowly. This could lead to new phenomena in consensus formation and polarization, especially when coupled with value alignment and semantic drift.

6.5 Mathematical and computational extensions

Several extensions of the present framework are mathematically natural.

High-probability and large-deviation bounds. The inequalities in Theorem 4.5 are in expectation or almost-sure averages. One could seek high-probability bounds or large-deviation principles for T_n^i/n under stronger mixing conditions, providing quantitative control on fluctuations of internal-time velocity.

Multiple internal times per agent. Agents may maintain different internal times for different functions, such as short-term working memory and long-term consolidation. This suggests internal-time vectors $(T_n^{i,(k)})_k$ with coupling functions $g^{i,(k)}$ and possibly different drift inequalities, leading to a richer multi-timescale theory.

Dynamic interaction graphs. In many systems, the interaction graph G changes over time (e.g., social networks, mobile ad-hoc networks). Incorporating a stochastic process for G_n into the framework would allow the study of how evolving connectivity patterns affect internal-time synchronization and de-synchronization.

Continuous-time limits. Consider a rescaled process where the discrete step size is $\Delta t = 1/n$ and the dynamics are accelerated appropriately. Under suitable scaling of Δ_{self}^i and I_n^i, the internal-time process may converge to a continuous additive functional

$$T_t^i = \int_0^t a^i(X_s)\, ds,$$

where a^i is a non-negative intensity process. The drift inequality (3.1) becomes a stochastic differential inequality for $\Phi^i(X_t)$, and social proper-time velocities correspond to

$$\lim_{t\to\infty} \frac{T_t^i}{t} = \mathbb{E}_\pi[a^i(X)]$$

under an invariant measure π. This connects directly to the theory of time-changed diffusions and subordinators [3,12], and to scaling-limit results for Markov chains converging to diffusions.

Computational aspects. Estimating λ_i in large systems requires approximating long-run averages of ΔT_n^i. Even for finite-state Markov chains, exact computation of invariant measures and Lyapunov functions can be costly in the dimension of the state space. In practice, Monte Carlo estimation combined with structure-exploiting approximations (e.g., mean-field limits or low-rank representations of interaction networks) will be needed. Internal time nevertheless provides a well-defined target quantity for such approximations and for algorithm design focused on local statistics. Developing efficient computational methods for estimating internal-time velocities in large-scale systems lies outside the scope of the present work, but the framework here provides a clear mathematical target for such methods.

Conclusion

We have introduced a general framework for internal time in multi-agent self-modifying systems, where internal time is constructed from self-change and axiomatized interaction intensity, constrained by semantically realized value potentials. Under ergodicity assumptions, internal-time velocities exist and obey natural-law inequalities that couple them to long-run averages of self-change, interaction, and semantic drift. In value-aligned and symmetric systems, internal time synchronizes; in heterogeneous systems, de-synchronization is typical.

The analysis shows that internal time can be treated as a mathematically precise, value-regulated quantity rather than a purely metaphorical construct, opening the way to applications in distributed learning, human-AI collaboration, and social simulations. Many extensions-including multi-timescale internal times, dynamic networks, continuous-time limits, and computational methods for estimating internal-time velocities-remain open.

References

[1] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability.* 2nd edition, Cambridge University Press, 2009.

[2] E. B. Dynkin. *Markov Processes*, Volumes I and II. Springer, 1965.

[3] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* 3rd edition, Springer, 1999.

[4] Y. Sun, I. Kadota, R. Talak, and E. Modiano. Age of Information: A new metric for information freshness. *Synthesis Lectures on Communication Networks*, 12(2):1-224, 2019.

[5] R. D. Yates, Y. Sun, D. R. Brown III, S. K. Kaul, E. Modiano, and S. Ulukus. Age of Information: An introduction and survey. *IEEE Journal on Selected Areas in Communications*, 39(5):1183-1210, 2021.

[6] K. J. Friston, J. Daunizeau, and S. J. Kiebel. Reinforcement learning or active inference? *PLOS ONE*, 4(7):e6421, 2009.

[7] Z. Fountas, A. Sylaidi, K. Nikiforou, A. K. Seth, M. Shanahan, and W. Roseboom. A predictive processing model of episodic memory and time perception. *Neural Computation*, 34(7):1501-1544, 2022.

[8] J. Gibbon. Scalar expectancy theory and Weber's law in animal timing. *Psychological Review*, 84(3):279-325, 1977.

[9] M. Treisman. Temporal discrimination and the indifference interval: Implications for a model of the "internal clock". *Psychological Monographs: General and Applied*, 77(13):1-31, 1963.

[10] M. J. Allman and W. H. Meck. Pathophysiological distortions in time perception and timed performance. *Brain*, 135(3):656-677, 2012.

[11] F. Bullo. *Lectures on Network Systems*. Version 1.7, 2022. Available online at http://motion.me.ucsb.edu/book-lns.

[12] S. N. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley, 1986.