

# Universal $\pm 1$ Congruence Speed Invariant in Any Numeral System

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*A self-contained statement of the most elegant constant congruence speed identity characterizing tetration in all numeral systems with squarefree radix  $r > 2$ .*

Let  $r > 2$ ,  $a > 1$ ,  $b > 1$ ,  $c > 0$ ,  $k \geq 0$  be integers, and consider the radix- $r$  numeral system. Denote by  $\nu_r(c)$  the maximum number of times  $r$  divides  $c$  (i.e.,  $\nu_r(c)$  equals the maximum integer  $m$  such that  $r^m$  divides  $c$ , and  $\nu_r$  is a valuation when  $r$  is prime) and let  $\text{rad}(r) := \prod_{p|r} p$  denote the product of the distinct prime factors of  $r$ .

Let

$${}_b a := \underbrace{a^{a^{\cdots a}}}_{b \text{ times}}$$

be the height- $b$  tetration of  $a$  (e.g.,  ${}_4 3 = 3^{7625597484987}$ ).

Let  $r$  be given and assume that  $s_b$  is the largest integer such that

$${}^{b+1}a \equiv {}^b a \pmod{r^{s_b}} \quad \text{and} \quad {}^{b+1}a \not\equiv {}^b a \pmod{r^{s_b+1}}.$$

We define the *radix- $r$  congruence speed of  $a$  at height  $b$*  as

$$V_b^{[r]}(a) := s_b - s_{b-1}.$$

Then, for every integer  $b > 1$ , for every integer  $c > 1$ , for every squarefree integer  $r > 2$  (or even for non-squarefree  $r > 2$  such that  $\text{rad}(r) \nmid c$  or  $c \mid r$ ), for every integer  $k \geq 0$ , and for every integer  $t > \nu_r(c) + 1$ , the following identity holds:

$$V_b^{[r]}((k \cdot r^{t+1} + r^{t-\nu_r(c)} + 1)^c) = t = V_b^{[r]}((k \cdot r^{t+1} + r^{t-\nu_r(c)} - 1)^c). \quad (1)$$

Hence, we can compactly rewrite the universal identity (1) as

$$\boxed{V_b^{[r]}((k \cdot r^{t+1} + r^{t-\nu_r(c)} \pm 1)^c) = t.} \quad (2)$$

Since, in (2), the “+1” case arises from the  $r$ -adic solution  $1_r$  of  $y^3 = y$  while the “−1” case corresponds to its symmetric solution  $-1_r$ , and both are valid in any ring of  $r$ -adic integers ( $r > 1$ ),

$$V_b^{[r]}(r^t \pm 1) = V_2^{[r]}(r^t \pm 1) = t \quad (3)$$

is true in each radix- $r$  numeral system as long as  $r$  and  $t$  are integers greater than 1 (we note that if  $t \geq 2$ , (3) is also true for the binary numeral system).

If  $3 \leq r = c$  and  $k = 0$  are given, from (2), we conclude that

$$V_b^{[r]}((r^{t-1} + 1)^r) = V_b^{[r]}((r^{t-1} - 1)^r) = t \quad (4)$$

is true for all integers  $b > 1$  and  $t > 2$ .