

## V.—SYMBOLIC REASONING (VII.).<sup>1</sup>

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### SYLLOGISTIC VALIDITY.

1. THE validity tests of the traditional logic turn mainly upon the question whether or not a syllogistic "term" (i.e., class X, Y or Z) is "distributed" or "undistributed". In ordinary language these words rarely, if ever, lead to any ambiguity or confusion of thought; but logicians have somehow managed to work them into a perplexing tangle. In the proposition "All X is Y," the class X is said to be "distributed," and the class Y "undistributed". In the proposition "No X is Y," the class X and the class Y are said to be both "distributed". In the proposition "Some X is Y," the class X and the class Y are said to be both "undistributed". Finally, in the proposition "Some X is not Y," the class X is said to be "undistributed" and the class Y "distributed".

2. Let us examine some consequences of this tangle of technicalities. Take the leading syllogism Barbara, the validity of which no one will question, provided it be expressed in its *conditional* form, namely, "If all Y is Z and all X is Y, then all X is Z". Being admittedly valid, this syllogism must hold good whatever values (or meanings) we give to its constituents X, Y, Z. It must therefore hold good (as every logician will surely admit) when X, Y and Z are synonyms, and therefore all denote *the same class*. In this case also the two premisses and the conclusion will be three truisms which no one would dream of denying. Consider now one of these truisms, say "All X is Y". Here, by the usual logical convention, the class X is said to be "distributed," and the class Y "undistributed". But when X and Y are synonyms, they denote *the same class*, so that the same class may, at the same time and in the same proposition, be both "*distributed*" and "*undistributed*". Does not this sound like a contradiction? Speaking of a certain

<sup>1</sup> For VI., see MIND, January, 1905.

concrete collection of apples in a certain concrete basket, can we consistently and in the same breath assert that "All the apples are already *distributed*" and that "All the apples are still *undistributed*"? Do we get out of the dilemma and secure consistency if on every apple in the basket we stick a ticket X and also a ticket Y? Can we then consistently assert that all the X apples are *distributed*, but that all the Y apples are *undistributed*? Clearly not, for every X apple is also a Y apple, and every Y apple an X apple. In ordinary language the classes which we can respectively qualify as *distributed* and *undistributed* are mutually exclusive; in the logic of our text-books this is evidently not the case. Students of the traditional logic should therefore disabuse their minds of the idea that the words "distributed" and "undistributed" of their text-books necessarily refer to classes mutually exclusive, as they do in everyday speech; or that there is anything but a forced and fanciful connexion between the "distributed" and "undistributed" of current English and the technical "distributed" and "undistributed" of logicians.

3. To make the traditional logic symmetrical, as well as more widely applicable, we must extend our *Symbolic Universe*, or 'Universe of Discourse,' so as to include not only the three syllogistic classes X, Y, Z, but also their *complementary* classes 'X, 'Y, 'Z (see my preceding paper); these being so related to the former that if we take any class X and its complement 'X, the two are, on the one hand, mutually exclusive, and, on the other, make up together the whole symbolic universe S. That is to say, all the  $m$  individuals  $X_1, X_2, X_3$ , etc., of the class X, together with the  $n$  individuals 'X<sub>1</sub>, 'X<sub>2</sub>, 'X<sub>3</sub>, etc., of the class 'X, make up the  $m + n$  individuals  $S_1, S_2, S_3$ , etc., which constitute the whole symbolic universe S. The same thing may, of course, be said of any other two complementary classes Y and 'Y. It is evident that if any two positive classes X and Y are mutually exclusive, the complementary negative classes 'X and 'Y overlap; and, *vice versa*, if 'X and 'Y are mutually exclusive, X and Y overlap. With this convention it follows that if S be any individual taken at random out of our Symbolic Universe (or Universe of Discourse)  $S_1, S_2, S_3$ , etc., neither the statement  $S^X$  nor the statement  $S'^X$  is impossible, and neither of them is a certainty; that is to say, we shall always have  $x^0$  and  $(x')^0$ , and never  $x^1$  nor  $x'$ , where  $x$  means  $S^X$  and  $x'$  means  $S'^X$  or its synonym  $S^{-X}$ . In other words, simplicity, logical consistency, and unrestricted generality require the convention that the recognition of any class A in our *Symbolic Universe* necessitates also the recognition in

it of the complementary class 'A; each forming a portion only, and both constituting the whole.

4. With these conventions we shall always have

(1) All X is Y =  $(xy)^\eta$ ; (2) No X is Y =  $(xy)^\eta$ ;

(3) Some X is Y =  $(xy)^{-\eta}$ ; (4) Some X is not Y =  $(xy)^{-\eta}$ .

When, in any of these four propositions, a letter  $x$  or  $y$  is affected by *one* negation (only) or by *three* negations, the class (X or Y) to which it refers is said (in text-book language) to be "distributed"; but if it is affected by *two* negations (only), it is said to be "undistributed". Let the symbol  $X^d$  assert that X is (in the text-book sense) "distributed," while  $X^u$  asserts that X is "undistributed". In (1) we have  $X^d Y^\eta$ ; for here  $x$  is affected by *one* negation only, namely, the exponent or predicate  $\eta$  outside the bracket, while  $y$  is affected by *two* negations, namely, the exponent  $\eta$  outside the bracket and the accent of denial inside the bracket. In (2) we have evidently  $X^d Y^d$ . In (3) we have  $X^u Y^\eta$ ; for here each letter is affected by *two* negations (only), namely, the negation  $\eta$  and the *minus* sign preceding it. In (4) we have  $X^u Y^d$ ; for here  $x$  is affected by *two* negations (only), namely, the negation  $\eta$  and the *minus* sign preceding it; while  $y$  is affected by *three* negations, namely, the negation  $\eta$ , the *minus* sign, and the accent inside the bracket. Thus *one* and *three* negations indicate a class "distributed"; while *two* indicates a class "*undistributed*". Evidently  $X^d$  implies  $(X)^\eta$ , and  $X^u$  implies  $(X)^d$ .

5. If we change  $y$  into  $x$  in proposition (1) of § 4, we get

All X is  $\bar{X} = (xx)^\eta$ .

Here we have  $X^d X^u$ . This shows that there is no necessary antagonism between  $X^d$  and  $X^u$ ; that (in the text-book sense) the same class may be both "distributed" and "undistributed" at the same time.

6. Instead of the six customary canons of the traditional logic, some of which are not quite reliable, and others not quite self-consistent, I propose the following methods of testing the validity of syllogisms:—

Let *unaccented* capitals denote *implications*, and let *accented* capitals denote *non-implications* (or the denials of implications). Thus, if A denote the implication  $x : y$ , then A' will denote its denial, the non-implication  $(x : y)'$ . Let A, B, C denote any syllogistic implications, while A', B', C' denote their respective denials. Every valid syllogism in *general* logic (with *unrestricted* values of  $x, y, z$ ), and every valid syllogism of the *traditional* logic, except Darapti, Felapton, Fesapo and Bramantip, must have one or other of the two forms

(1) AB : C, (2) AB' : C'.

That is to say, either the two premisses and the conclusion are all three implications (or "universals"), as in (1); or else one premiss only and the conclusion are both non-implications. If, in *general* logic, any syllogism fails to comply with one or other of these two forms, it can only be valid conditionally; that is to say, it will not be a formal certainty for all values  $\epsilon$  or  $\eta$  or  $\theta$  of  $x, y, z$ . The second form may be reduced to the first form by transposing the premiss  $B'$  and the conclusion  $C'$ , and changing their signs; for  $AB' : C'$  is equivalent to  $AC : B$ . When thus transformed, the validity of  $AB' : C'$ , that is of  $AC : B$ , may be tested in the same way as the validity of  $AB : C$ . The test is easy. Suppose the conclusion  $C$  to be  $x : z$ , in which  $z$  may be affirmative or negative. If, for example,  $z = \text{He is a soldier}$ ; then  $z' = \text{He is not a soldier}$ . But if  $z = \text{He is not a soldier}$ ; then  $z' = \text{He is a soldier}$ . The conclusion  $C$  being, by hypothesis,  $x : z$ , the syllogism  $AB : C$ , if valid, becomes either

$$(x : y : z) : (x : z)$$

or else

$$(x : y' : z) : (x : z),$$

in which  $y$  refers to the class  $Y$  (or "middle term") not mentioned in the conclusion  $x : z$ .

7. To take a concrete example, let it be required to test the validity of

$$(y : z') (y : x') : (x : z').$$

Let  $Q$  denote this syllogism. Transposing the non-implications, we get

$$\begin{aligned} Q &= (y : z') (x : z) : (y : x'), \\ &= (y : z') (z' : x') : (y : x'), \\ &= (y : z' : x') : (y : x'). \end{aligned}$$

Thus,  $Q$  satisfies the necessary condition of validity as laid down in § 6. It is therefore valid both in *general* logic and in the *traditional* logic.

8. As an instance of a *non-valid* syllogism of the form  $AB : C$ , we may give

$$(x : y') (y : z') : (x : z');$$

for since the  $y$ 's in the two premisses have *different signs*, the one being negative and the other affirmative, the combined premisses can neither take the form  $x : y : z'$  nor the form  $x : y' : z'$ , which are respective abbreviations for  $(x : y) (y : z')$  and  $(x : y') (y' : z')$ . The syllogism is therefore not valid.

9. Syllogisms of the form  $AB : C'$  include *Darapti*, *Felapton*, *Fesapo* and *Bramantip*. In *general* logic these are not formal certainties, and are therefore only valid conditionally. With the conventions of § 3, however, the required conditions sometimes hold good in the traditional logic. The following is a

valid syllogism, though the traditional logic would not recognise it as valid because it violates the canon that "the middle term must be distributed at least once in the premisses":—

If all who approve of *Protection* are *Conservatives*, and all who approve of fiscal *Retaliation* are also *Conservatives*; then somebody (one person at least) who does not approve of fiscal *Retaliation* does not approve of *Protection*.

Speaking of an individual taken at random from our 'Universe of Discourse,' let  $P$  = "He approves of *Protection*"; let  $C$  = "He is a *Conservative*"; and let  $R$  = "He approves of fiscal *Retaliation*". Then, for our two premisses we have  $(P : C)(R : C)$ , and for our conclusion  $(R' : P)$ . Thus, putting  $\phi$  for the whole syllogism, we have

$$\phi = (P : C)(R : C) : (R' : P),$$

which may be read "If  $P$  implies  $C$ , and  $R$  also implies  $C$ ; then  $R'$  does not imply  $P$ ". In other words, the conclusion asserts that "One may disapprove of fiscal *Retaliation* without approving of *Protection*".

Now, let us first consider the formula  $\phi$  from the standpoint of *general* logic, in which the symbols  $P$ ,  $C$ ,  $R$  may denote any statements whatever, possible or impossible. By a general method (which it would take too long here to explain) for testing the validity of formulæ, it may be shown that the formula  $\phi$  fails in the case  $C'(R + P)^*$ . The failure may be easily verified as follows. We have

$$\begin{aligned}\phi &= (P + R : C) : (R'P)^{-} \\ &= (R + P : C) : (R + P)^{-}.\end{aligned}$$

Now, suppose  $C = \epsilon_1$  and  $R + P = \epsilon_2$ . We get (see preceding paper)

$$\phi = (\epsilon_2 : \epsilon_1) : \epsilon_2^{-} = \epsilon_3 : \epsilon_2^{-} = \epsilon_3 : \eta_1 = \eta_2.$$

This shows that in *general* logic  $\phi$  fails in the case  $C'(R + P)^*$ . Let  $\phi_1$  now denote the general syllogism  $\phi$  when restricted to the *concrete* example about "conservatives," "retaliation," "protection". The failure of the general formula  $\phi$ , with *unrestricted* values of  $C$ ,  $R$ ,  $P$ , does not necessarily involve the failure of  $\phi_1$ , in which the values (or meanings) of the statements  $C$ ,  $R$ ,  $P$  are *restricted* by the condition  $C'R'P^*$  (see § 3). The *restriction*  $C'$  of the *traditional* logic is inconsistent with the factor  $C'$  in the failure case of *general* logic. Thus we have  $\phi^{-} \phi_1^*$ , which asserts that the particular formula  $\phi_1$  is valid though the general formula  $\phi$  (with unrestricted values of  $C$ ,  $R$ ,  $P$ ) is not. To assert that the concrete syllogism  $\phi_1$  fails in the case  $C'(R + P)^*$  would be to assert that it fails on the supposition that *everybody* is a *Conservative*, and that also *everybody* is either a *Retaliationist*

or a *Protectionist*. This supposition is not only contrary to fact, but, by the convention of § 3, it cannot even *arise* in the traditional logic, since the existence of C (conservatives) in our 'Universe of Discourse' necessarily implies the existence (the *symbolic* existence) also of 'C (non-conservatives) in the same universe, even if the latter should be mere "men of straw".

10. The syllogism discussed in § 9 may also be expressed in the form

$$(PC')^*(RC')^*(R'P')^{-*}.$$

Let  $C' = y$ , let  $P = z'$ , and let  $R = x'$ . It then becomes

$$(yz')^*(yx')^*(xz)^{-*}$$

which is equivalent to

$$(y : z)(y : x) : (x : z')',$$

and may be read "If all *non-conservatives* (Y) are *non-protectionists* (Z), and all *non-conservatives* (Y) are also *non-retaliationists* (X); then some *non-retaliationist* (one at least) is a *non-protectionist*. In this form it becomes a valid syllogism of the type Darapti, with the "middle term" no longer "undistributed". Yet the two syllogisms are necessarily equivalent since  $\alpha : \beta$  is always equivalent to  $\beta' : \alpha'$ ; so that the canon about "middle term" distribution refuses admittance to a syllogism when it presents itself under one form, but lets it pass as valid when it disguises itself under another.

11. Consider the syllogism "If all *non-existences* are *fictitious*, and all *non-existences* are represented by the symbol *zero*; then some *fictitious* things (or thing) are represented by the symbol *zero*".

Speaking of something S taken at random out of our Symbolic Universe, or Universe of Discourse, let the symbol 0 denote the statement "It is *non-existent*"; let  $f$  denote "It is *fictitious*"; and let  $z$  denote "It is represented by the symbol *zero*". Putting  $\phi$  for the syllogism, we have

$$\begin{aligned}\phi &= (0 : f)(0 : z) : (f : z')' \\ &= (0 : fz) : (fz : \eta)'\end{aligned}$$

Now, since the Symbolic Universe contains both real existences ( $\epsilon$ ) and non-existences (0), the statement 0 (which is short for  $S^0$ ) is not impossible, so that  $0^{-*}$ , or its synonym  $(0 : \eta)'$ , is always understood among our data, and may be expressed whenever convenient. Hence, we get

$$\begin{aligned}\phi &= (0 : fz)(0 : \eta)' : (fz : \eta)' \\ &= (0 : fz)(fz : \eta) : (0 : \eta), \text{ by transposing.}\end{aligned}$$

Thus, the syllogism  $\phi$ , like all valid syllogisms, comes ultimately under the formula  $(\alpha : \beta)(\beta : \gamma) : (\alpha : \gamma)$ , and is therefore a formal certainty.

12. Next, take the syllogism "If no *centaurs* are really

*existent*, and no *fairies* are really *existent*; then some things (or thing) that are not *centaurs* are not *fairies*".

This syllogism is perfectly valid, though, in the above form, it violates the traditional canon that no conclusion can be drawn from negative premisses.

Speaking of an entity  $S$  taken at random out of our Symbolic Universe, let  $c =$  "It is a *centaur*"; let  $e =$  "It *exists* really"; and let  $f =$  "It is a *fairy*". Also let  $\phi$  denote the syllogism. We have

$$\begin{aligned}\phi &= (c : e') (f : e') : (c' : f)' = (e : c) (e : f') : (c' : f)' \\ &= (e : c') : (c' : f)' = (e : c') (e : \eta)' : (c' : f)' \\ &= (e : c') (c' : f)' : (e : \eta)' \text{ by transposition;}\end{aligned}$$

so that here also  $\phi$  is a formal certainty, as it is a particular case of the formula  $(a : \beta) (\beta : \gamma) : (a : \gamma)$ . The factor  $(e : \eta)'$ , which we introduced in the last complex implication but one, is, of course, understood throughout; as, by our convention of a 'Symbolic Universe,' the statement  $e$  (which is short for  $S^e$ ) is not impossible, just as its denial, the statement  $e'$ , or its synonym  $0$  or  $S^0$ , is not impossible. For our convention (see § 3) implies both  $e^e$  and  $0^e$ , which respectively imply  $e^{-\eta}$  and  $0^{-\eta}$ , these last being synonymous with  $(e : \eta)'$  and  $(0 : \eta)'$ .

13. The three common-sense canons of the traditional logic, (1) that "All  $X$  is  $Y$ " implies "Some  $X$  is  $Y$ ," (2) that "No  $X$  is  $Y$ " implies "Some  $X$  is not  $Y$ ," and (3) that "All  $X$  is  $Y$ " and "No  $X$  is  $Y$ " are incompatible have been (somewhat paradoxically) called in question by some logicians; but on the assumption of a Symbolic Universe, including, when needed, both the real existences  $e_1, e_2, e_3$ , etc., and the non-existences or unrealities  $0_1, 0_2, 0_3$ , etc., they can be formally proved as follows:—

Speaking of an entity  $S$ , taken at random out of our Symbolic Universe,  $S_1, S_2, S_3$ , etc., let  $x$  and  $y$  respectively denote the two statements, "It belongs to the class  $X$ " and "It belongs to the class  $Y$ ". Let  $\phi_1, \phi_2, \phi_3$  denote the three canons respectively. We have

$$\begin{aligned}\phi_1 &= (x : y) : (x : y')' = (xy)^\eta : (xy)^{-\eta} = (xy)^\eta (xy)^\eta : \eta \\ &= (xy' : \eta) (xy : \eta) : \eta = (xy' + xy : \eta) : \eta \\ &= \{x(y' + y) : \eta\} : \eta = (x\epsilon : \eta) : \eta = (x : \eta) : \eta \\ &= (\theta : \eta) : \eta = \eta : \eta = \epsilon.\end{aligned}$$

This proof is, of course, far fuller than is necessary, and I only give it thus in all its details to show how the same statement may sometimes be presented under different forms. The canon  $\phi_2$  may be proved similarly by merely interchanging  $y$  and  $y'$ . To prove  $\phi_3$ , we have

$$\phi_3 = (x : y) (x : y') : \eta = (xy)^\eta (xy)^\eta : \eta,$$

which (as already shown) is equivalent to  $\phi_1$ .

14. As already explained, my symbolic system assumes that any class  $A$  may be divided into individuals, or mutually exclusive divisions,  $A_1, A_2, A_3$ , etc. This convention may, when convenient or necessary, be carried farther. Any of these divisions, say  $A_8$ , may again be subdivided into  $(A_8)_1, (A_8)_2, (A_8)_3$ , etc.; which, to avoid brackets, may be denoted respectively by  $A_{8.1}, A_{8.2}, A_{8.3}$ , etc. And these again may be subdivided, so that, taking any subdivision  $A_{8.3}$ , we may have  $A_{8.3.1}, A_{8.3.2}, A_{8.3.3}$ , etc., *ad libitum*.

15. The convention that the symbol of non-existence  $0$  should be treated as a class symbol just like others, and that this supposed indivisible atom of symbolic logic may after all be broken up into individuals or elements  $0_1, 0_2, 0_3$ , etc., like any other class symbol, appears to have caused as much astonishment as the modern discovery (or convention?) that the physical atom may be broken up into electrons. But the convention will help us to get rid of some too hastily accepted paradoxes, not only in logic but also in mathematics, in metaphysics, and even in physics. We are all more or less subject to a certain mental disorder which, for want of a better name, I may call *symbolatry*. We mistake the symbol for the reality. We worship the formulæ of our own invention as if they were living oracles whose infallibility it would be impious to call in question. Yet all formulæ, being founded more or less on arbitrary conventions or definitions, must necessarily have their limits of applicability. When we force them beyond those limits, as we too often do, they evolve strange paradoxes, which some eminent logicians and mathematicians accept with surprising readiness, born of over-confidence in symbolic reasoning, but which the plain man of common sense stubbornly refuses to believe. Having myself been a victim more than once to symbolic hallucination, I have now become thoroughly sceptical. When rigorous symbolic reasoning brings me face to face with a startling paradox, I carefully scrutinise the fundamental assumptions, including definitions and conventions, in search of some lurking ambiguity; and, in nine cases out of ten, the search is successful. That is why I cannot accept some of the paradoxes of the non-Euclidean geometry, such as that "two straight lines may enclose a space," and that "a point moving always in a straight line and in the same direction may, finally, after an infinitely long journey, find itself at the point of starting". The path of the moving point may be "straight" *symbolically*, but it can hardly be so *really*.