Note on the Equation $y^2 = x(x^4-1)$. *By* W. Burnside.

Received November 1st, and read November 10th, 1892.

The Cartesian co-ordinates of a point on a curve whose deficiency is greater than one, cannot be expressed as rational or as elliptic functions of a single parameter; but it has recently been shown that, in a large number of cases, including certainly all curves whose equations are of hyper-elliptic form with real coefficients, the co-ordinates of a variable point on the curve can be expressed as uniform functions of a single parameter with an infinite number of essentially singular points, the functions being automorphic with respect to a certain group of substitutions.

Although, however, the possibility of this mode of representing the coefficients has been proved, no instance has ever been given, so far as I know, in which the representation has actually been carried out for a particular equation. Indeed, to do this it would generally be necessary to find a group having the required relation to the equation, and there is, I believe, no known method of doing this. If, however, from independent considerations, the relation in question between a given equation and a known group has once been established, it will become a matter of calculation to carry out the process referred to.

In the case of the equation

$$y^2 = x(x^4-1) \qquad \text{(i.)},$$

the actual expressions for x and y, as functions of a single parameter, can, by such considerations as those just mentioned, be obtained with remarkably simple analysis, and the result is perhaps of sufficient interest to justify me in communicating it to the Society.

It is convenient to begin by giving two well-known formulæ in elliptic functions that will be required. These are

$$\wp 2u - e = \left[\frac{\wp^2 u - 2e\wp u - e^2 - e'e''}{\wp' u}\right]^2,$$

and

$$\wp(u+\omega) - e = \frac{(e-e')(e-e'')}{\wp u - e};$$

whence, by differentiation,

$$\frac{\wp'(u+\omega)}{\wp' u} = -\frac{(e-e')(e-e'')}{(\wp u - e)^2}.$$

If 2ω, $2\omega'$ are any pair of primitive periods of the elliptic functions, these formulæ are equivalent to

$$\wp 2u - \wp\omega = \left\{ \frac{\left(\wp u - \wp\frac{\omega}{2}\right)\left[\wp u - \wp\left(\omega' + \frac{\omega}{2}\right)\right]}{\wp'(u)} \right\}^2 \quad \text{......(ii.)}$$

and $$\frac{\wp'(u+\omega)}{\wp' u} = -\left\{ \frac{\wp\frac{\omega}{2} - \wp\omega}{\wp u - \wp\omega} \right\}^2 \quad \text{..................(iii.).}$$

In Klein-Fricke, *Theorie der Elliptischen Modulfunctionen*, I., p. 652, it is shown that the x and y of equation (i.) are expressible as modular functions for that sub-group of the modular group which is formed of all substitutions

$$\omega, \left(\frac{\alpha\omega+\beta}{\gamma\omega+\delta}\right), \quad \alpha\delta - \beta\gamma = 1,$$

such that $\begin{pmatrix}\alpha, \beta \\ \gamma, \delta\end{pmatrix}$ is congruent to modulus 8 with one of the forms

$$\begin{pmatrix}1, 0 \\ 0, 1\end{pmatrix}, \quad \begin{pmatrix}1, 4 \\ 4, 1\end{pmatrix}, \quad \begin{pmatrix}5, 4 \\ 0, 5\end{pmatrix}, \quad \begin{pmatrix}5, 0 \\ 4, 5\end{pmatrix};$$

and moreover that x is then that modular function which Prof. Klein calls the octahedral irrationality. It does not, however, enter into the plan of Prof. Klein's work to exhibit x and y explicitly as functions of ω, which is the object of the present note.

There is no difficulty in showing that the twenty-four values of Prof. Klein's octahedral irrationality are given in terms of elliptic functions of submultiples of the periods by the expression

$$\frac{\wp\frac{\omega}{2} - \wp\omega}{\wp\frac{\omega'}{2} - \wp\omega},$$

and the twenty-three expressions derivable from it by the substitutions

$$\omega_1 = \omega + \omega', \quad \omega_1' = \omega'$$

and $$\omega_2 = -\omega', \quad \omega_2' = \omega.$$

If now $$x = \frac{\wp\frac{\omega}{2} - \wp\omega}{\wp\frac{\omega'}{2} - \wp\omega},$$

then, by (iii.),
$$x^2 = -\frac{\wp'\left(\frac{\omega'}{2}+\omega\right)}{\wp'\frac{\omega'}{2}}.$$

The addition equation gives
$$\wp\omega'+2\wp\frac{\omega'}{2} = \frac{1}{4}\frac{\wp'^2\frac{\omega'}{2}}{\left(\wp\omega'-\wp\frac{\omega'}{2}\right)^2},$$

and
$$\wp\omega'+2\wp\left(\frac{\omega'}{2}+\omega\right) = \frac{1}{4}\frac{\wp'^2\left(\frac{\omega'}{2}+\omega\right)}{\left[\wp\omega'-\wp\left(\frac{\omega'}{2}+\omega\right)\right]^2},$$

or since, as may be deduced at once from (ii.),
$$2\wp\omega' = \wp\frac{\omega'}{2}+\wp\left(\frac{\omega'}{2}+\omega\right),$$
$$\wp\omega'+2\wp\left(\frac{\omega'}{2}+\omega\right) = \frac{1}{4}\frac{\wp'^2\left(\frac{\omega'}{2}+\omega\right)}{\left[\wp\omega'-\wp\frac{\omega'}{2}\right]^2}.$$

Hence
$$x^4 = \frac{\wp\omega'+2\wp\left(\frac{\omega'}{2}+\omega\right)}{\wp\omega'+2\wp\frac{\omega'}{2}},$$

and
$$x^4-1 = 2\frac{\wp\left(\frac{\omega'}{2}+\omega\right)-\wp\frac{\omega'}{2}}{\wp\omega'+2\wp\frac{\omega'}{2}}$$
$$= 4\frac{\wp\omega'-\wp\frac{\omega'}{2}}{\wp\omega'+2\wp\frac{\omega'}{2}}$$
$$= 16\frac{\left(\wp\omega'-\wp\frac{\omega'}{2}\right)^3}{\wp'^2\frac{\omega'}{2}}.$$

Therefore $$x(x^4-1) = -16\,\frac{\left(\wp\frac{\omega}{2}-\wp\omega\right)\left(\wp\frac{\omega'}{2}-\wp\omega'\right)^3}{\left(\wp\frac{\omega'}{2}-\wp\omega\right)\wp'^2\frac{\omega'}{2}}.$$

Taking account of equation (ii.), this can be written in the form

$$x(x^4-1)$$
$$= -16\left\{\frac{\left(\wp\frac{\omega'}{2}-\wp\omega'\right)\left(\wp\frac{\omega'}{4}-\wp\frac{\omega'}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega'}{2}+\omega\right)\right]\times\left(\wp\frac{\omega}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]}{\left(\wp\frac{\omega'}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]\wp'\frac{\omega}{4}\,\wp'\frac{\omega'}{2}}\right\}^2;$$

and therefore, finally, if

$$y = 4i\,\frac{\left(\wp\frac{\omega'}{2}-\wp\omega'\right)\left(\wp\frac{\omega'}{4}-\wp\frac{\omega'}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega'}{2}+\omega\right)\right]\times\left(\wp\frac{\omega}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]}{\left(\wp\frac{\omega'}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]\wp'\frac{\omega}{4}\,\wp'\frac{\omega'}{2}},$$

and $$x = \frac{\wp\frac{\omega}{2}-\wp\omega}{\wp\frac{\omega'}{2}-\wp\omega},$$

then $$y^2 = x(x^4-1).$$

Taking account of homogeneity, x and y are thus expressed as uniform functions of the parameter ω'/ω, and they each have the real axis for an essentially singular line.