



XLIV. On the problem of two and that of three electrified spherical conductors

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the flame, thus causing a rise in temperature of the latter. The temperature is highest along the path of the oxygen, and when the latter flows in the direction of the substance to be vaporized, the full effect of the temperature is, as it were, concentrated upon it. The effect is greatest near the base of the flame, where combustion is only beginning, and least near the tip. It is consequently essential that the oxygen, prior to reaching the substance, should pass through a region of the flame containing unburnt gases.

In conclusion, I wish to thank Lord Rayleigh for having provided me with the opportunity of paying a modest tribute to the work of one who came so near to discovering spectrum analysis.

Manchester, Feb. 18, 1918.

XLIV. *On the Problem of Two and that of Three Electrified Spherical Conductors.* By Prof. A. ANDERSON, M.A.*

WHEN an insulated conducting sphere of radius a is charged to potential A , the potential V due to the charge at any external point, P , whose distance from the centre of the sphere is r , is given by

$$rV = aA.$$

If now, charged bodies are brought into the field, this equation no longer holds: we have, instead,

$$rV + aV' = aA,$$

where V' is the potential that the introduced bodies have at P' the inverse point or, as we may call it for shortness, the image of P in the sphere. A has, of course, altered in value and V is, as before, the potential due to the charge on the sphere.

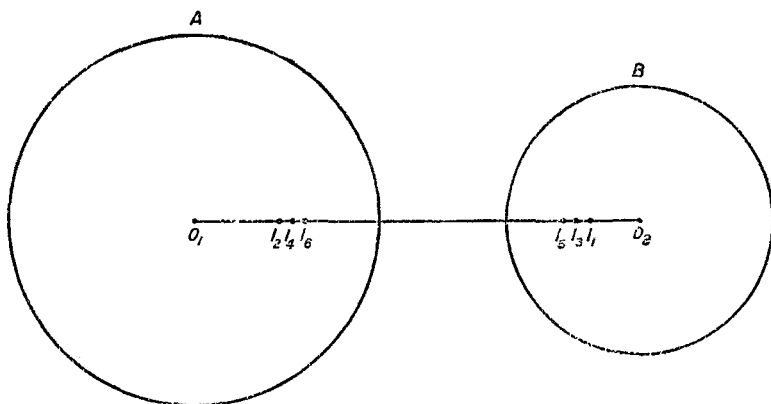
This equation may be used to find the coefficients of capacity and induction of two conducting spheres.

Let the potentials of the spheres be A and B , their centres O_1 and O_2 , their radii a and b , and the distance apart of their centres c . Also, in fig. 1, let I_1 be the image of O_1 in B , I_2 the image of I_1 in A , I_3 the image of I_2 in B , and so on, I_{n+1} in either sphere being the image of I_n in that sphere. We have thus a series of points I_1, I_3, I_5, I_7 , &c. inside the

* Communicated by the Author.

sphere B, and a series I_2, I_4, I_6, I_8 , &c. inside A. Let A_n, B_n denote the potentials due to the charges on the spheres A and B at the point I_n , and B_0 the potential due to the charge on B at the centre of A.

Fig. 1.



We have $cB_0 + bA_1 = bB,$

$$O_1I_1 \cdot A_1 + aB_2 = aA.$$

$$\frac{c}{b} \cdot B_0 - \frac{a}{O_1I_1} \cdot B_2 = B - \frac{aA}{O_1I_1},$$

and, likewise,

$$\frac{O_2I_2}{b} \cdot B_2 - \frac{a}{O_1I_3} \cdot B_4 = B - \frac{aA}{O_1I_3},$$

$$\frac{O_2I_4}{b} \cdot B_4 - \frac{a}{O_1I_5} \cdot B_6 = B - \frac{aA}{O_1I_5}.$$

.....

$$\frac{O_2I_{2n}}{b} \cdot B_{2n} - \frac{a}{O_1I_{2n+1}} \cdot B_{2n+2} = B - \frac{aA}{O_1I_{2n+1}}.$$

Hence we have

$$\begin{aligned} \frac{c}{b} \cdot B_0 = & B - \frac{aA}{O_1I_1} + \left(B - \frac{aA}{O_1I_3} \right) \frac{ab}{O_1I_1 \cdot O_2I_2} \\ & + \left(B - \frac{aA}{O_1I_5} \right) \frac{a^2b^2}{O_1I_1 \cdot O_1I_3 \cdot O_2I_2 \cdot O_2I_4} \\ & + \left(B - \frac{aA}{O_1I_7} \right) \frac{a^3b^3}{O_1I_1 \cdot O_1I_3 \cdot O_1I_5 \cdot O_2I_2 \cdot O_2I_4 \cdot O_2I_6} \\ & + \dots \dots \dots \\ & + \left(B - \frac{aA}{O_1I_{2n+1}} \right) \frac{a^nb^n}{O_1I_1 \dots O_1I_{2n-1} \cdot O_2I_2 \dots O_2I_{2n}} \\ & + \dots \dots \dots \end{aligned}$$

But if E is the charge on A , $A = B_0 + \frac{E}{a}$, and therefore

$$E = aA - aB_0.$$

Thus

$$\begin{aligned} E = A \left\{ a + \frac{a^2 b}{c} \left(\frac{1}{O_1 I_1} + \frac{ab}{O_1 I_1 \cdot O_1 I_3 \cdot O_2 I_2} \right. \right. \\ \left. \left. + \frac{a^2 b^2}{O_1 I_1 \cdot O_1 I_3 \cdot O_1 I_5 \cdot O_2 I_2 \cdot O_2 I_4} \right. \right. \\ \left. \left. + \dots + \frac{a^n b^n}{O_1 I_1 \dots O_1 I_{2n+1} \cdot O_2 I_2 \dots O_2 I_{2n}} + \dots \right) \right\}, \\ - B \frac{ab}{c} \left\{ 1 + \frac{ab}{O_1 I_1 \cdot O_2 I_2} + \frac{a^2 b^2}{O_1 I_1 \cdot O_1 I_3 \cdot O_2 I_2 \cdot O_2 I_4} + \dots \right. \\ \left. + \frac{a^n b^n}{O_1 I_1 \cdot O_1 I_3 \dots O_1 I_{2n-1} \cdot O_2 I_2 \dots O_2 I_{2n}} + \dots \right\}. \end{aligned}$$

Thus q_{11} and q_{12} have been found, and, of course, also q_{22} , by a simple application of the above equation.

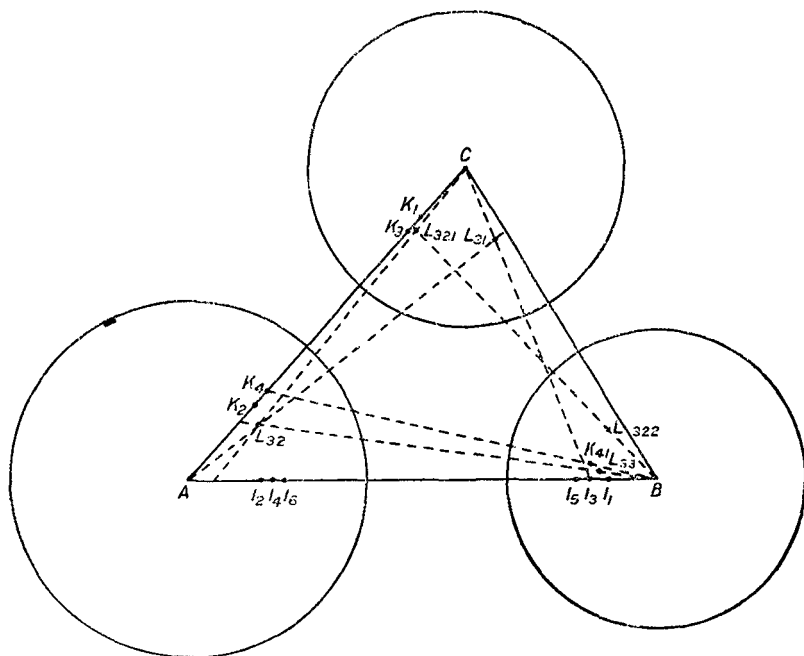
The same method is applicable to the case of several conducting spheres. For three spheres whose centres are at the corners of a triangle the work is necessarily much longer than that for two, but it is possible to find the values of the coefficients of induction and capacity to any degree of approximation.

We require for the solution of the problem a set of points I_1, I_3, I_5, \dots inside B , and a set I_2, I_4, I_6, \dots inside A , as in the problem for two spheres, and corresponding to these, for A and C , a set K_1, K_3, K_5, \dots inside C , and a set K_2, K_4, K_6, \dots inside A . But other points besides these are needed. Take one of the points, say I_3 , inside B . We take its image in C and denote it by L_{31} , the image of this in A by L_{32} , the image of L_{32} in B by L_{33} , and so on, going round again and again in the positive direction.

For the I points inside A we do a similar thing, the image of I_4 in C being L_{41} , and that of this in B L_{42} , and so on, going round in the negative direction. We do the same thing for the K points. Thus the image of K_5 in B is M_{51} , and the image of this in A is M_{52} , and so on. But this does not exhaust all the points required. Starting from any L or M point, we reverse the direction and find an infinite series of points for it. Thus, taking the point M_{52} , its image in C is M_{53} , but we also take its image in B and call it M_{521} , the image of this in C , M_{522} , and so on. A few of the points are shown in fig. 2.

The centres of the spheres are the points A, B, C, and we shall use these letters to denote also their potentials; α, β, γ are the lengths of the radii of the spheres, and a, b, c the

Fig. 2.



lengths of the sides of the triangle. The potential due to the sphere A at any point, say K_{41} , will be denoted by $A_{K_{41}}$, the potential due to B at L_{31} by $B_{L_{31}}$, and so on. The object of the problem is to find $B_a + C_a$.

We have

$$c \cdot B_a + \beta(A_{i_1} + C_{i_1}) = \beta B,$$

$$b \cdot C_a + \gamma(A_{K_1} + B_{K_1}) = \gamma C.$$

$$\therefore B_a + C_a + \frac{\beta}{c} A_{i_1} + \frac{\gamma}{b} A_{K_1} + \frac{\beta}{c} C_{i_1} + \frac{\gamma}{b} B_{K_1} = \frac{\beta}{c} B + \frac{\gamma}{b} C.$$

Thus the first two terms of the expression for $B_a + C_a$ are

$$\frac{\beta}{c} B + \frac{\gamma}{b} C.$$

$$\text{Again,} \quad AI_1 \cdot A_{i_1} + \alpha(B_{i_2} + C_{i_2}) = \alpha A,$$

$$AK_1 \cdot A_{k_1} + \alpha(B_{k_2} + C_{k_2}) = \alpha A ;$$

hence

$$\begin{aligned} B_a + C_a + \frac{\beta}{c} C_{i_1} + \frac{\gamma}{b} \cdot B_{k_1} - \frac{\alpha\beta}{c \cdot AI_1} (B_{i_2} + C_{i_2}) - \frac{\alpha\gamma}{b \cdot AK_1} (B_{k_2} + C_{k_2}) \\ = \frac{\beta}{c} B + \frac{\gamma}{b} \cdot C - \left(\frac{\alpha\beta}{c \cdot AI_1} + \frac{\alpha\gamma}{b \cdot AK_1} \right) A. \end{aligned}$$

$$\text{Also,} \quad BK_1 \cdot B_{k_1} + \beta(A_{m_{11}} + C_{m_{11}}) = \beta B,$$

$$CI_1 \cdot C_{i_1} + \gamma(A_{l_{11}} + B_{l_{11}}) = \gamma C,$$

$$\text{and} \quad AL_{11} \cdot A_{l_{11}} + \alpha(B_{l_{12}} + C_{l_{12}}) = \alpha A,$$

$$AM_{11} \cdot A_{m_{11}} + \alpha(B_{m_{12}} + C_{m_{12}}) = \alpha A,$$

from which we get, by substitution,

$$\begin{aligned} B_a + C_a - \frac{\alpha\beta}{c \cdot AI_1} (B_{i_2} + C_{i_2}) - \frac{\alpha\gamma}{b \cdot AK_1} (B_{k_2} + C_{k_2}) - \frac{\beta\gamma}{b \cdot BK_1} \cdot C_{m_{11}} \\ - \frac{\beta\gamma}{c \cdot CI_1} \cdot B_{l_{11}} + \frac{\alpha\beta\gamma}{b \cdot BK_1 \cdot AM_{11}} (B_{m_{12}} + C_{m_{12}}) \\ + \frac{\alpha\beta\gamma}{c \cdot CI_1 \cdot AL_{11}} (B_{l_{12}} + C_{l_{12}}) \\ = \frac{\beta}{c} B + \frac{\gamma}{b} \cdot C - \frac{\alpha\beta}{c \cdot AI_1} A \\ - \frac{\alpha\gamma}{b \cdot AK_1} \cdot A - \frac{\beta\gamma}{b \cdot BK_1} \cdot B - \frac{\beta\gamma}{c \cdot CI_1} \cdot C \\ + \alpha\beta\gamma \left[\frac{1}{b \cdot BK_1 \cdot AM_{11}} + \frac{1}{c \cdot CI_1 \cdot AL_{11}} \right] A. \end{aligned}$$

Thus, as far as the sixth term,

$$\begin{aligned} B_a + C_a = -\alpha \left(\frac{\beta}{c \cdot AI_1} + \frac{\gamma}{b \cdot AK_1} \right) \\ + B \left(\frac{\beta}{c} - \frac{\beta\gamma}{b \cdot BK_1} \right) + C \left(\frac{\gamma}{b} - \frac{\beta\gamma}{c \cdot CI_1} \right). \end{aligned}$$

Proceeding in this way, we find that as far as the fourteenth term,

$$\begin{aligned} B_a + C_a = & -\alpha \left[\frac{\beta}{c \cdot AI_1} + \frac{\gamma}{b \cdot AK_1} - \alpha\beta\gamma \left(\frac{1}{b \cdot BK_1 \cdot AM_{11}} \right. \right. \\ & \left. \left. + \frac{1}{c \cdot CI_1 \cdot AL_{11}} \right) \right] A. \\ & + B \left[\frac{\beta}{c} - \frac{\beta\gamma}{b \cdot BK_1} + \frac{\alpha\beta^2}{c \cdot AI_1 \cdot BI_2} + \frac{\alpha\beta\gamma}{b \cdot AK_1 \cdot BK_2} \right. \\ & \left. + \frac{\beta^2\gamma}{c \cdot CI_1 \cdot BL_{11}} \right] \\ & + C \left[\frac{\gamma}{b} - \frac{\beta\gamma}{c \cdot CI_1} + \frac{\alpha\gamma^2}{b \cdot AK_1 \cdot CK_2} + \frac{\alpha\beta\gamma}{c \cdot AI_1 \cdot CI_2} \right. \\ & \left. + \frac{\gamma^2\beta}{b \cdot BK_1 \cdot CM_{11}} \right]. \end{aligned}$$

By continuing this process we may find any number of terms of the expression for $B_a + C_a$.

Now, if E is the charge on A , we have

$$A = B_a + C_a + \frac{E}{\alpha},$$

$$\text{or} \quad E = A\alpha - (B_a + C_a).$$

$$\begin{aligned} \therefore E = A & \left[\alpha + \alpha^2 \left(\frac{\beta}{c \cdot AI_1} + \frac{\gamma}{b \cdot AK_1} \right) \right. \\ & \left. - \alpha^2\beta\gamma \left(\frac{1}{b \cdot BK_1 \cdot AM_{11}} + \frac{1}{c \cdot CI_1 \cdot AL_{11}} \right) + \dots \right] \\ & - B \left[\frac{\alpha\beta}{c} - \frac{\alpha\beta\gamma}{b \cdot BK_1} + \frac{\alpha^2\beta^2}{c \cdot AI_1 \cdot BI_2} \right. \\ & \left. + \frac{\alpha^2\beta\gamma}{b \cdot AK_1 \cdot BK_2} + \frac{\alpha\beta^2\gamma}{c \cdot CI_1 \cdot BL_{11}} + \dots \right] \\ & - C \left[\frac{\alpha\gamma}{c} - \frac{\alpha\beta\gamma}{c \cdot CI_1} + \frac{\alpha^2\gamma^2}{b \cdot AK_1 \cdot CK_2} \right. \\ & \left. + \frac{\alpha^2\beta\gamma}{c \cdot AI_1 \cdot CI_2} + \frac{\alpha\gamma^2\beta}{b \cdot BK_1 \cdot CM_{11}} + \dots \right]. \end{aligned}$$

Thus the first terms of the expressions for q_{11} , q_{12} , q_{13} have been determined and the corresponding terms of q_{22} , q_{23} , q_{33} may be got by changing the letters. We can get the formula for two spheres from this by making b , AK_1 , BK_1 , AL_{11} , BL_{11} all infinite and $C=0$.

Thus

$$E = A \left[\alpha + \frac{\alpha^2 \beta}{c \cdot AI_1} + \dots \right] - B \left[\frac{\alpha \beta}{c} + \frac{\alpha^2 \beta^2}{c \cdot AI_1 \cdot BI_2} + \dots \right],$$

or, since $AI_1 = \frac{c^2 - \beta^2}{c}$, and $BI_2 = c - \frac{\alpha^2 c}{c^2 - \beta^2}$,

$$E = A \left[\alpha + \frac{\alpha^2 \beta}{c^2 - \beta^2} + \dots \right] - B \left[\frac{\alpha \beta}{c} + \frac{\alpha^2 \beta^2}{c(c^2 - \beta^2 - \alpha^2)} + \dots \right].$$

As an example, suppose we have three small equal conducting spheres, whose centres are at the corners of an equilateral triangle the length of whose side is c . The terms we have found will give q_{11} , q_{12} , q_{13} correctly to $\left(\frac{\alpha}{c}\right)^3$.

$$q_{11} = q_{22} = q_{33} = \alpha \left(1 + \frac{2\alpha^2}{c^2} - \frac{2\alpha^3}{c^3} \right),$$

$$q_{12} = q_{13} = q_{23} = -\frac{\alpha^2}{c} \left(1 - \frac{\alpha}{c} + \frac{2\alpha^3}{c^2} \right),$$

from which we obtain the coefficients of potential

$$p_{11} = p_{22} = p_{33} = \frac{1}{3\alpha} \left(1 - \frac{2\alpha^2}{c^2} + \frac{2\alpha^3}{c^3} \right),$$

$$p_{12} = p_{13} = p_{23} = \frac{1}{3\alpha} \left(1 + \frac{3\alpha}{c} + \frac{\alpha^2}{c^2} - \frac{16\alpha^3}{c^3} \right).$$

The energy of the system, each sphere being supposed to have unit charge, is

$$\frac{1}{\alpha} \left(\frac{3}{2} + \frac{3\alpha}{c} - \frac{15\alpha^3}{c^3} \right),$$

and the force acting on one of them is

$$\sqrt{3} \left(\frac{1}{c^2} - \frac{15\alpha^2}{c^4} \right),$$

the force being $\frac{\sqrt{3}}{c^2}$ in the case of point charges.

By the above method the potential due to two charged conducting spheres at an external point can be written down easily. Let the centres of the spheres be A and B, their radii a and b , the distance between the centres c , and the potentials U and V . Let the image of any external point P in B be P_1 , the image of P_1 in A, P_2 , the image of P_2 in B, P_3 , and so on. Also, let the image of P in A be Q_1 , the image of Q_1 in B, Q_2 , the image of Q_2 in A, Q_3 , and so on.

The potential at P is

$$\begin{aligned}
 U & \left(\frac{a}{AP} - \frac{ab}{BP \cdot AP_1} + \frac{a^2b}{AP \cdot BQ_1 \cdot AQ_2} \right. \\
 & \quad - \frac{a^2b^2}{BP \cdot AP_1 \cdot AP_3 \cdot BP_2} + \frac{a^3b^2}{AP \cdot BQ_1 \cdot BQ_3 \cdot AQ_2 \cdot AQ_4} \\
 & \quad \left. - \frac{a^3b^3}{BP \cdot AP_1 \cdot AP_3 \cdot AP_5 \cdot BP_2 \cdot BP_6} + \dots \right) \\
 + V & \left(\frac{b}{BP} - \frac{ab}{AP \cdot BQ_1} + \frac{ab^2}{BP \cdot AP_1 \cdot BP_2} \right. \\
 & \quad - \frac{a^2b^2}{AP \cdot BQ_1 \cdot BQ_3 \cdot AQ_2} + \frac{a^2b^3}{BP \cdot AP_1 \cdot AP_3 \cdot BP_2 \cdot BP_4} \\
 & \quad \left. - \frac{a^3b^3}{AP \cdot BQ_1 \cdot BQ_3 \cdot BQ_5 \cdot AQ_2 \cdot AQ_6} + \dots \right).
 \end{aligned}$$

If, now, the image of A in B is I_1 , the image of I_1 in A, I_2 , the image of I_2 in B, I_3 , and so on, and if the image of B in A is J_1 , the image of J_1 in B, J_2 , the image of J_2 in A, J_3 , and so on, it is easy to show from similar triangles that

$$\begin{aligned}
 BP \cdot AP_1 &= c \cdot PI_1, \\
 AP \cdot BQ_1 \cdot AQ_2 &= c \cdot PI_2 \cdot AI_1, \\
 BP \cdot AP_1 \cdot BP_2 \cdot AP_3 &= c \cdot PI_3 \cdot BI_2 \cdot AI_1, \\
 AP \cdot BQ_1 \cdot AQ_2 \cdot BQ_3 \cdot AQ_4 &= c \cdot PI_4 \cdot AI_3 \cdot BI_2 \cdot AI_1, \\
 &\dots \dots \dots \&c.
 \end{aligned}$$

Hence the potential is

$$\begin{aligned}
 U & \left(\frac{a}{AP} - \frac{ab}{c \cdot PI_1} + \frac{a^2b}{c \cdot PI_2 \cdot AI_1} \right. \\
 & \quad - \frac{a^2b^2}{c \cdot PI_3 \cdot BI_2 \cdot AI_1} + \frac{a^3b^2}{c \cdot PI_4 \cdot AI_3 \cdot BI_2 \cdot AI_1} \\
 & \quad \left. - \frac{a^3b^3}{c \cdot PI_5 \cdot BI_4 \cdot AI_3 \cdot BI_2 \cdot AI_1} + \dots \right) \\
 + V & \left(\frac{b}{AP} - \frac{ab}{c \cdot PJ_1} + \frac{ab^2}{c \cdot PJ_2 \cdot BJ_1} - \frac{a^2b^2}{c \cdot PJ_3 \cdot AJ_2 \cdot BJ_1} + \dots \right),
 \end{aligned}$$

which shows that the potential outside the spheres has the same value as that due to a series of point charges at

A, B, I₁, J₁, I₂, J₂, I₃, J₃, I₄, J₄, &c., equal respectively to

$$Ua, Vb, -\frac{Uab}{c}, -\frac{Vab}{c}, \frac{Ua^2b}{c \cdot AI_1}, \frac{Vab^2}{c \cdot BJ_1}, -\frac{Ua^2b^2}{c \cdot BI_2 \cdot AI_1}, \\ -\frac{Va^2b^2}{c \cdot AJ_2 \cdot BJ_1}, \frac{Ua^3b^2}{c \cdot AI_3 \cdot BI_2 \cdot AI_1}, \frac{Va^2b^3}{c \cdot BJ_3 \cdot AJ_2 \cdot BJ_1}, \&c., \dots,$$

which are the image charges in the usual way of treating the subject.

Note.—The above paper was written before the one that appeared in the March number of the *Philosophical Magazine* on the same subject. It is, perhaps, unfortunate that the word “image” has been used for “inverse point.” The method has, of course, nothing to do with electrical images.

XLV. Some Two-Dimensional Potential Problems connected with the Circular Arc. By W. G. BICKLEY, B.Sc.*

§ 1. **I**N this paper a method of dealing with potential problems in two dimensions, depending on the use of functions of a complex variable and of the method of images, is applied to the solution of problems connected with an infinitely long lamina, the section of which is a circular arc. The results obtained are interpreted in terms of electricity and hydrodynamics.

§ 2. The first step in the investigation is the determination of the transformation by which the two sides of the arc in the z -plane become the real axis in the plane of an auxiliary variable ζ ($=\xi+i\eta$). The arc is taken as that part of the circle $z=-\iota e^{i\theta}$ for which $-\alpha \leq \theta \leq \alpha$, so that the angle subtended at the centre is 2α . For any point on this circle the ratio $(1+\iota z)/(z+\iota)$ is purely real, its value being $\cot \frac{\theta}{2}$, so that when θ lies within the above limits, the values of

$$\tan \frac{\alpha}{2} \frac{1+\iota z}{z+\iota} + \sqrt{\tan^2 \frac{\alpha}{2} \left(\frac{1+\iota z}{z+\iota} \right)^2 - 1} \quad . \quad . \quad (1)$$

are purely real, but become complex when θ lies outside them. Also, when $z \rightarrow -\iota$, the expression (1) tends to 0 or ∞ according as the root is taken negatively or positively; and when $\theta = \pm \alpha$, the expression has the values ± 1 .

* Communicated by the Author.