ON REPEATED INTEGRALS

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IN my paper* "On Absolutely Convergent Improper Integrals," I considered the relation of Lebesgue's definition of integration to those definitions which have been obtained by generalizing Riemann's definition in such a manner that they are applicable to improper double integrals. I also considered the relation of double integrals as defined, for the case of unlimited functions, by de la Vallée-Poussin and by Jordan, with the corresponding repeated integrals. I did not, however, consider the case of functions for which the double integral exists only when the definition of Lebesgue is adopted. This I propose to do in the present communication. It appears that the theory of Lebesgue's integrals throws light upon those cases in which one of the repeated integrals, or both of them, exist in accordance with the definition of Riemann or its generalization, but in which the double integral according to the definitions of Jordan and de la Vallée-Poussin has no existence. Even in the case of limited functions, the consideration of the Lebesgue double integral helps to fill up gaps in the ordinary theory of integration with respect to a parameter under the sign of integration.

I shall assume, as in my former paper, that a function $\phi(x, y)$, defined for a limited domain G, is replaced by a function $f(x, y)$, defined everywhere in a rectangle bounded by $x=a$, $x=b$, $y=c$, $y=d$, which contains the domain G; the function $f(x, y)$ being defined to be equal to $\phi(x, y)$ at every point of G, and to be zero at every point of the rectangle which does not belong to G. It is assumed that the frontier of G has the plane measure zero.

1. Let $f(x, y)$ be a limited summable function.

It has been pointed out by Lebesgue that, if E be a set of points measurable in the plane, it does not necessarily follow that the section of

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the set E by a straight line parallel to one of the axes is necessarily linearly measurable; unless, indeed, the unproved assumption be adopted that all linear sets of points have a linear measure. Accordingly, although $\int f(x, y)(dx\,dy)$ necessarily exists, the single integrals

$$\int_a^b f(x, y)\,dx, \qquad \int_c^d f(x, y)\,dy,$$

and therefore also the repeated integrals

$$\int_a^b dx \int_c^d f(x, y)\,dy, \qquad \int_c^d dy \int_a^b f(x, y)\,dx,$$

cannot be assumed necessarily to have definite meanings, Lebesgue has, for the case of a single integral of a function $\phi(x)$ which is not summable, defined an upper and a lower integral, denoted by $\int_{\text{sup}} \phi(x)\,dx$, and by $\int_{\text{inf}} \phi(x)\,dx$ respectively. These must not be confused with the upper and lower integrals as defined by Darboux. For such a summable function as is not integrable in accordance with Riemann's definition, the upper and lower integrals as defined by Lebesgue are identical in value, whereas the upper and lower integrals, as defined by Darboux, have different values.

Lebesgue* has then proved the general theorem,

$$\int f(x, y)(dx\,dy) = \underset{\text{sup}}{\int_a^b} dx \underset{\text{sup}}{\int_c^d} f(x, y)\,dy = \underset{\text{inf}}{\int_a^b} dx \underset{\text{inf}}{\int_c^d} f(x, y)\,dy$$

$$= \underset{\text{sup}}{\int_c^d} dy \underset{\text{sup}}{\int_a^b} f(x, y)\,dx = \underset{\text{inf}}{\int_c^d} dy \underset{\text{inf}}{\int_a^b} f(x, y)\,dx.$$

This theorem reduces to the theorem that

$$\int f(x, y)(dx\,dy) = \int_a^b dx \int_c^d f(x, y)\,dy = \int_c^d dy \int_a^b f(x, y)\,dx,$$

whenever the repeated integrals have definite meanings, the double integral always existing, since $f(x, y)$ is a limited summable function.

In case $\int f(x, y)(dx\,dy)$ exist in accordance with the ordinary definition, which is an extension of Riemann's definition of a single integral, it is

* See his memoir "Intégral, Longueur, Aire," *Annali di Mat.*, Ser. 3, Vol. VII., 1902.

well known that both the repeated integrals exist, and are equal to the double integral.

Various examples have, however, been given of functions for which only one of the repeated integrals exists, according to Riemann's definition, or for which both of them exist, and yet for which no double integral exists in accordance with the ordinary definition. The above theorem of Lebesgue throws light on such cases; for the following results follow immediately from it:—

If the two repeated integrals exist, in accordance with the ordinary definition, they must have equal values, if $f(x, y)$ be summable as a function in the plane, and they are equal to the Lebesgue double integral of the function.

If only one of the repeated integrals exist, in accordance with the ordinary definition, then the other exists as a repeated Lebesgue integral, and the two are equal to the Lebesgue double integral; it being assumed that the function is summable in the plane, and also on any straight line parallel to either axis.

All functions defined by any of the ordinary means are summable; it is, in fact, not definitely known whether it is possible to define a function which is not summable. Accordingly, all cases which arise in practice are covered by the two theorems here given.

The following examples will illustrate the utility of these remarks in the direction of completing the ordinary theory:—

(1) For the rectangle* bounded by $x=0$, $x=1$, $y=0$, $y=1$, let $f(x,y)=1$, for all rational values of x; and let $f(x,y)=2y$, for all irrational values of x. We have then $\int_0^1 f(x,y)\,dy=1$, whatever value x may have; and hence $\int_0^1 dx \int_0^1 f(x,y)\,dy$ exists, in accordance with the ordinary definition, and $=1$. The integral $\int_0^1 f(x,y)\,dx$ does not exist as a Riemann integral, except when $y=\frac{1}{2}$; for every point is a point of discontinuity. Consequently $\int_0^1 dy \int_0^1 f(x,y)\,dx$ does not exist in accordance with the ordinary definition. The double integral does not exist in accordance with the ordinary definition. The Lebesgue integral $\int f(x,y)(dx\,dy)$, however, exists, and is equal to 1. For the set of points at which $f(x,y)=1$ has the

* This function was given by Thomae, *Schlömilch's Zeitschrift*, Vol. XXIII., p. 67.

measure zero; and therefore $f(x, y)$ has the same Lebesgue integral as the function $\phi(x, y)$, which is defined by $\phi(x, y) = 2y$ at all points. For the functional values at a set of points of zero measure are irrelevant in a Lebesgue integral. The repeated integral $\int_0^1 dy \int_0^1 f(x, y)\,dx$ also exists as a Lebesgue repeated integral, and is equal to 1; this may easily be verified directly.

(2) For the same rectangle as in (1), let a set* K of points be defined as follows:—The numbers x, y are expressed in the dyad scale, and only those values of x and y are taken which are expressed by terminating radix-fractions, the number of digits being the same for x as for y. Let the function $f(x, y)$ be defined as equal to c' at every point of K, and equal to c at every other point.

If x' denotes a terminating radix-fraction, there are only a finite number of points of K on the straight line $x = x'$; similarly there are only a finite number of points of K on the straight line $y = y'$, where y' denotes a terminating radix-fraction. It thus appears that both repeated integrals exist, in accordance with Riemann's definition, and that they are both equal to c. The double integral does not exist in accordance with the ordinary definition; for it can be shewn that the set K is everywhere-dense, and therefore the function $f(x, y)$ is totally discontinuous. Consider the straight line $y = x+a$, where a is a positive or negative radix-fraction with a finite number of digits; we see that, corresponding to any number x expressed by a finite number of digits greater than the number by which a is expressed, there is a point (x, y) on the straight line belonging to K. The component of K on this straight line is consequently everywhere-dense; thus, since the values of a are everywhere-dense in the interval $(-1, 1)$, it follows that the points of K are everywhere-dense in the rectangle. The Lebesgue double integral exists, and $= c$; for, as in (1), the points at which $f(x, y) = c'$ form a set of zero measure, and therefore the integral is the same as for a function which has everywhere the value c.

2. Let $f(x, y)$ be summable, but not limited. It has been established by Lebesgue that the equalities

$$\int f(x, y)(dx\,dy) = \int dx \int f(x, y)\,dy = \int dy \int f(x, y)\,dx$$

* See Pringsheim, *Munich Sitzungsberichte*, Vol. XXIX., p. 48.

still hold whenever the integrals have meanings in accordance with his definition.

This theorem throws light upon the validity of the process of integration with respect to a parameter, under the integral sign, the limits of integration in both cases being finite. We may state the theorem as follows:—

Integration with respect to a parameter, under the integral sign, is always a valid process, provided the function is summable and integrable in the plane, if the result of the process have a definite meaning.

It must, however, be remembered that, even if $\int_c^{y_1} dy \int_a^b f(x, y)\, dx$ exist, in accordance with Riemann's definition, or one of its extensions to absolutely convergent integrals, $\int_a^b dx \int_c^{y_1} f(x, y)\, dy$ may exist, if it exist at all, only when Lebesgue's definition is employed.

The only case in which the repeated integrals of an unlimited function can both exist, but have unequal values, is when the function is either not summable in the plane, or else when it is summable but does not possess a Lebesgue integral.

In order to find sufficient conditions for the existence and equality of the repeated integrals of an unlimited function in cases when the corresponding double integral does not exist, either in accordance with the definition of Jordan, or with that of de la Vallée-Poussin, it is useful to introduce a definition of a double integral of a less stringent character than that of Jordan. This definition differs from that of Jordan, as given in my former paper (p. 137), in the one respect, that the domains D_n are restricted each to consist of a finite set of rectangles, the sides of each of which are parallel to those of the fundamental rectangle in which the function is defined.

A double integral which exists in accordance with this modified definition, I propose to speak of as a *restricted Jordan double integral.* Such a double integral may exist for a function which does not possess a Jordan double integral.

Assuming that the integral $\int_A f(x, y)(dx\,dy)$ exists, as a restricted Jordan double integral, let $f_n(x, y)$ be that limited function which, in the domain D_n, consisting of a finite set of rectangles, is equal to $f(x, y)$, and is zero in the complementary domain $C(D_n)$ which contains all the points of infinite discontinuity of $f(x, y)$.

We have then

$$\int_A f(x, y)(dx\,dy) = \lim_{n=\infty} \int_A f_n(x, y)(dx\,dy)$$
$$= \lim_{n=\infty} \int_a^b dx \int_c^d f_n(x, y)\,dy;$$

and therefore we have

$$\int_A f(x, y)(dx\,dy) = \int_a^b dx \int_c^d f(x, y)\,dy,$$

provided
$$\lim_{n=\infty} \int_a^b dx \int_{\Lambda_n(x)} f(x, y)\,dy = 0,$$

where $\Lambda_n(x)$ denotes that finite set of intervals which forms the section of $C(D_n)$ by the ordinate corresponding to the abscissa x. From this result, the following theorem, very similar to one given by Jordan,* and specifying a particular mode of satisfying the last condition, may be deduced :—

For the existence and equality of the two repeated integrals

$$\int_a^b dx \int_c^d f(x, y)\,dy, \qquad \int_c^d dy \int_a^b f(x, y)\,dx,$$

it is sufficient :

(1) *That the function $f(x, y)$ possess a restricted Jordan double integral in the fundamental rectangle.*

(2) *That the points of infinite discontinuity of $f(x, y)$ be distributed on a limited number of arcs of continuous curves representing monotone functions.*

(3) *That, corresponding to any positive number ϵ, positive numbers h_1, k_1 exist, such that*

$$\left| \int_x^{x+h} f(x, y)\,dx \right| < \epsilon, \qquad \left| \int_y^{y+k} f(x, y)\,dy \right| < \epsilon,$$

for $|h| < h_1$, $|k| < k_1$, and for every value of x and y in the fundamental rectangle.

To shew that, under the conditions stated,

$$\lim_{n=\infty} \int_a^b dx \int_{\Lambda_n(x)} f(x, y)\,dy = 0,$$

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* *Cours d'Analyse*, Vol. II., p. 67.

it is clear that the points of any one such curve can be enclosed in the interiors of a finite set of rectangles, the height of each of which is $< k_1$. Then $\Lambda_n(x)$ consists of a number of intervals not exceeding the number r of the curves on which the points of infinite discontinuity lie.

We have then $$\left| \int_{\Lambda_n(x)} f(x, y)\, dy \right| < r\epsilon,$$

and therefore $$\left| \int_a^b dx \int_{\Lambda_n(x)} f(x, y)\, dy \right|$$

is less than the arbitrarily small number $r\epsilon(b-a)$. Thus

$$\int_a^b dx \int_c^d f(x, y)\, dy$$

exists, and is equal to the restricted Jordan integral. Similarly it can be shewn that the other repeated integral has the same value.

The two examples given in my former paper (pp. 156, 157) will serve as illustrations of the greater completeness given to the theory by taking account of the existence of Lebesgue integrals.

(1) If $\mathsf{f}(x)$ be defined for the rectangle bounded by $x=0, x=1, y=0, y=1$, by the rule that $\mathsf{f}(x) = \frac{1}{2^n}$, for $x = \frac{2m+1}{2^n}$ $(n \geqslant 0)$, and $\mathsf{f}(x) = 0$, for all other values of x, then

$$\int \left| \frac{1}{y} \sin \frac{1}{y} \right| \mathsf{f}(x)\,(dx\,dy) = 0.$$

The repeated integral $\int_0^1 dy \int_0^1 \left| \frac{1}{y} \sin \frac{1}{y} \right| \mathsf{f}(x)\, dx$ exists, and $= 0$. The other repeated integral $\int_0^1 dx \int_0^1 \mathsf{f}(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$, which was shewn not to exist in accordance with the earlier definitions, exists in accordance with Lebesgue's definition, and is also equal to zero. For, although $\int_0^1 \mathsf{f}(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$ diverges for an everywhere-dense set of values of x, that set of points has zero measure. Hence, since the Lebesgue integral is independent of the functional values at a set of points of zero measure, $\int_0^1 \mathsf{f}(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$ is integrable with respect to x in the interval (0, 1), and has the value zero.

(2) If $\mathsf{f}(x, y) = 0$ at all points in the rectangle bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, except at the points $x = \frac{2m+1}{2^s}$, $y \leqslant \frac{1}{2^s}$, where it has the improper value $+\infty$.

The repeated integral $\int_0^1 dx \int_0^1 f(x, y)\, dy$, which was shewn not to exist in accordance with Harnack's definition, exists in accordance with that of Lebesgue, and is equal to zero, the value of the double integral.

3. The question of the validity of differentiation under the integral sign has been treated by reducing the question to one of the validity of reversing the order of a repeated integral. Denoting by $Df(x, y)$ the derivative on the right, of $f(x, y)$ with respect to y, we have, under certain conditions,

$$f(x, y_0+h)-f(x, y_0) = \int_{y_0}^{y_0+h} Df(x, y)\, dy.$$

It may happen that $Df(x, y)$, although limited in the domain bounded by $x = a$, $x = b$, $y = y_0$, $y = y_0+h$, is not integrable with respect to y in accordance with Riemann's definition. Even when a differential coefficient everywhere exists, it is known that this may be the case. But the integral certainly exists; for $Df(x, y)$ is summable if $f(x, y)$ be so. It is here assumed that $Df(x, y)$ has a definite value for each value of x, for all values of y in (y_0, y_0+h) with the possible exception of a set of values of y of zero measure.

We have now, if u_y denote $\int_a^b f(x, y)\, dy$,

$$\frac{u_{y_0+h}-u_{y_0}}{h} = \frac{1}{h}\int_a^b dx \int_{y_0}^{y_0+h} Df(x, y)\, dy.$$

In accordance with Lebesgue's theorem, if $f(x, y)$ be summable as a function of the two variables (x, y), the order of integration in the repeated integral may be reversed, or

$$\frac{u_{y_0+h}-u_{y_0}}{h} = \frac{1}{h}\int_{y_0}^{y_0+h} dy \int_a^b Df(x, y)\, dx,$$

provided the repeated integral have a definite meaning, and this is the case, since $\left| \int_a^b Df(x, y)\, dx \right|$ cannot exceed $b-a$ multiplied by the upper limit of $| Df(x, y) |$ in the two-dimensional domain.

We have now $$Du_{y_0} = \int_a^b Df(x, y_0)\,dx,$$

provided $\int_a^b Df(x, y)\,dx$ be continuous with respect to y at y_0 on the right.

We have then the following theorem:—

If $Df(x, y)$ be limited in the domain bounded by $x = a$, $x = b$, $y = y_0$, $y = y_0 + a$, then, provided the Lebesgue integral $\int_a^b Df(x, y)\,dx$ be continuous at y_0 on the right, the derivative of $\int_a^b f(x, y)\,dx$ at y_0 on the right is $\int_a^b Df(x, y_0)\,dx$; it being assumed that $\int_{y_0}^{y_0+a} Df(x, y)\,dy$ exists at least as a Lebesgue integral.

In case $Df(x, y)$ be unlimited in the domain bounded by $x = a$, $x = b$, $y = y_0$, $y = y_0 + a$, it may still have a Lebesgue integral with respect to y in $(y_0, y_0 + a)$. In case the points of infinite discontinuity form, for each value of x, a reducible set in the interval $(y_0, y_0 + a)$, the equation

$$f(x, y_0 + h) - f(x, y_0) = \int_{y_0}^{y_0+h} Df(x, y)\,dy \quad (0 < h < a)$$

is still valid. If $Df(x, y)$ have a Lebesgue double integral in the two-dimensional domain, and $\int_{y_0}^{y_0+h} dy \int_a^b Df(x, y)\,dx$ have a definite meaning, as a repeated Lebesgue integral, then the above process is still valid. We therefore obtain the following theorem:—

If the points of infinite discontinuity of $f(x, y)$, considered as a function of y, form a reducible set, for each value of x, and $\int_{y_0}^{y_0+h} Df(x, y)\,dy$ exist as a Lebesgue integral, for each value of x; if, further, $Df(x, y)$ have a Lebesgue double integral in the two-dimensional domain, and

$$\int_{y_0}^{y_0+h} dy \int_a^b Df(x, y)\,dx$$

exist as a repeated Lebesgue integral; and, if, lastly, $\int_a^b Df(x, y)\,dx$ be continuous on the right at $y = y_0$, then the derivative of $\int_a^b f(x, y)\,dx$ at y_0 on the right is $\int_a^b Df(x, y_0)\,dx$.

In any particular case, the Lebesgue integrals may exist in accordance with the earlier definitions.

As an example* of the application of this theorem, we may take the case of the differentiation of $\int_0^X (x-y)^{\frac{1}{3}}\,dx$ with respect to y. It can easily be shewn that $(x-y)^{-\frac{2}{3}}$ possesses a double integral in the domain bounded by $x=0$, $x=X$, $y=0$, $y=h$, and that the other conditions of the theorem are also satisfied.

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* This example has been given by Mr. Hardy, *Messenger of Math.*, Vol. XXXIII., p. 63, as a case not covered by the ordinary criteria.