

An Account of Cauchy's Theory of Reflection and Refraction of Light

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moreover, the resistance of the copper wire of the former was a comparatively large fraction, at least one sixth or one seventh, of the whole resistance in circuit with it, while the copper resistance of the larger primary coil was not much more than one fifteen-hundredth of the whole resistance in circuit with it—the ratio of the resistances was no doubt somewhat disturbed by the unequal heating of the two primary circuits, and was in reality rather less than what was inferred from the marked values of the coils used. That this was the case was shown by the fact that the apparent ratio decreased progressively from 40·3 to 39·45, as the strength of the testing current was diminished from its first value to rather less than one sixth. A better arrangement of the apparatus would have been to put the two primary wires in series with the battery, and to have connected the two secondaries in parallel circuit; but the matter was not thought important enough to require a repetition of the measurements.

The method of measuring coefficients of mutual induction described in this paper may perhaps be of use in the experimental study of dynamo-electric machines, whose whole action depends upon the variation of the coefficient of mutual induction between the field-magnet coils and the armature coils, as the latter take various positions during the course of a revolution.

XXIV. *An Account of Cauchy's Theory of Reflection and Refraction of Light.* By JAMES WALKER, M.A., Demonstrator at the Clarendon Laboratory, Oxford*.

THE theory of reflection and refraction of light holds such an important place among the problems of Optics which await their solution that it is advantageous to have a clear idea of the work which has been previously done in the subject.

The theory advanced by Green has been so thoroughly discussed by Lord Rayleigh and Sir W. Thomson that all questions connected with it may be considered as completely settled. But this is by no means the case with Cauchy's work on the subject; and some account of it may be of in-

* Read December 11, 1886.

terest, even though the theory cannot be said to contribute much towards a solution of the problem.

Several "reproductions"* of Cauchy's work have indeed appeared in French and German, but in most of them the elegance, and therewith the clearness, of Cauchy's method have been given up; while they leave in more or less obscurity the reasoning which led him to enunciate his "principle of continuity," and make no mention of a point of considerable interest, viz. the *mistake* which originally led to his adoption of a theory involving the strange assumption of a negative value for the coefficient of compressibility of the æther.

I.

Cauchy, at different periods, gave three distinct theories of reflection: the first two, however, require only a passing notice, as they were afterwards rejected by him as in no respect affording a complete solution of the problem.

The *first* theory was published in the *Bulletin de Férussac* of 1830. It rested on the true dynamical basis of the equality of pressures† at the interface of the media; but was vitiated by the neglect of the pressural waves, which must take part in the act of reflection and refraction. The method led, on the assumption of the equality of the density of the æther in the two media, to the formulæ given by Fresnel‡.

The *second* theory was based on a method of obtaining the equations of condition at the interface, which was given in a lithographed memoir published in 1836. This method assumes a change in the equations of motion near the interface to a distance comparable with the radius of the sphere of activity of a molecule, and leads to the following theorem:—

"Etant donnés deux milieux ou deux systèmes de molécules séparés l'un de l'autre par le plan de yz , supposons que des équations d'équilibre ou de mouvement généralisées de manière à subsister pour tous les points de l'un et de l'autre système

* A. v. Ettingshausen, Pogg. *Ann.* l. p. 409; *Sitzb. der Wien. Akad.* xviii. p. 369. Beer, Pogg. *Ann.* xci. pp. 268, 467, 561; xcii. p. 402. Eisenlohr, Pogg. *Ann.* civ. p. 346. Briot, *Liouv. Journ.* (2nd) xi. p. 305; xii. p. 185. Lundquist, Pogg. *Ann.* clii. pp. 177, 398, 565.

† Cauchy's reasons for rejecting the principle of the equality of pressures at the interface are given in *Comptes Rendus*, xxviii. p. 60.

‡ Cauchy, *Mémoire sur la Dispersion*, § 10.

et même pour les points situés sur la surface de séparation, l'on puisse déduire une equation de la forme

$$\frac{d^2g}{dx^2} = \Theta,$$

g , Θ désignant deux quantités finies, mais variables avec les coordonnées $x y z$. On aura, pour $x=0$,

$$\frac{dg}{dx} = \frac{dg'}{dx}, \quad g = g',$$

en admettant que l'on prenne pour premier et pour second membre de chacune des formules les resultâts que fournit la réduction de x à zéro, dans les deux valeurs de la fonction $\frac{dg}{dx}$ ou g relatives aux points intérieurs du premier et du second système."

The equations of condition resulting from the application of this theorem were published in Cauchy's memoir on Dispersion in the same year*. They express that the linear dilatation of the æther normal to the interface is the same for both the media, and that the rotations in the three coordinate planes of a particle at the interface is the same, whether the particle is considered as belonging to the first or second medium.

The method of deducing these conditions was given in a memoir presented to the French Academy on October 29, 1838†. This memoir has never been published; and all we know is that the method involved the assumption that the velocity of propagation of the pressural waves is very great compared with that of the distortional waves‡. In 1842 Cauchy showed that these conditions lead to Fresnel's formulæ§.

The final theory was published in detail|| in the years 1838 and 1839, and is contained in the 8th and 9th volumes of the *Comptes Rendus*, and in the *Exercices d'Analyse et de Physique*. Later volumes of the *Comptes Rendus* contain re-statements of it; and in 1850 an extension of the method

* *Mém. sur la Dispersion*, § 10.

† *Comptes Rendus*, vii. p. 751.

‡ *Ibid.* x. p. 905.

§ *Ibid.* xv. p. 418.

|| The idea seems to be prevalent that we are indebted to the German reproductions for our knowledge of the details of Cauchy's method.

was made to rotatory isotropic media* and to anisotropic media†; but this later work was never completed.

II.

Cauchy's final method‡ of determining the conditions at the interface of the media depended on finding the relations which must exist between the known values of the displacements in the interior of the medium, and the values, consistent with the conditions of the problem, which these displacements take when the change in the form of the equations of motion near the interface is taken into account.

Treating the æther as an isotropic elastic solid, for which the density is ρ , and the coefficients of compressibility and rigidity are k, n , the equations of motion are

$$\left. \begin{aligned} \rho \frac{d^2 \xi}{dt^2} &= m \frac{d\delta}{dx} + n \nabla^2 \xi, \\ \rho \frac{d^2 \eta}{dt^2} &= m \frac{d\delta}{dy} + n \nabla^2 \eta, \\ \rho \frac{d^2 \zeta}{dt^2} &= m \frac{d\delta}{dz} + n \nabla^2 \zeta, \end{aligned} \right\} \dots \dots (1)$$

where

$$\delta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}, \quad \text{and } m = k + \frac{1}{3}n.$$

Sir W. Thomson§ has shown that all possible solutions of these equations are included in

$$\xi = \frac{d\phi}{dx} + u, \quad \eta = \frac{d\phi}{dy} + v, \quad \zeta = \frac{d\phi}{dz} + w,$$

where ϕ, u, v, w are some functions of x, y, z, t and u, v, w such that $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$; further that, making these substitutions, equations (1) may be replaced by

$$\begin{aligned} \rho \frac{d^2 \phi}{dt^2} &= (m+n) \nabla^2 \phi, & \rho \frac{d^2 u}{dt^2} &= n \nabla^2 u, & \rho \frac{d^2 v}{dt^2} &= n \nabla^2 v, \\ \rho \frac{d^2 w}{dt^2} &= n \nabla^2 w. \end{aligned}$$

So that there are two modes of waves possible: a condensa-

* C. R. xxxi. pp. 160, 225.

† Ibid. xxxi. pp. 257, 297.

‡ Ibid. viii. pp. 374, 432, 459.

§ Baltimore Lectures, p. 32.

tional wave, propagated with velocity $\sqrt{\frac{m+n}{\rho}}$, and for which the velocity-potential is ϕ ; and a distortional wave, propagated with the velocity $\sqrt{\frac{n}{\rho}}$, and for which the components of the displacement are u, v, w .

Let the interface of the media be the plane of yz , and suppose the first medium on the side of positive x .

Considering only plane waves, which have the same period of vibration $2\pi/\omega$ and the same trace $by + cz = 0$ on the interface of the media, the values of ϕ, u, v, w satisfying the equations may be taken as

$$\phi = \frac{B_{II}}{\sqrt{-1}} e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} + \frac{C_{II}}{\sqrt{-1}} e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}*},$$

$$u = A e^{(ax + by + cz - \omega t)\sqrt{-1}} + A_I e^{(-ax + by + cz - \omega t)\sqrt{-1}},$$

$$v = B e^{(ax + by + cz - \omega t)\sqrt{-1}} + B_I e^{(-ax + by + cz - \omega t)\sqrt{-1}},$$

$$w = C e^{(ax + by + cz - \omega t)\sqrt{-1}} + C_I e^{(-ax + by + cz - \omega t)\sqrt{-1}};$$

where

$$a_{II} = +\sqrt{\frac{\rho\omega^2}{m+n} - b^2 - c^2}, \quad a = +\sqrt{\frac{\rho\omega^2}{n} - b^2 - c^2},$$

and $A, B, C; A_I, B_I, C_I$ are connected by the relations

$$Aa + Bb + Cc = 0, \quad -A_I a + B_I b + C_I c = 0.$$

The corresponding values of ξ, η, ζ are accordingly

$$\left. \begin{aligned} \xi &= A e^{(ax + by + cz - \omega t)\sqrt{-1}} + A_I e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II} a_{II} e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} \\ &\quad - C_{II} a_{II} e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}, \\ \eta &= B e^{(ax + by + cz - \omega t)\sqrt{-1}} + B_I e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II} b e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} \\ &\quad + C_{II} b e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}, \\ \zeta &= C e^{(ax + by + cz - \omega t)\sqrt{-1}} + C_I e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II} c e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} \\ &\quad + C_{II} c e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}. \end{aligned} \right\} (2)$$

Now within the medium the displacements are those due to the distortional waves alone, and hence the values of ξ, η, ζ are

* The $\sqrt{-1}$ is inserted for convenience.

$$\left. \begin{aligned} \xi &= A e^{(ax+by+cz-\omega t)\sqrt{-1}} + A_1 e^{(-ax+by+cz-\omega t)\sqrt{-1}}, \\ \eta &= B e^{(ax+by+cz-\omega t)\sqrt{-1}} + B_1 e^{(-ax+by+cz-\omega t)\sqrt{-1}}, \\ \zeta &= C e^{(ax+by+cz-\omega t)\sqrt{-1}} + C_1 e^{(-ax+by+cz-\omega t)\sqrt{-1}}, \end{aligned} \right\} \quad (3)$$

whence, comparing these values with equations (2),

$$B_{11} = C_{11} = 0^*.$$

Near the interface, and for values of x less than the small quantity ϵ , the differential equations change form by the addition of terms whose coefficients are functions of x , which vanish when x exceeds the small quantity ϵ . These additional terms may be reduced to linear functions of ξ , η , ζ , and their differential coefficients with respect to x^\dagger , since the equations will still be satisfied by taking the displacements proportional to the same exponential $e^{(by+cz-\omega t)\sqrt{-1}}$.

We require now to determine the values of ξ , η , ζ which satisfy these altered equations.

Cauchy's method of doing this depends, as v. Ettingshausen[†] has pointed out, on the method of the variation of parameters: by this method the constants A , A_1 , ... are treated as functions of x ; and a first condition imposed upon them is that $\frac{d\xi}{dx}$, $\frac{d\eta}{dx}$, $\frac{d\zeta}{dx}$ must remain unaltered in form, so that

$$\left. \begin{aligned} \frac{d\xi}{dx} &= \left\{ A a e^{(ax+by+cz-\omega t)\sqrt{-1}} - A_1 a e^{(-ax+by+cz-\omega t)\sqrt{-1}} \right. \\ &\quad \left. + B_{11} a_{11} e^{(a_{11}x+by+cz-\omega t)\sqrt{-1}} + C_{11} a_{11} e^{(-a_{11}x+by+cz-\omega t)\sqrt{-1}} \right\} \sqrt{-1}, \\ \frac{d\eta}{dx} &= \left\{ B a e^{(ax+by+cz-\omega t)\sqrt{-1}} - B_1 a e^{(-ax+by+cz-\omega t)\sqrt{-1}} \right. \\ &\quad \left. + B_{11} a_{11} b e^{(a_{11}x+by+cz-\omega t)\sqrt{-1}} - C_{11} a_{11} b e^{(-a_{11}x+by+cz-\omega t)\sqrt{-1}} \right\} \sqrt{-1}, \\ \frac{d\zeta}{dx} &= \left\{ C a e^{(ax+by+cz-\omega t)\sqrt{-1}} - C_1 a e^{(-ax+by+cz-\omega t)\sqrt{-1}} \right. \\ &\quad \left. + B_{11} a_{11} c e^{(a_{11}x+by+cz-\omega t)\sqrt{-1}} - C_{11} a_{11} c e^{(-a_{11}x+by+cz-\omega t)\sqrt{-1}} \right\} \sqrt{-1}. \end{aligned} \right\} \quad (4)$$

Consider now any one of the parameters, say B_{11} ; its value deduced from equations (2) and (4) is of the form

$$B_{11} = \left(\lambda \xi + \mu \eta + \nu \zeta + \pi \frac{d\xi}{dx} + \rho \frac{d\eta}{dx} + \sigma \frac{d\zeta}{dx} \right) e^{-(a_{11}x+by+cz-\omega t)\sqrt{-1}}.$$

Differentiating this equation with respect to x , and substituting

* C. R. viii. p. 440.

† *Tom. cit.* p. 461.

† Pogg. *Ann.* l. p. 409.

ting for $\frac{d^2\xi}{dx^2}$, $\frac{d^2\eta}{dx^2}$, $\frac{d^2\zeta}{dx^2}$ from the changed differential equations, all the terms will cancel out except those which depend on the change of form; and we shall have

$$\frac{dB_{II}}{dx} = \left(L\xi + M\eta + N\zeta + P\frac{d\xi}{dx} + R\frac{d\eta}{dx} + S\frac{d\zeta}{dx} \right) e^{-(a_{II}x + by + cz - \omega t)\sqrt{-1}}$$

where L, M, \dots vanish for finite values of x .

Now the values of ξ, η, ζ will differ but slightly from those given by (3), so that this last expression may be written

$$\frac{dB_{II}}{dx} = (AL + BM + \dots)e^{(a - a_{II})x\sqrt{-1}} + (A_1L + B_1M + \dots)e^{-(a + a_{II})x\sqrt{-1}},$$

whence the variable part of B_{II} is

$$B_{II} = \int (AL + BM + \dots)e^{(a - a_{II})x\sqrt{-1}} dx \\ + \int (A_1L + B_1M + \dots)e^{-(a + a_{II})x\sqrt{-1}} dx.$$

Similar values are obtained for the parts of A, A_1, \dots which depend on x . Now L, M, \dots vanish for finite values of x ;

so that if $\int_0^x L dx, \int_0^x M dx, \dots$ are very small relatively to

$\lambda, \mu, \nu, \dots^*$, the variable part of B_{II} may be neglected if $-(a + a_{II})\sqrt{-1}$, $(a - a_{II})\sqrt{-1}$ have no real positive part; so that those among the coefficients A, A_1, \dots will remain unaltered, when the change in the medium near the interface is taken into account, which have the coefficient of x in their exponential factor with a real part *not less than* that of $a\sqrt{-1}$.

In the present case this will be so for all the parameters except B_{II} ; and hence, calling $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ the corrected values of ξ, η, ζ , we have

$$\bar{\xi} = Ae^{(ax + by + cz - \omega t)\sqrt{-1}} + A_1e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II}a_{II}e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}},$$

$$\bar{\eta} = Be^{(ax + by + cz - \omega t)\sqrt{-1}} + B_1e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II}b_{II}e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}},$$

$$\bar{\zeta} = Ce^{(ax + by + cz - \omega t)\sqrt{-1}} + C_1e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II}c_{II}e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}},$$

* This necessitates, first, that the coefficients of the added terms in the altered differential equations are all finite, and their product by ϵ very small; secondly, that the thickness of the modified layer is small compared with the wave-length (*Comptes Rendus*, viii. p. 439; ix. p. 5).

$$\begin{aligned}\frac{d\bar{\xi}}{dx} &= \{Aae^{(ax+by+cz-\omega t)\sqrt{-1}} - A_1ae^{(-ax+by+cz-\omega t)\sqrt{-1}} \\ &\quad + B_{II}a_{II}e^{(a_{II}x+by+cz-\omega t)\sqrt{-1}}\}\sqrt{-1}, \\ \frac{d\bar{\eta}}{dx} &= \{Bae^{(ax+by+cz-\omega t)\sqrt{-1}} - B_1ae^{(-ax+by+cz-\omega t)\sqrt{-1}} \\ &\quad + B_{II}a_{II}be^{(a_{II}x+by+cz-\omega t)\sqrt{-1}}\}\sqrt{-1}, \\ \frac{d\bar{\xi}}{dx} &= \{Cae^{(ax+by+cz-\omega t)\sqrt{-1}} - C_1ae^{(-ax+by+cz-\omega t)\sqrt{-1}} \\ &\quad + B_{II}a_{II}ce^{(a_{II}x+by+cz-\omega t)\sqrt{-1}}\}\sqrt{-1};\end{aligned}$$

or, if

$\left. \begin{matrix} \xi, \eta, \zeta \\ \xi_I, \eta_I, \zeta_I \\ \xi_{II}, \eta_{II}, \zeta_{II} \end{matrix} \right\}$ are the components of the displacements in the
 $\left. \begin{matrix} \text{incident} \\ \text{reflected} \\ \text{pressural} \end{matrix} \right\}$ wave, $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ are such that

$$\begin{aligned}\bar{\xi} &= \xi + \xi_I + \xi_{II}, & \bar{\eta} &= \eta + \eta_I + \eta_{II}, & \bar{\zeta} &= \zeta + \zeta_I + \zeta_{II}; \\ \frac{d\bar{\xi}}{dx} &= \frac{d\xi}{dx} + \frac{d\xi_I}{dx} + \frac{d\xi_{II}}{dx}, & \frac{d\bar{\eta}}{dx} &= \frac{d\eta}{dx} + \frac{d\eta_I}{dx} + \frac{d\eta_{II}}{dx}, & \frac{d\bar{\zeta}}{dx} &= \frac{d\zeta}{dx} + \frac{d\zeta_I}{dx} + \frac{d\zeta_{II}}{dx}.\end{aligned}$$

In the same way, for the second medium the corrected values of the displacements are such that

$$\begin{aligned}\bar{\xi}' &= \xi' + \xi'', & \bar{\eta}' &= \eta' + \eta'', & \bar{\zeta}' &= \zeta' + \zeta''; \\ \frac{d\bar{\xi}'}{dx} &= \frac{d\xi'}{dx} + \frac{d\xi''}{dx}, & \frac{d\bar{\eta}'}{dx} &= \frac{d\eta'}{dx} + \frac{d\eta''}{dx}, & \frac{d\bar{\zeta}'}{dx} &= \frac{d\zeta'}{dx} + \frac{d\zeta''}{dx},\end{aligned}$$

where

$\left. \begin{matrix} \eta', \zeta' \\ \eta'', \zeta'' \end{matrix} \right\}$ are the components of the displacements in the $\left. \begin{matrix} \text{refracted} \\ \text{pressural} \end{matrix} \right\}$ wave.

Finally, *assuming**, that for $x=0$,

$$\bar{\xi} = \bar{\xi}', \quad \bar{\eta} = \bar{\eta}', \quad \bar{\zeta} = \bar{\zeta}', \quad \frac{d\bar{\xi}}{dx} = \frac{d\bar{\xi}'}{dx}, \quad \frac{d\bar{\eta}}{dx} = \frac{d\bar{\eta}'}{dx}, \quad \frac{d\bar{\zeta}}{dx} = \frac{d\bar{\zeta}'}{dx}, \quad (5)$$

we have as the interfacial conditions, that for $x=0$,

$$\left. \begin{aligned}\xi + \xi_I + \xi_{II} &= \xi' + \xi'', & \eta + \eta_I + \eta_{II} &= \eta' + \eta'', \\ \zeta + \zeta_I + \zeta_{II} &= \zeta' + \zeta'', \\ \frac{d\xi}{dx} + \frac{d\xi_I}{dx} + \frac{d\xi_{II}}{dx} &= \frac{d\xi'}{dx} + \frac{d\xi''}{dx}, & \frac{d\eta}{dx} + \frac{d\eta_I}{dx} + \frac{d\eta_{II}}{dx} &= \frac{d\eta'}{dx} + \frac{d\eta''}{dx}, \\ & & \frac{d\zeta}{dx} + \frac{d\zeta_I}{dx} + \frac{d\zeta_{II}}{dx} &= \frac{d\zeta'}{dx} + \frac{d\zeta''}{dx}.\end{aligned}\right\} (6)$$

* C. R. ix. p. 94.

These equations express Cauchy's principle of the continuity of the motion of the æther, according to which the incident wave passes into the reflected and refracted waves "sans transition brusque."

Judging from the historical sequence of Cauchy's papers, there can be little doubt that he enunciated this principle as the physical interpretation of the result arrived at by reasoning analogous to the above; it is, however, impossible to agree with v. Ettingshausen that "Cauchy hat diese Gleichungen (6) anfänglich aus Gründen gerechtfertigt, die sich auf das Verfahren der Variationen der Constanten zurückführen lassen;"* as the principle is already involved in the assumption (5)†.

All that the above analysis really leads to, and all that Cauchy‡ claimed to have established by it, is the necessity for including the pressural waves in the problem of reflection and refraction.

Since the true dynamical equations of condition, given by the equality of displacements and pressures, are that for $x=0$,

$$\left. \begin{aligned} \bar{\xi} &= \bar{\xi}', & \bar{\eta} &= \bar{\eta}', & \bar{\xi} &= \bar{\xi}', \\ (m-n)\bar{\delta} + 2n\frac{d\bar{\xi}}{dx} &= (m'-n')\bar{\delta}' + 2n'\frac{d\bar{\xi}'}{dx}, \\ n\left(\frac{d\bar{\xi}}{dy} + \frac{d\bar{\eta}}{dx}\right) &= n'\left(\frac{d\bar{\xi}'}{dy} + \frac{d\bar{\eta}'}{dx}\right), & n\left(\frac{d\bar{\xi}}{dx} + \frac{d\bar{\xi}}{dz}\right) &= n'\left(\frac{d\bar{\xi}'}{dx} + \frac{d\bar{\xi}'}{dz}\right), \end{aligned} \right\} (7)$$

it is clear, as has been often pointed out, that Cauchy's assumption involves that of the identity of the statical properties of the æther in the two media. Lundquist§, however, considers that "Cauchy has established his principle of continuity by the aid of analysis, the exactitude of which it is

* *Sitzb. der Wien. Akad.* xviii. p. 371.

† I do not think Cauchy contemplated a continuous rapid transition of one medium into the other (cf. *C. R.* x. p. 347); neither does v. Ettingshausen in his paper. Supposing the assumption justified on these grounds, yet, as von der Mühl has pointed out, the former assumption respecting the coefficients of the additional terms in the modified equations precludes the assumption of a finite change in the statical properties of the media (*Matt. Ann.* v. p. 477).

‡ *C. R.* x. p. 347.

§ *Pogg. Ann.* clii. p. 185.

not easy to contest ;” and hence that this result, combined with the dynamically exact conditions (7), proves “the legitimacy of Green’s assumption of the equality of the compressibility and the rigidity of the æther in the two media.”

Cauchy himself did not see that this was involved in his conditions ; and so in what follows the compressibilities and rigidities of the two media will be considered as unequal.

III.*

Taking, as before, the interface of the media as the plane of yz , and the first medium on the side of positive x , let the axis of z be parallel to the plane of the waves, so that the plane of xy is the plane of incidence ; then, if $\xi \eta \zeta$ and $\xi' \eta' \zeta'$ denote the components of the displacements in the first and second medium respectively, $\xi \eta \zeta$, $\xi' \eta' \zeta'$ will be independent of z .

(1) Let the incident vibrations be perpendicular to the plane of incidence.

The general equations of motion are in this case

$$\rho \frac{d^2 \zeta}{dt^2} = n \left(\frac{d^2 \zeta}{dx^2} + \frac{d^2 \zeta}{dy^2} \right), \quad \rho' \frac{d^2 \zeta'}{dt^2} = n' \left(\frac{d^2 \zeta'}{dx^2} + \frac{d^2 \zeta'}{dy^2} \right),$$

and the principle of continuity gives for the interfacial conditions that for $x=0$,

$$\zeta = \zeta', \quad \frac{d\zeta}{dx} = \frac{d\zeta'}{dx}.$$

Assuming

$$\zeta = C e^{(ax+by-\omega t)\sqrt{-1}} + C' e^{(-ax+by-\omega t)\sqrt{-1}},$$

$$\zeta' = e^{(a'x+by-\omega t)\sqrt{-1}},$$

we get at once

$$C = \frac{a+a'}{2a}, \quad C' = \frac{a-a'}{2a},$$

$$\therefore C' = \frac{a-a'}{a+a'} = -\frac{\sin(i-r)}{\sin(i+r)},$$

since

$$\frac{b}{a} = \tan i, \quad \frac{b}{a'} = \tan r;$$

where i , r are the angles of incidence and refraction.

* C. R. viii. p. 985; ix. pp. 1, 59, 91, 676, 726, 727; x. p. 347. *Ex. d'An. et de Phys.* i. pp. 133, 212.

(2) Let the incident vibrations be in the plane of incidence. The equations of motion in the first medium are

$$\begin{cases} \rho \frac{d^2 \xi}{dt^2} = m \frac{d}{dx} \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} \right) + n \left(\frac{d^2 \xi}{dx^2} + \frac{d^2 \xi}{dy^2} \right), \\ \rho \frac{d^2 \eta}{dt^2} = m \frac{d}{dy} \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} \right) + n \left(\frac{d^2 \eta}{dx^2} + \frac{d^2 \eta}{dy^2} \right). \end{cases}$$

Using Green's* method of separating the distortional and condensational parts of the solution, and assuming

$$\xi = \frac{d\phi}{dx} + \frac{d\psi}{dy}, \quad \eta = \frac{d\phi}{dy} - \frac{d\psi}{dx},$$

the equations of motion become

$$\frac{d^2 \phi}{dt^2} = g^2 \left(\frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right), \quad \frac{d^2 \psi}{dt^2} = \gamma^2 \left(\frac{d^2 \psi}{dx^2} + \frac{d^2 \psi}{dy^2} \right),$$

where

$$g^2 = (m+n)/\rho, \quad \gamma^2 = n/\rho.$$

Similar equations apply to the second medium.

The principle of continuity gives for the interfacial conditions that for $x=0$,

$$\left. \begin{aligned} \frac{d\phi}{dx} + \frac{d\psi}{dy} &= \frac{d\phi'}{dx} + \frac{d\psi'}{dy} \\ \frac{d\phi}{dy} - \frac{d\psi}{dx} &= \frac{d\phi'}{dy} - \frac{d\psi'}{dx} \end{aligned} \right\} \dots \dots (8)$$

$$\left. \begin{aligned} \frac{d^2 \phi}{dx^2} + \frac{d^2 \psi}{dx dy} &= \frac{d^2 \phi'}{dx^2} + \frac{d^2 \psi'}{dx dy} \\ \frac{d^2 \phi}{dx dy} - \frac{d^2 \psi}{dx^2} &= \frac{d^2 \phi'}{dx dy} - \frac{d^2 \psi'}{dx^2} \end{aligned} \right\} \dots \dots (9)$$

Since these equations are true for all values of y , we may differentiate with respect to it, and hence, by means of the equations of motion, replace (9) by

$$\frac{1}{g^2} \frac{d^2 \phi}{dt^2} = \frac{1}{g'^2} \frac{d^2 \phi'}{dt^2}, \quad \frac{1}{\gamma^2} \frac{d^2 \psi}{dt^2} = \frac{1}{\gamma'^2} \frac{d^2 \psi'}{dt^2} \dots \dots (9a)$$

It may here be noted, that if we take the general equations of condition (7) and assume the equality of the rigidities of the æther in the two media with no assumption respecting the

* Collected Works, p. 261.

compressibilities, we get, instead of (9a),

$$\rho \frac{d^2 \phi}{dt^2} = \rho' \frac{d^2 \phi'}{dt^2}, \quad \rho \frac{d^2 \psi}{dt^2} = \rho' \frac{d^2 \psi'}{dt^2} \dots \dots \dots (9b)$$

Assume

$$\begin{aligned} \phi &= B_{//} e^{(a_{//}x + by - \omega t)\sqrt{-1}}, \\ \psi &= A e^{(ax + by - \omega t)\sqrt{-1}} + A_{/} e^{(-ax + by - \omega t)\sqrt{-1}}, \\ \phi' &= B'' e^{(a''x + by - \omega t)\sqrt{-1}}, \\ \psi' &= e^{(a'x + by - \omega t)\sqrt{-1}}. \end{aligned}$$

The equations of motion give

$$\omega^2 = \gamma^2(a^2 + b^2) = g^2(a_{//}^2 + b^2) = \gamma'^2(a'^2 + b^2) = g'^2(a'^2 + b^2);$$

whence

$$a_{//} = b \sqrt{1 - \frac{\omega^2}{g^2 b^2}} \cdot \sqrt{-1} = bu_{//} \sqrt{-1}, \text{ say;}$$

$$a'' = -b \sqrt{1 - \frac{\omega^2}{g'^2 b^2}} \cdot \sqrt{-1} = -bu'' \sqrt{-1}, \text{ say,}$$

the negative sign being taken, as the second medium corresponds to negative x .

From equations (8) and (9a) we get

$$\left. \begin{aligned} u_{//} B_{//} \sqrt{-1} + (A + A_{/}) &= -u'' B'' \sqrt{-1} + 1, \\ b B_{//} - a(A - A_{/}) &= b B'' - a', \\ \frac{1}{g^2} B_{//} &= \frac{1}{g'^2} B'', \\ \frac{1}{\gamma^2} (A + A_{/}) &= \frac{1}{\gamma'^2}. \end{aligned} \right\} \quad (10)$$

The last two of these equations give

$$A + A_{/} = \frac{\gamma^2}{\gamma'^2} = \mu^2,$$

where μ is the refractive index, and

$$B_{//} = \frac{g^2}{g'^2} B'' = \frac{a'^2 + b^2}{a_{//}^2 + b^2} B'' = \mu_0^2 B'', \text{ say.}$$

Substituting in the first two of equations (10), we get

$$(u_{//} B_{//} + u'' B'') \sqrt{-1} = 1 - \mu^2,$$

whence

$$B'' = \frac{\mu^2 - 1}{\mu_0^2 u_{//} + u''} \sqrt{-1},$$

and

$$\begin{aligned} A - A_1 &= \frac{a'}{a} + \frac{b}{a}(B_{//} - B'') \\ &= \frac{a'}{a} + \frac{b}{a}(\mu^2 - 1)M\sqrt{-1}, \end{aligned}$$

where

$$M = \frac{\mu_0^2 - 1}{\mu_0^2 u_{//} + u''}.$$

Hence

$$\begin{aligned} 2A &= \mu^2 + \frac{a'}{a} + \frac{b}{a}(\mu^2 - 1)M\sqrt{-1} \\ &= \frac{a'^2 + b^2}{a^2 + b^2} + \frac{a'}{a} + \frac{b}{a} \cdot \frac{a'^2 - a^2}{a^2 + b^2} \cdot M\sqrt{-1} \\ &= \frac{(aa' + b^2) + b(a' - a)M\sqrt{-1}}{a(a^2 + b^2)}(a' + a) = 2R e^{\delta\sqrt{-1}}, \text{ say;} \\ 2A_1 &= \frac{(aa' - b^2) - b(a' + a)M\sqrt{-1}}{a(a^2 + b^2)} \cdot (a' - a) = 2R_1 e^{\delta_1\sqrt{-1}}, \text{ say.} \end{aligned}$$

Then R, R_1 denote the amplitudes of the incident and reflected vibrations, and δ, δ_1 the difference of phase between the incident and refracted, the reflected and refracted waves respectively.

Hence, if α is the azimuth with respect to the plane of incidence of the incident vibration, the reflected vibration will in general be elliptical with a difference of phase $\delta_1 - \delta$ between the components in and perpendicular to the plane of incidence; and if this difference of phase is destroyed, the azimuth β of the resulting rectilinear vibration will be given by

$$\cot \beta = R_1/C_1.$$

Hence

$$\frac{\cot \beta}{\cot \alpha} e^{(\delta_1 - \delta)\sqrt{-1}} = - \frac{(aa' - b^2) - b(a' + a)M\sqrt{-1}}{(aa' + b^2) + b(a' - a)M\sqrt{-1}}; \quad (11)$$

and

$$\begin{aligned} \frac{\cot^2 \beta}{\cot^2 \alpha} &= \frac{(aa' - b^2)^2 + b^2(a' + a)^2 M^2}{(aa' + b^2)^2 + b^2(a' - a)^2 M^2} \\ &= \frac{\cos^2(i + r) + M^2 \sin^2(i + r)}{\cos^2(i - r) + M^2 \sin^2(i - r)}; \end{aligned}$$

also

$$\tan(\delta_i - \delta) = \frac{M\{\tan(i+r) + \tan(i-r)\}}{1 - M^2 \tan(i+r) \tan(i-r)}.$$

Total Reflection.*

If μ is less than unity, we may write $\mu = \sin I$, and we get

$$a'^2 = \frac{4\pi^2}{\lambda'^2} \cos^2 r = \frac{4\pi^2}{\lambda^2} \sin(I-i) \sin(I+i).$$

Hence, if $i > I$, the value of a' becomes imaginary, and the refracted ray will die out as it leaves the refracting surface.

Writing

$$U = \sin^{\frac{1}{2}}(i-I) \sin(i+I),$$

we must substitute in the formulæ obtained above

$$a' = -\frac{2\pi}{\lambda} U \cdot \sqrt{-1},$$

the negative sign being taken, as the second medium is on the side of negative x .

Substituting this value, we find that the reflection is total both for the vibration in the plane of incidence, and for the vibration perpendicular to the plane of incidence, and for the difference of phase between the components of the reflected ray we get from (11)

$$\frac{\cot \beta}{\cot \alpha} \cdot e^{(\delta_i - \delta)\sqrt{-1}} = \frac{\sin i (\sin i + UM) + \cos i (M \sin i + U) \sqrt{-1}}{\sin i (\sin i + UM) - \cos i (M \sin i + U) \sqrt{-1}};$$

whence

$$\begin{aligned} \tan \frac{\delta_i - \delta}{2} &= \frac{\cos i \{M \sin i + \sin^{\frac{1}{2}}(i-I) \sin^{\frac{1}{2}}(i+I)\}}{\sin i \{\sin i + M \sin^{\frac{1}{2}}(i-I) \sin^{\frac{1}{2}}(i+I)\}} \\ &= \frac{M \frac{\sin^2 I}{\sin i} + \sin^{\frac{1}{2}}(i-I) \sin^{\frac{1}{2}}(i+I)}{\sin^2 i} \cdot \cos i, \end{aligned}$$

if the square and higher powers of the small quantity M are neglected.

Cauchy has $\sin^2 i$ instead of $\sin^2 I$ in the numerator of the

* *C. R.* ix. p. 764; *xxx.* p. 465.

last expression; the correct formula was first given by Beer*.

IV.†

Before proceeding further, it will be as well to discuss the value of the expression denoted above by M.

Cauchy, not seeing that his equations of condition involved the assumption of the identity of the statical properties of the æther in the two media, adopted the following relations,

$$m+n=-\epsilon^2 n, \quad m'+n'=-\epsilon'^2 n',$$

where ϵ, ϵ' are very small numerics.

These relations give

$$\mu_0^2 = \frac{m+n}{\rho} \times \frac{\rho'}{m'+n'} = \mu^2 \frac{\epsilon^2}{\epsilon'^2},$$

$$u_{II} = \sqrt{1 - \frac{n}{(m+n) \sin^2 i}} = \sqrt{1 + \frac{1}{\epsilon^2 \sin^2 i}} \simeq \frac{1}{\epsilon \sin i},$$

$$u'' \simeq \frac{1}{\epsilon' \sin r}.$$

Hence

$$M = \frac{\mu^2 \frac{\epsilon^2}{\epsilon'^2} - 1}{\mu^2 \frac{\epsilon^2}{\epsilon'^2} \cdot \frac{1}{\epsilon \sin i} + \frac{1}{\epsilon' \sin r}} = \epsilon \sin i - \epsilon' \sin r = E \sin i,$$

if $E = \epsilon - \frac{\epsilon'}{\mu}$ ‡.

No attempt has been made, so far as I am aware, to indicate the reasons which led to Cauchy's adoption of the above remarkable relations between the coefficients of compressibility and rigidity of the æther in a medium.

In order to find a relation between the coefficients, Cauchy considered the condition which must be fulfilled if the incident light is completely polarized by reflection.

This condition is that $M=0$, giving since

$$M = \frac{\mu_0^2 - 1}{\mu_0^2 u_{II} + u''} = \frac{u_{II} - u''}{1 - u_{II} u''},$$

where u_{II}, u'' are both positive, that

$$u_{II} = u'', \quad \text{or} \quad \frac{\rho}{m+n} = \frac{\rho'}{m'+n'}.$$

* Pogg. Ann. xci. p. 274.

† C. R. ix. pp. 691, 727.

‡ C. R. xxviii. p. 64. Originally Cauchy took $\epsilon=0$.

In his first memoir on the subject, Cauchy*, forgetting to take into account the fact of the media being on different sides of the plane of yz ; wrote

$$a'' = bu'' \sqrt{-1},$$

where u'' is positive.

Hence he obtained

$$M = \frac{\mu_0^2 - 1}{\mu_0^2 u_{II} - u''} = \frac{u_{II} + u''}{1 + u_{II} u''},$$

where u_{II} , u'' are both positive, giving as the condition for complete polarization

$$u_{II} = u'' = \infty, \text{ or } m + n = 0 = m' + n'.$$

He then argued that incomplete polarization must be due to the fact that these expressions differ slightly from zero, and that their value must be negative, in order that the pressural waves should be insensible at a distance from the interface for all angles of incidence.

In a memoir published in 1840 and in the *Exercices d'Analyse et de Physique*†, this mistake was corrected, and the true condition $\rho/(m+n) = \rho'/(m'+n')$ was given; but, apparently led astray by his original mistake and by a desire‡ (afterwards given up, *Compt. Rend.* xxviii. p. 125) to make complete polarization depend on the properties of the refracting medium alone, and not on any relation between the two media, he still adopted the solution

$$m + n = 0 = m' + n';$$

though he mentioned§ also the true solution, viz. that the coefficient of compressibility of the æther is infinite, and the wave-lengths of the pressural waves in the two media are equal.

Assuming that the æther is incompressible, the polarization of the reflected ray will be elliptical when the wave-lengths of the pressural waves are unequal, and we get

$$M = \left(\frac{\lambda_{II}^2}{\lambda''^2} - 1 \right) / \left(\frac{\lambda_{II}^2}{\lambda''^2} + 1 \right),$$

where λ_{II} , λ'' are the wave-lengths of the pressural waves in the two media. This is Eisenlohr's suggestion; but the

* C. R. ix. p. 94. *Ex. d'An. et de Phys.* i. p. 167.

† C. R. x. p. 357. *Ex. d'An. et de Phys.* i. p. 233.

‡ C. R. ix. p. 727. § *Ibid.* x. p. 358.

form in which he made it does not show that it involves the absolute incompressibility of the æther. If $\lambda_{//}/\lambda'' = \lambda/\lambda'$, we get Green's formula. Eisenlohr* says that this assumption is absolutely untenable: it is, however, as Green shows, a direct consequence of the assumption made by him, and involved in Cauchy's conditions, viz. the identity of the statical properties of the æther in the two media.

Further than that, if we assume only the equality of the rigidities, the equations of condition become (8), (9b); whence

$$\frac{B_{//}}{B''} = \frac{\rho'}{\rho} = \mu^2;$$

and if the æther is incompressible,

$$M = \frac{\mu^2 - 1}{\mu^2 + 1}.$$

Haughton's† suggestion that the coefficient of compressibility is very great, but not infinite, does not help matters; so that it would appear that the only way to escape the difficulty is by one of Lord Rayleigh's‡ suggestions:—

(1) That “although the transition between the two media is so sudden that the principal waves of transverse vibrations are affected nearly in the same way as if it were instantaneous, yet we may readily imagine that the case is different for the surface-waves, whose existence is almost confined to the layer of variable density.”

(2) That “the densities concerned in the propagation of the so-called longitudinal waves are unknown, and may possibly not be the same as those on which transverse vibrations depend.”

Eisenlohr§ gives another (it appears entirely empirical) value for M : it involves, as Cauchy's, a negative value for the coefficient compressibility of the æther, and leads to formulæ closely agreeing with experiment; as, however, they contain a third disposable constant, this close agreement is hardly to be wondered at.

V.

Cauchy's formulæ for metallic reflection were originally published on April 15, 1839||, and thus were obtained from his

* Pogg. *Ann.* civ. p. 358.

† Phil. Mag. [4] iv. p. 81.

‡ *Ibid.* xlii. pp. 96, 97.

§ Pogg. *Ann.* civ. p. 356.

|| *C. R.* viii. p. 553.

second set of equations of condition, in which the pressural waves were neglected. The formulæ were republished on January 17, 1848*, and apparently no attempt was made to obtain equations in which the influence of the pressural waves was included.

Cauchy considers the peculiarities of metallic reflection to be due to a complex value of the refractive index.

Writing

$$\mu = \theta e^{e\sqrt{-1}},$$

we get

$$a'^2 = \frac{4\pi^2}{\lambda^2} (\theta^2 e^{2e\sqrt{-1}} - \sin^2 i) = \frac{4\pi^2}{\lambda^2} \cdot U^2 e^{2u\sqrt{-1}}, \text{ say;}$$

whence

$$U^2 \sin 2u = \theta^2 \sin 2\epsilon, \quad \cot \overline{2u - \epsilon} = \cot \epsilon \cos \left(2 \tan^{-1} \frac{\sin i}{\theta} \right). \quad (12)$$

Substituting

$$a' = -\frac{2\pi}{\lambda} U e^{u\sqrt{-1}}, \quad a = \frac{2\pi}{\lambda} \cos i, \quad b = \frac{2\pi}{\lambda} \sin i,$$

in the values of $C//C$, $A//A$, and making $M=0$, we get at once Cauchy's well-known formulæ.

Making these same substitutions in (11), we get

$$\frac{\cot \beta}{\cot \alpha} e^{\Delta\sqrt{-1}} = \frac{\sin^2 i + \cos i U e^{u\sqrt{-1}}}{\sin^2 i - \cos i U e^{u\sqrt{-1}}};$$

whence

$$\tan \Delta = \frac{2U \sin u \cos i \sin^2 i}{\sin^4 i - U^2 \cos^2 i} = \sin u \tan \left(2 \tan^{-1} \frac{U \cos i}{\sin^2 i} \right),$$

$$\frac{\cot^2 \beta}{\cot^2 \alpha} = \frac{\sin^4 i + \cos^2 i U^2 - 2 \sin^2 i \cos i U \cos u}{\sin^4 i + \cos^2 i U^2 + 2 \sin^2 i \cos i U \cos u},$$

$$= \cot(\psi - 45^\circ),$$

where

$$\cot \psi = \cos u \sin \left(2 \tan^{-1} \frac{U \cos i}{\sin^2 i} \right),$$

or, if $\alpha = 45^\circ$,

$$\cot 2\beta = \cos u \left(\sin 2 \tan^{-1} \frac{U \cos i}{\sin^2 i} \right).$$

At the polarizing angle I , for which $\Delta = \pi/2$, we have

$$U = \tan I \sin I, \quad u = 2\beta,$$

* *C. R.* xxvi. p. 86.

where β is the azimuth of the reflected vibrations, when the incident vibrations are in an azimuth 45° with respect to the plane of incidence.

These values substituted in equations (12) give the values of the constants θ , ϵ , and then these same equations serve for the determination of u , U for any other angle of incidence.

While the above equations can at the best be only considered incomplete, objections have also been made to the complex value of the refractive index involved in them.

Lord Rayleigh's criticism* that the real part of μ^2 should be positive, while the results of experiment substituted in Cauchy's equations give a value of μ^2 with its real part negative, seems not so much an argument against Cauchy's idea, as an "argument against the attempt to account for the effects on a purely elastic solid theory" †.

The value of μ^2 resulting from Sir W. Thomson's theory of light is a real negative quantity; this value substituted in Green's equations gives the reflection total at all angles of incidence. For this result there is no experimental evidence at present, except in the case of silver. The same will result from Lord Rayleigh's extension of Green's theory, unless, as seems scarcely probable, the refractive index of the pressural waves is a complex quantity.

VI.

In August 1850‡ Cauchy published the outlines of the result of applying his method to the case of reflection at the surface of an isotropic medium which possesses rotatory power.

The displacements in the upper medium are taken as

$$\xi = A b e^{(ax+by-\omega t)\sqrt{-1}} + A_1 b e^{(-ax+by-\omega t)\sqrt{-1}} + B_1 a_1 e^{(a_1x+by-\omega t)\sqrt{-1}},$$

$$\eta = -A a e^{(ax+by-\omega t)\sqrt{-1}} + A_1 a e^{(-ax+by-\omega t)\sqrt{-1}} + B_1 b e^{(a_1x+by-\omega t)\sqrt{-1}},$$

$$\zeta = \frac{2\pi}{\lambda} C e^{(ax+by-\omega t)\sqrt{-1}} + \frac{2\pi}{\lambda} C_1 e^{(-ax+by-\omega t)\sqrt{-1}};$$

* Phil. Mag. [4] xliii. p. 325.

† Eisenlohr, Wied. Ann. i. p. 204; Glazebrook, Brit. Assoc. Report, 1885, p. 197.

‡ C. R. xxxi. pp. 160, 225.

and those in the lower medium, since there will be two refracted waves circularly polarized in opposite directions,

$$\begin{aligned}\xi &= A_1' b e^{(a_1'x + by - \omega t)\sqrt{-1}} + A_2' b e^{(a_2'x + by - \omega t)\sqrt{-1}} + B'' a'' e^{(a''x + by - \omega t)\sqrt{-1}}, \\ \eta &= -A_1' a_1' e^{(a_1'x + by - \omega t)\sqrt{-1}} - A_2' a_2' e^{(a_2'x + by - \omega t)\sqrt{-1}} + B'' b e^{(a''x + by - \omega t)\sqrt{-1}}, \\ \zeta &= -\sqrt{-1} \cdot A_1' \frac{2\pi}{\lambda_1'} e^{(a_1'x + by - \omega t)\sqrt{-1}} + \sqrt{-1} A_2' \frac{2\pi}{\lambda_2'} e^{(a_2'x + by - \omega t)\sqrt{-1}}.\end{aligned}$$

Substituting these values in the equations of condition resulting from the principle of continuity, we get

$$\left. \begin{aligned}b(A + A_1 - A_1' - A_2') &= B'' a'' - B_{II} a'', \\ -(A - A_1)a + A_1' a_1' + A_2' a_2' &= b(B'' - B_{II}), \\ b\{(A - A_1)a - A_1' a_1' - A_2' a_2'\} &= (B'' a''^2 - B_{II} a_{II}^2), \\ -(A + A_1)a^2 + A_1' a_1'^2 + A_2' a_2'^2 &= b(B'' a'' - B_{II} a''), \\ C + C_1 &= \left(-\frac{\lambda}{\lambda_1'} A_1' + \frac{\lambda}{\lambda_2'} A_2'\right) \sqrt{-1}, \\ (C - C_1)a &= \left(-\frac{\lambda}{\lambda_1'} a_1' A_1' + \frac{\lambda}{\lambda_2'} a_2' A_2'\right) \sqrt{-1}.\end{aligned}\right\} \quad (13)$$

The last two of these equations give

$$\left. \begin{aligned}2aC &= \left\{ (a + a_2') \frac{\lambda}{\lambda_2'} A_2' - (a + a_1') \frac{\lambda}{\lambda_1'} A_1' \right\} \sqrt{-1}, \\ 2aC_1 &= \left\{ (a - a_2') \frac{\lambda}{\lambda_2'} A_2' - (a - a_1') \frac{\lambda}{\lambda_1'} A_1' \right\} \sqrt{-1}.\end{aligned}\right\} \quad (14)$$

From the first and fourth we get

$$A + A_1 = \frac{\lambda^2}{\lambda_1'^2} A_1' + \frac{\lambda^2}{\lambda_2'^2} A_2', \quad . \quad . \quad . \quad (15)$$

and from the second and third

$$B_{II} = \frac{a'^2 + b^2}{a_{II}^2 + b^2} B'' = \mu_0^2 B'';$$

whence, writing as before,

$$\frac{\mu_0^2 - 1}{\mu_0^2 a_{II} - a''} b = -M \sqrt{-1},$$

where M is the coefficient of ellipticity, and eliminating B_{II} B'' between the first two of equations (13),

$$(a - Mb \sqrt{-1})A - (a + Mb \sqrt{-1})A_1 = (a_1' - Mb \sqrt{-1})A_1' + (a_2' - Mb \sqrt{-1})A_2',$$

and from (15)

$$2aA = U_1A_1' + U_2A_2', \quad 2aA_r = V_1A_1' + V_2A_2', \quad (16)$$

where

$$U_1 = \{ (aa_1' + b^2) + Mb(a_1' - a) \sqrt{-1} \} \frac{a_1' + a}{a^2 + b^2} \\ = \frac{2\pi}{\lambda_1'} \{ \cos(i - r_1) + M \sin(i - r_1) \sqrt{-1} \} \frac{\sin(i + r_1)}{\sin r_1},$$

$$V_1 = \{ (aa_1' - b^2) - Mb(a_1' + a) \sqrt{-1} \} \frac{a_1' - a}{a^2 + b^2} \\ = \frac{2\pi}{\lambda_1'} \{ \cos(i + r_1) - M \sin(i + r_1) \sqrt{-1} \} \frac{\sin(i - r_1)}{\sin r_1},$$

and U_2, V_2 are similar expressions with (2) written instead of (1).

First, consider the case in which the incident vibrations are perpendicular to the plane of incidence.

Then $A = 0$, and equation (15) and the first of equations (16) give

$$A_r = \frac{\lambda^2}{\lambda_1'^2} A_1' + \frac{\lambda^2}{\lambda_2'^2} A_2', \quad U_1A_1' + U_2A_2' = 0,$$

whence from (14)

$$\frac{A_r}{\frac{\lambda^2}{\lambda_1'^2} U_2 - \frac{\lambda^2}{\lambda_2'^2} U_1} = \frac{2a \sqrt{-1} C}{\frac{\lambda}{\lambda_2'} (a + a_2') U_1 + \frac{\lambda}{\lambda_1'} (a + a_1') U_2} \\ = \frac{2a \sqrt{-1} C_r}{\frac{\lambda}{\lambda_2'} (a - a_2') U_1 + \frac{\lambda}{\lambda_1'} (a - a_1') U_2},$$

and writing for $U_1, U_2, a, a_1', a_2', b$ their values in terms of the angles of incidence and refraction,

$$A_r = -\sqrt{-1} \frac{\sin 2i \sin \frac{r_1 - r_2}{2} \cdot [\cos 2R - M \sin 2R \sqrt{-1}]}{D \left[\sin^2(i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot C_r, \\ C_r = - \frac{D \sin(i - R) \sin(i + R) + D' \sin^2 \frac{r_1 - r_2}{2}}{D \left[\sin^2(i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot C;$$

where

$$D = \cos(i - R) + M \sin(i - R) \sqrt{-1},$$

$$D' = \cos(i + R) - M \sin(i + R) \sqrt{-1},$$

$$R = \frac{r_1 + r_2}{2}, \text{ the mean angle of refraction.}$$

Omitting squares and products of the small quantities M , $\sin \frac{r_1 - r_2}{2}$, the formulæ become

$$A_1 = -\sqrt{-1} \frac{\sin 2i \sin \frac{r_1 - r_2}{2} \cdot \cos 2R}{[\cos(i - R) + M \sin(i - R) \sqrt{-1}] \sin^2(i + R)} C,$$

$$C_1 = -\frac{\sin(i - R)}{\sin(i + R)} C.$$

Hence the reflected ray will be in general elliptically polarized, except for an angle of incidence such that the angle of mean refraction is $\pi/4$, in which case the reflected ray will be plane-polarized with vibrations perpendicular to the plane of incidence. In all cases the component perpendicular to the plane of incidence is practically the same as if the medium had no rotating power, the other component being very small.

Next consider the case in which the incident vibrations are in the plane of incidence.

Then $C=0$, and from equation (14)

$$A_1' = \frac{\lambda_1'}{\lambda_2'} \cdot \frac{a + a_2'}{a + a_1'} \cdot A_2',$$

and hence

$$\begin{aligned} \frac{C_1}{(a_1' - a_2')} &= \frac{\frac{\lambda^2}{\lambda_1' \lambda_2'} a A \sqrt{-1}}{\frac{\lambda}{\lambda_2'} (a + a_2') V_1 + \frac{\lambda}{\lambda_1'} (a + a_1') V_2} \\ &= \frac{\frac{\lambda^2}{\lambda_1' \lambda_2'} a A \sqrt{-1}}{\frac{\lambda}{\lambda_2'} (a + a_2') U_1 + \frac{\lambda}{\lambda_1'} (a + a_1') U_2}. \end{aligned}$$

Whence

$$C_1 = \sqrt{-1} \cdot \frac{\sin 2i \sin \frac{r_1 - r_2}{2}}{D \left[\sin^2 (i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot A,$$

$$A_1 = \frac{D' \sin (i + R) \sin (i - R) + D \sin^2 \frac{r_1 - r_2}{2}}{D \left[\sin^2 (i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot A;$$

or, to the same degree of approximation as in the former case,

$$C_1 = \sqrt{-1} \sin 2i \frac{\sin \frac{r_1 - r_2}{2}}{[\cos (i - R) + M \sin (i - R) \sqrt{-1}] \sin^2 (i + R)}.$$

$$A_1 = \frac{\cos (i + R) - M \sin (i + R) \sqrt{-1}}{\cos (i - R) + M \sin (i - R) \sqrt{-1}} \cdot \frac{\sin (i - R)}{\sin (i + R)} \cdot A.$$

Hence the reflected ray will be in general elliptically polarized, the component of the vibration in the plane of incidence being practically the same as if the refracting medium had no rotating power, the component of the vibration perpendicular to the plane of incidence being extremely small. At the polarizing angle for which $R + i = \pi/2$, the reflected vibration is plane-polarized, and the vibrations will be at an azimuth with respect to the plane of incidence given by

$$\tan \beta = \tan 2i \cdot \frac{\sin \frac{r_1 - r_2}{2}}{M}.$$

VII.

In the same year (1850) Cauchy extended his method to the problem of crystalline reflection: the complete solution was given in a memoir presented to the French Academy on September 16, 1850*.

This memoir was never published, though it was announced† to appear in the 23rd volume of the *Mémoires de l'Académie*; and we have only slight indications of Cauchy's manner of dealing with the problem.

* *C. R.* xxxi. p. 422.

† *Tom. cit.* p. 509.

In accordance with the results of his theory of double refraction, Cauchy does not suppose the vibrations to be necessarily strictly transversal and longitudinal*. In order to eliminate the amplitudes of the latter vibrations, he assumes as an approximation the strict transversality of the former, and thus obtains† four equations between the quasi-transversal amplitudes, which contain three coefficients, whose values are known when coordinate axes are taken depending on the refracting surface and the plane of incidence.

A second memoir‡ is devoted to the determination of the value of these coefficients, when fixed directions in the crystal are taken as the axes. The value of this determination is lessened by the fact, that at the very commencement an approximation is made depending on the peculiar relation between the coefficients of elasticity, which we have considered above.

This is all that has been published, except some notes indicating a few of the results of his analysis ; it is, however, *probable*§ that Cauchy first obtained a solution on the assumption of the strict transversality of the luminous vibrations, and then proceeded to apply corrections to the values thus obtained, and it is *possible*|| that he adopted in the solution MacCullagh's idea of uniradial directions.

There is no need to enter further into this part of Cauchy's work, as Briot¶ has employed both these methods in his excellent adaptation of Cauchy's theory to the problem of Crystalline Reflection.

* C. R. xxxi. pp. 258, 299.

† Tom. cit. p. 297.

|| Tom. cit. p. 532.

† Tom. cit. p. 257.

§ Tom. cit. p. 160.

¶ Liouv. Journ. [2] xii. p. 185