

XIII.—*The General Form of the Involutive 1-1 Quadric Transformation in a Plane.* By CHARLES TWEEDIE, M.A., B.Sc.

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§ 1. In a communication read before the Society, 3rd December 1900, Dr Muir discusses the generalisation, for more than two pairs of variables, of the proposition that : If

$$x = (\eta^2 - \xi)(1 - \xi\eta); \quad y = (\xi^2 - \eta)(1 - \xi\eta);$$

then

$$\xi = (y^2 - x)(1 - xy); \quad \eta = (x^2 - y)(1 - xy).$$

If we interpret  $(x, y)$  and  $(\xi, \eta)$  as points in a plane, it is manifest that the transformation thereby obtained is a Cremona transformation. It has the special property of being reciprocal or involutive in character; *i.e.*, if the point P is transformed into Q, then the repetition of the same transformation on Q transforms Q into P. Symbolically, if the transformation is denoted by T,  $T(P) = Q$ , and  $T(Q) = T^2(P) = P$ ; so that  $T^2 = 1$ , and  $T = T^{-1}$ . Moreover, if the locus of P  $(x, y)$  is a straight line, the locus of Q  $(\xi, \eta)$  is in general a conic.

§ 2. The object of this note is to find the most general bilinear transformation connecting two points  $(x, y)$ ,  $(\xi, \eta)$  of the form

$$\left. \begin{aligned} L_1\xi + M_1\eta + N_1 &= 0 \\ L_2\xi + M_2\eta + N_2 &= 0 \end{aligned} \right\} \text{I.}$$

( $L_1$ , etc., being linear functions of  $x$  and  $y$ ) which possesses this property; *i.e.*, the most general 1-1 transformation which is involutive in character and in which to a straight line corresponds a conic.

This problem has already been discussed by CZUBER (*Monatshefte für Mathematik und Physik*, 1894), but unfortunately his discussion is not free from error, and one of the best-known transformations of the kind—the so-called Hirst transformation—entirely escapes his observation. Moreover, he describes the above transformation (I.) as the most general 1-1 point transformation, which is by no means the case, as it is not difficult to frame 1-1 point transformations in which to a straight line corresponds a curve of higher degree than the second (*v.* SALMON'S *Higher Plane Curves*). In his paper, however, he discusses a very large variety of degenerate cases, and this enables me to dispense with these entirely and to discuss only the leading case in which to a straight line corresponds a conic.

§ 3. The transformation (I.) would appear to be the most general 1-1 quadric transformation. On solving for  $\xi$  and  $\eta$  we deduce

$$\begin{aligned} \xi &= \frac{M_1N_2 - M_2N_1}{L_1M_2 - L_2M_1} \\ \eta &= \frac{N_1L_2 - N_2L_1}{L_1M_2 - L_2M_1}. \end{aligned}$$

The conics  $M_1N_2 - M_2N_1 = 0$ ;  $N_1L_2 - N_2L_1 = 0$ ;  $L_1M_2 - L_2M_1 = 0$  have three points in common, and this is the characteristic of the quadric Cremona transformation (c. SALMON).

Conversely, any three conics having three common points may be so represented. Let  $ABC PQ$ ,  $ABC QR$ ,  $ABC RP$  be three such conics (where no two of the points  $PQR$  are coincident). Let  $L_1 = 0$ ,  $L_2 = 0$  denote two lines through  $P$ ;  $M_1 = 0$ ,  $M_2 = 0$  two lines through  $Q$ ;  $N_1 = 0$ ,  $N_2 = 0$  two lines through  $R$ .

The conic  $ABC PQ$  may be represented as the intersection of corresponding rays of the pencils

$$\begin{aligned} L_1 - aL_2 &= 0 \\ M_1 - aM_2 &= 0. \end{aligned}$$

Similarly, the conic  $ABC QR$  may be obtained from

$$\begin{aligned} M_1 - aM_2 &= 0 \\ N_1 - aN_2 &= 0, \end{aligned}$$

$N_1$  and  $N_2$  being lines suitably chosen through  $R$ .

Now the two pencils of lines

$$\begin{aligned} L_1 - aL_2 &= 0 \\ N_1 - aN_2 &= 0 \end{aligned}$$

furnish a conic which clearly passes through  $A, B, C$ , also through the centres  $P$  and  $R$ , and therefore through the five points  $ABC PR$ , i.e., they furnish the conic  $ABC PR$ . The values of  $a$  corresponding to  $A, B, C$  are found by expressing the condition that the three equations in  $x$  and  $y$

$$L_1 - aL_2 = 0; \quad M_1 - aM_2 = 0; \quad N_1 - aN_2 = 0$$

be consistent. When the cubic for  $a$  has two equal roots, two of the points coincide and the conics have contact of the first order; when all three roots coincide the conics have contact of the second order—all at a common point. A point  $P$  may also coincide with  $A$ , say, in a given direction, but  $Q$  cannot coincide with  $A$  at the same time.

It may be noted that a linear construction for any number of points on the third conic is hereby indicated, the first two conics being given. Let  $S$  be any point on the first conic, and let  $QS$  meet the second conic in  $T$ . Then  $RT$  and  $PS$  intersect on the third conic in  $U$ .

§ 4. Let the bilinear equations in  $x, y, \xi, \eta$  be

$$\begin{aligned} (1) \quad & \xi(A_1x + B_1y + C_1) + \eta(A_2x + B_2y + C_2) + A_3x + B_3y + C_3 = 0 \\ (2) \quad & \xi(a_1x + \beta_1y + \gamma_1) + \eta(a_2x + \beta_2y + \gamma_2) + a_3x + \beta_3y + \gamma_3 = 0. \end{aligned}$$

If these equations are involutive, then the interchange of  $\xi$  and  $x$ ,  $\eta$  and  $y$ , must give two equations which lead to identical solutions with (1) and (2). Hence the result of the substitutions must be to replace (1) and (2) by two equations,

$$\begin{aligned} l(1) + l(2) &= 0 \\ m(1) + n(2) &= 0 \end{aligned}$$

where  $ln - lm$  is distinct from zero.

The transformed equations are

$$\begin{aligned}(1)' & x(A_1\xi + B_1\eta + C_1) + y(A_2\xi + B_2\eta + C_2) + A_3\xi + B_3\eta + C_3 = 0 \\ (2)' & x(a_1\xi + \text{etc.}) + \text{etc.} = 0.\end{aligned}$$

CASE 1. If

$$\begin{aligned}(1) & \equiv (1)' \\ (2) & \equiv (2)'\end{aligned}$$

there must exist the following equations in the coefficients of (1)—

$$\begin{aligned}A_1 &= A_1, & A_2 &= B_1, & A_3 &= C_1, \\ B_1 &= A_2, & B_2 &= B_2, & B_3 &= C_2, \\ C_1 &= A_3, & C_2 &= B_3, & C_3 &= C_3,\end{aligned}$$

so that (1) may be written as

$$\text{I.} \quad A_1x\xi + B_2y\eta + C_3 + C_1(x + \xi) + C_2(y + \eta) + B_1(\xi y + x\eta) = 0.$$

But if  $A_1 = B_1 = C_1 = 0$ , we may reverse signs throughout and deduce

$$\text{II.} \quad C_1(x \pm \xi) + C_2(y \pm \eta) + B_1(\xi y \pm x\eta) = 0.$$

Similar results hold for the second equation. (I. is practically the only case discussed by CZUBER.)

CASE 2. Nothing new is gained by supposing

$$(1)' \equiv \pm(2); \quad (2) \equiv \pm(1).$$

For the solutions of  $(1) = 0$  and  $(2) = 0$  are those of

$$(1) + (2) = 0; \quad (1) - (2) = 0,$$

and the transformation transforms these latter equations into themselves, so that there is a reduction to the preceding case.

CASE 3. More generally, no new result is obtained by supposing

$$\begin{aligned}(1)' & \equiv k(1) + l(2) & \text{(i.)} \\ (2)' & \equiv m(1) + n(2) & \text{(ii.)}\end{aligned}$$

for, since the repetition of the transformation gives again  $(1) = 0$  and  $(2) = 0$ , there would result

$$\begin{aligned}(1) & \equiv k\{k(1) + l(2)\} + l\{m(1) + n(2)\} \\ (2) & \equiv m\{k(1) + l(2)\} + n\{m(1) + n(2)\}\end{aligned}$$

$$\begin{aligned}(1) & \equiv (k^2 + lm)(1) + l(k + n)(2) \\ (2) & \equiv m(k + n)(1) + (lm + n^2)(2),\end{aligned}$$

giving

$$\left. \begin{aligned}k^2 + lm &= 1 \\ l(k + n) &= 0 \\ lm + n^2 &= 1 \\ m(k + n) &= 0\end{aligned} \right\} \text{(iii.)}$$

If  $l \neq 0$ ,  $m \neq 0$ , these equations reduce to

$$k + n = 0; \quad k^2 + lm = 1.$$

Now it is possible to determine  $a$  and  $b$  such that

$$\begin{aligned} a\{k(1)+l(2)\}+b\{m(1)-k(2)\} &\equiv a(1)+b(2), \\ a'\{k(1)+l(2)\}+b'\{m(1)-k(2)\} &\equiv -\{a'(1)+b'(2)\}; \end{aligned}$$

for these lead to

$$\begin{aligned} \left. \begin{aligned} ak+bm &= a \\ al-bk &= b \end{aligned} \right\} (A), \quad \left. \begin{aligned} a'k+b'm &= -a' \\ a'l-b'k &= -b' \end{aligned} \right\} (B), \end{aligned}$$

pairs of equations which are consistent, since  $k^2+lm=1$ . Also  $\frac{a}{a'}$  can not be equal to  $\frac{b}{b'}$ .

Hence the original equations may be replaced by

$$\begin{aligned} a(1)+b(2) &= 0 \\ a'(1)+b'(2) &= 0, \end{aligned}$$

so that the discussion again reduces to that of Case (1).

§ 5. If  $l=0$ ,  $m=0$ , then  $k=\pm 1$ ;  $n=\pm 1$ , a case already discussed.

If  $l \neq 0$ ,  $m=0$ , there result, when  $k=+1$ ,

$$\begin{aligned} (1)' &\equiv (1)+l(2) \\ (2)' &\equiv -(2). \end{aligned}$$

From the identity  $(2)' \equiv -(2)$  it follows that (2) has the form

$$x-\xi+K(y-\eta)+L(x\eta-y\xi)=0, \quad (K \text{ and } L \text{ constants}).$$

The other identity leads to

$$\begin{aligned} &\xi(A_1x+B_1y+C_1)+\eta(A_2x+B_2y+C_2)+A_3x+B_3y+C_3 \\ &\equiv \Sigma x(A_1\xi+B_1\eta+C_1)+l[x-\xi+K(y-\eta)+L(x\eta-y\xi)]. \end{aligned}$$

The comparison of coefficients shows that the corresponding equation is

$$(1) \quad \xi(A_1x+B_1y+C_1)+\eta(B_1x+B_2y+C_2)+C_1x+C_2y+C_3+l(Lx\eta+x+Ky)=0;$$

while (2) is

$$x-\xi+K(y-\eta)+L(x\eta-y\xi)=0.$$

Now, so far as solutions of (1) and (2) are concerned,

$$\begin{aligned} x+Ky+Lx\eta &= \xi+K\eta+Ly\xi \\ &= \frac{1}{2}\{x+\xi+K(y+\eta)+L(x\eta+y\xi)\}. \end{aligned}$$

Therefore for (1) may be substituted

$$\xi(A_1x+\dots)+\text{etc.}+\frac{l}{2}[x+\xi+K(y+\eta)+L(x\eta+y\xi)]=0,$$

an equation collaterally symmetrical in  $xy$ ;  $\xi\eta$ , and therefore of a type already discussed.

Finally, if  $l \neq 0$ ,  $m=0$ ,  $k=-1$ , we obtain

$$\begin{aligned} (1)' &= -(1)+l(2) \\ (2)' &= +(2), \end{aligned}$$

the solutions for which are the same as for

$$\begin{aligned} (1)-\frac{l}{2}(2) &= 0 \\ (2) &= 0, \end{aligned}$$

which are of a type already discussed.

§ 6. The analysis therefore leads to the conclusion that when the transformation is involutive the bilinear equations may be reduced to one or other of the types I. and II.

If we consider  $(\xi, \eta)$  as a fixed point, the equation I. is simply its polar with respect to the conic

$$A_1x^2 + 2B_1xy + B_1y^2 + 2C_1x + 2C_2y + C_3 = 0;$$

whereas the equation

$$C_1(x - \xi) + C_2(y - \eta) + B_1(\xi y - x\eta) = 0$$

is simply the equation to the straight line joining  $(\xi, \eta)$  to the fixed point  $\left(-\frac{C_2}{B_1}, \frac{C_1}{B_1}\right)$ .

Hence the theorem:—

*The most general transformation of the nature of a quadric inversion, in which to a straight line corresponds a conic, may be obtained as a point transformation in which:—*

First.—*To a point corresponds the intersection of its polars with respect to two fixed conics; or*

Second.—*To a point corresponds the intersection of its polar with respect to a fixed conic with the straight line joining the point to a fixed point.*

The case in which there are two equations of the form II. simply corresponds to the identical transformation. Naturally there are various degenerate cases, for many of which CZUBER's paper may be profitably consulted. The ordinary inversion is a particular case of the second transformation in which the conic is a circle, while the fixed point is its centre.

Both transformations have already been discussed geometrically, the first by BELTRAMI in 1863, in his well-known memoir, "Intorno alle coniche di nove Punti" (*Mem. della Acad. di Bologna*, Tomo II.); the second by HIRST ("Quadric Inversion of Plane Curves," *Proc. R. S. L.*, 1865).

HIRST never mentions the Beltrami transformations, although BELTRAMI had already shown how to obtain certain Hirst transformations, such as the ordinary inversion. CZUBER (*l.c.*), in his analytical discussion, has omitted the Hirst transformations entirely.

§ 7. The two transformations present several points of contrast, and that of BELTRAMI would appear to be the more symmetrical.

In the Beltrami transformation let the two conics be represented by

$$\begin{aligned} ax^2 + by^2 + cz^2 &= 0 \\ lx^2 + my^2 + nz^2 &= 0, \end{aligned}$$

as referred to their common self-conjugate triangle X Y Z.

To any point  $(\xi, \eta, \zeta)$  corresponds the point given by

$$\begin{aligned} ax\xi + by\eta + cz\zeta &= 0 \\ lx\xi + my\eta + nz\zeta &= 0, \end{aligned}$$

therefore

$$x : y : z = (bn - cm)\eta\zeta : (cl - an)\xi\zeta : (am - bl)\xi\eta$$

and similarly

$$\xi : \eta : \zeta = (bn - cm)gz : \quad \text{etc.} \quad : \quad \text{etc.}$$

There are *four* self-corresponding points, the points A, B, C, D, in which the two

conics cut, so that  $X, Y, Z$  are the intersections of pairs of opposite sides of this quadrangle.

To a straight line

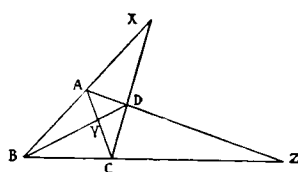
$$px + qy + rz = 0$$

corresponds the conic

$$\Sigma p(bu - cm)\eta\xi = 0,$$

*i.e.*, a conic through the principal points  $X, Y, Z$ . This conic is not degenerate unless a coefficient is zero. (This would always happen if  $b/c = m/n$ .)

If  $p = 0$ , *i.e.*, if the line passes through  $X$ , the conic breaks up into the line  $YZ$  (which corresponds to  $X$ ) and a line through  $X$ . Thus, if to the point  $P$  corresponds  $P'$ , to  $XP$  corresponds  $XP'$ , and inversely; so that the lines through a principal point



$X$  are paired in involution, the two self-corresponding lines of the involution being simply the two sides of  $ABCD$  that pass through  $X$ , viz.,  $AB$  and  $CD$ . Any two corresponding lines through  $X$  are therefore harmonically separated by  $XAB$  and  $XDC$ .

If a line  $XP$  cuts  $BC$  in  $P$ , the ray  $XP'$  corresponding to  $XP$  therefore cuts  $BC$  in  $P'$ , which is the harmonic conjugate of  $P$  with respect to  $B$  and  $C$ . Hence BELTRAMI'S theorem: To a straight line not passing through a principal point corresponds a conic through the following nine points—the three points  $X, Y, Z$ , and the harmonic conjugates of all such points  $P$  in which the straight line cuts the six sides of the quadrangle  $ABCD$ .

The discussion is simplified by noting that the conics of reference may be replaced by any two conics of the system through  $ABCD$ , and in particular by two pairs of opposite sides of  $ABCD$ , especially when all these points are real.

BELTRAMI also proves that to a curve of degree  $n$ , passing  $\alpha$  times through  $X$ ,  $\beta$  times through  $Y$ ,  $\gamma$  times through  $Z$ , there corresponds a curve of degree  $n'$ , passing  $\alpha'$  times through  $X$ ,  $\beta'$  times through  $Y$ ,  $\gamma'$  times through  $Z$ , where

$$\begin{aligned} n' &= 2n - \alpha - \beta - \gamma; \\ n' - \alpha' &= n - \alpha; \quad n' - \beta' = n - \beta; \quad n' - \gamma' = n - \gamma; \\ n &= 2n' - \alpha' - \beta' - \gamma'. \end{aligned}$$

One case is worthy of note. To a conic through two principal points corresponds a conic through the same points. If the conics intersect in  $P$  and  $P'$ ,  $P$  and  $P'$  are corresponding points and are coincident only when the conic passes through a vertex of  $ABCD$ , in which case the conics touch at that point. Hence any non-degenerate conic through two principal points and through two of the points  $ABCD$  must correspond to itself, for two conics cannot have four common points and contact at two of these points without coinciding. The points on such a conic are paired in an involution, and therefore the joins of corresponding points are concurrent, the centre of the involution being the intersections of the tangents at these two self-corresponding points through which the conic passes.

§ 8. In the Hirst transformation all points on the fixed conic are self-corresponding points, and the three principal points are X, the given fixed point, and the points of contact Y, Z of the tangents from X to the conic. To a straight line through X corresponds a line through X, but to a line through Y a line through Z, and *vice versa*, while the numerical equations for two corresponding curves are

$$\begin{aligned}u' &= 2u - \alpha - \beta - \gamma; \\u' - \alpha' &= u - \alpha; \quad u' - \beta' = u - \gamma; \quad u' - \gamma' = u - \beta; \\u &= 2u' - \alpha' - \beta' - \gamma',\end{aligned}$$

so that the Beltrami transformation is the more symmetrical.

In the Hirst transformation the points on any line through X are paired in an involution which is hyperbolic or elliptic according as the line cuts or does not cut the fundamental conic.

Also in order that a conic shall transform into a conic it must pass through two of the points X, Y, Z. Hence, if a conic transforms into itself it must pass through the two points Y and Z, for to a conic through X and Y corresponds a conic through X and Z, so that, if self-corresponding, it would pass through X, Y, Z, which is impossible.

If the self-corresponding conic through Y and Z cut the fundamental conic again in P and Q, since P and Q are self-corresponding points, it follows that X P and X Q are tangents to the new conic. The points on it are paired in involution, the centre of involution being, of course, the point X.

§ 9. If we take the fundamental conic to be

$$x^2 - yz = 0$$

it is easy to prove that the correspondence gives

$$x : y : z = \frac{1}{\xi} : \frac{1}{\xi} : \frac{1}{\eta},$$

and the self-corresponding conics are given by

$$x^2 + yz + \alpha(By + Cz) = 0.$$

There are therefore a two-fold infinity of such conics. Such a conic is over-specified, for it passes through Y, Z, P, Q, and is tangent to X P and to X Q. This suggests the following theorem:—

“If two conics cut in four points Y Z P Q, and if the pole of Y Z with respect to one conic is on the tangent at P to the second conic, it is also on the tangent at Q to the second conic and is the pole of P Q with respect to the second conic.”

This proposition may be verified analytically as follows:

Let  $x^2 - yz = 0$  be the equation in trilinear co-ordinates to one of the conics, so that the pole of  $x = 0$  with respect to it is the point X.

Let P Q be the line  $2x + By + Cz = 0$ .

The equation to any conic through the four points is

$$h(x^2 - yz) + \alpha(2x + By + Cz) = 0.$$

Let  $(\xi, \eta, \zeta)$  be the co-ordinates of P. The equation to the tangent at P is

$$x(2k\xi + 4\xi + B\eta + C\zeta) + y(\text{etc.}) + z(\text{etc.}) = 0.$$

Hence, if this line pass through X,

$$\begin{array}{r} 2k\xi + 4\xi + B\eta + C\zeta = 0; \\ 2\xi + B\eta + C\zeta = 0 \\ \hline \therefore 2k\xi + 2\xi = 0 \end{array}$$

Hence  $k = -1$ , and the conic has for equation

$$x^2 + ayz + x(By + Cz) = 0,$$

while there is no distinction between the points P and Q. The theorem is therefore established. Various sub-cases arise according to the relative positions of the four common points.

The tangents at Y and Z to the second conic are given by

$$ax + By = 0 \quad (1)$$

$$ay + Cz = 0 \quad (2),$$

and the tangent at P to the first conic is

$$2x\xi - ay\zeta - ax\eta = 0 \quad (3).$$

These tangent lines are concurrent, provided

$$2\xi + B\eta + C\zeta = 0,$$

which is the case.

Hence the theorem :—

“If two conics cut in Y, Z, P, Q, and if the pole of YZ with respect to the first conic coincides with the pole of PQ with respect to the second conic, then the pole of YZ with respect to the second conic coincides with the pole of PQ with respect to the first conic.”

Naturally both statements admit of reciprocation. In a sense, they are particular cases of the theorem that the eight tangents to two conics at their common points in general envelop a curve of the second class.

§ 10. It may be noted that one of the three canonical forms of the 1-1 quadric transformation, as given in Miss SCOTT's *Modern Analytical Geometry*,

$$x : y : z = \frac{1}{\xi} : \frac{1}{\eta} : \frac{1}{\zeta}$$

is a Beltrami transformation and not really a Hirst transformation.

It corresponds to

$$x\xi = y\eta = z\zeta,$$

so that  $(x, y, z)$  is the point of intersection of the polars of  $(\xi, \eta, \zeta)$  with respect to the two degenerate conics

$$x^2 - y^2 = 0; \quad x^2 - z^2 = 0.$$

The other two are Hirst transformations.



§ 11. Numerous examples of either transformation are to be found in the elementary geometry. One example of each is given.

If the base  $AA'$  of a triangle  $AA'C$  is kept fixed, the orthocentre  $P$  of the triangle is such that  $C$  is the orthocentre of the triangle  $AA'P$ . Hence to  $C$  corresponds  $P$  and to  $P$  the point  $C$ . Moreover, if  $C$  moves in a straight line, the locus of  $P$  is in general a conic. The transformation is therefore one of the kind in question.

Take  $AA'$  as the  $x$ -axis, so that  $A$  and  $A'$  are the points  $(a, 0)$ ,  $(-a, 0)$ . Let  $C$  be the point  $(\xi, \eta)$ ; then  $P$  has for co-ordinates

$$x = \xi; \quad y = -(\xi^2 - a^2)/\eta.$$

Hence the two pairs of co-ordinates are connected by the relation

$$x - \xi = 0; \quad y\eta + x\xi - a^2 = 0.$$

Hence the straight line  $CP$  passes through the point at infinity in a direction perpendicular to  $AA'$ , and  $P$  is on the polar of  $C$  with respect to the circle whose diameter is  $AA'$ . Hence the transformation is a Hirst transformation.

The analysis also leads to the known proposition that the three lines found by taking the polar of each vertex of a triangle with respect to the circle which has for diameter the opposite side are concurrent in the orthocentre of the triangle.

§ 12. A Beltrami transformation is furnished by the following theorem of Professor CHRYSTAL'S, and its generalisation, viz.: "A circle meets the side  $BC$  of a triangle  $ABC$  in  $D$  and  $D'$ ,  $CA$  in  $E$  and  $E'$ , and  $AB$  in  $F$  and  $F'$ . If  $AD$ ,  $BE$ ,  $CF$  be concurrent, then  $AD'$ ,  $BE'$ ,  $CF'$  are also concurrent.

This is included in the following:  $P$  and  $P'$  are two points taken in the side  $OA$  of the triangle  $OAB$ , and  $Q$   $Q'$  on the side  $OB$  such that  $OP \cdot OP' = \rho$ ,  $OQ \cdot OQ' = \sigma$ , where  $\rho$  and  $\sigma$  are constants.  $AQ$  and  $BP$  meet in  $S$ ;  $AQ'$  and  $BP'$  meet in  $S'$ . If  $AB$  is met by  $OS$  in  $R$  and by  $OS'$  in  $R'$ , it follows that the six points  $PP'QQ'RR'$  lie on a conic. Moreover, to  $S$  corresponds a unique point  $S'$  and inversely to  $S'$  corresponds  $S$ , so that the transformation from  $S$  to  $S'$  is involutive. Also if  $S$  move on a straight line  $S'$  in general traces out a conic. The transformation ought therefore to be either a Hirst transformation or a Beltrami transformation.

To obtain the transformation, let  $OA$  and  $OB$  be taken as axes, and let  $OA = a$ ,  $OB = b$ ,  $OP = \alpha$ ,  $OQ = \beta$  (so that  $\alpha$  and  $\beta$  vary).

Then  $AQ$  and  $BP$  have for equations

$$\left. \begin{aligned} \frac{x}{a} + \frac{y}{\beta} &= 1 \\ \frac{x}{\alpha} + \frac{y}{b} &= 1 \end{aligned} \right\} \quad (\text{i.})$$

Hence, if  $S$  be the point  $(x, y)$ ,

$$\alpha = \frac{bx}{b-y}; \quad \beta = \frac{ay}{a-x}. \quad (\text{ii.})$$

Similarly if  $S'$  be the point  $(\xi, \eta)$ ,

$$\alpha = \rho \frac{b-\eta}{b\xi}; \quad \beta = \sigma \frac{a-\xi}{a\eta} \quad (\text{iii.})$$

$$\therefore \frac{bx}{b-y} = \rho \frac{b-\eta}{b\xi}; \quad \frac{ay}{a-x} = \sigma \frac{a-\xi}{a\eta} \quad (\text{iv.})$$

$$\text{i.e., } b^2x\xi - \rho(b-y)(b-\eta) = 0; \quad a^2y\eta - \sigma(a-x)(a-\eta) = 0. \quad (\text{v.})$$

Hence  $(x, y)$  is the point of intersection of the polars of  $(\xi, \eta)$  with respect to the two degenerate conics:—

$$b^2x^2 - \rho(b-y)^2 = 0; \quad a^2y^2 - \sigma(a-x)^2 = 0. \quad (\text{vi.})$$

These conics determine a quadrilateral  $XYZW$ , which is such that the intercepts cut off by its sides on  $OA$  and  $OB$  are bisected at  $O$ , while the two pairs of lines pass through  $A$  and  $B$  respectively. For these conics may be substituted any two conics through  $XYZW$ .

Two of the principal points are  $A$  and  $B$ . The third principal point is not  $O$ , but the intersection  $C$  of  $XZ$  and  $YW$  which are given by

$$x^2\sigma(\rho\sigma - a^2b^2) - y^2\rho(\rho\sigma - a^2b^2) - 2a\rho\sigma(\sigma - b^2)x + 2b\rho\sigma(\rho - a^2)y + \rho\sigma(a^2\sigma - b^2\rho) = 0.$$

( $ABC$  is the self-conjugate triangle of the system of conics.)

The lines through the origin parallel to these are given by

$$x^2\sigma - y^2\rho = 0.$$

They are therefore parallel to the sides of the parallelogram  $KLMN$ , where  $KLMN$  are the points in which  $OA$  and  $OB$  are cut by the sides of  $XYZW$ . Hence the third line-pair  $XZ$  and  $YW$  are such that each makes on the axes intercepts the square of whose ratio is  $\rho : \sigma$ . When  $\rho = \sigma$ , i.e., when the circles of inversion coincide, the parallelogram is a rectangle, and each line of the third line-pair makes equal intercepts on the axes.

In the transformation as a Beltrami transformation the point  $O$  has no important rôle. In the general Beltrami transformation the lines through  $B$  are paired in involution, and they therefore determine a point-range in involution on any straight line such as  $OA$ . Similarly, the lines through  $A$  determine an involution on  $OB$ , and we have here the particular case in which the centres of involution of the point-ranges coincide in the point common to the two ranges. If  $A$  and  $B$  are real, and if  $LN$  pass through  $B$ , the locus of its middle point  $O$  is a conic passing through  $A$ , and similarly for  $O$  as the middle point of  $KM$  through  $A$ . Hence when  $A$  and  $B$  are real there may exist real points in finite number possessing the property in question.

The transformation suggests the apparent generalisation:— $A$  and  $B$  are fixed points in the plane of two involutive point-ranges  $PP'$  and  $QQ'$ , where  $PP'$ ,  $QQ'$  denote corresponding points of the respective involutions. The lines joining  $A$  and  $B$  to  $PP'$  and  $QQ'$  determine a quadrilateral  $STS'T'$  in which  $S$  corresponds to  $S'$  (or  $T$  to  $T'$ ) in a 1-1 involutive quadric transformation.