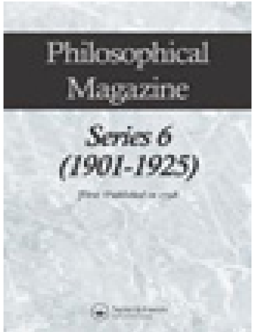

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LII. On expansion in Bessel's functions

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The two last equations reducing to those above for $y=0$ and $z=0$.

But our hypothesis requires that the equation

$$\xi^2+\eta^2+\zeta^2 = c^2\tau^2$$

shall be a result of the equation

$$x^2+y^2+z^2 = c^2t^2.$$

This is so if, and only if, c_1, c_2, d_1, d_2 are all zero.

Thus we have arrived exactly at the transformation as given above, and, as Einstein has shown (*loc. cit.*), in order that the electromagnetic equations may be invariant under this transformation the electric and magnetic vectors in the two systems must be correlated in the manner done in this paper.

Bücherer in the paper referred to does not take into account this necessary modification of coordinates, and therefore when in the latter part of it he evaluates the electromagnetic mass of the electron on the assumption that it is spherical he is in reality considering the Abraham electron, and so obtains Abraham's expression for its mass.

LII. *On Expansion in Bessel's Functions.* *By* ANDREW STEPHENSON *.

IN the ordinary Fourier expansion in sine series

$$f(x) = \Sigma \mathrm{A} \sin \alpha x$$

for the range of x from 0 to c the determination of the coefficients depends upon the vanishing of the integral $\int_0^c \sin \alpha_k x \sin \alpha_m x dx$ when $k \neq m$. I have shown, however, that the coefficients can also be found readily in the more general case when

$$\int_0^c \sin \alpha_k x \sin \alpha_m x \, dx = p \sin \alpha_k c \sin \alpha_m c,$$

where $k \neq m$ and p is some constant †. Similarly the cosine expansion can be effected if

$$\int_0^c \cos \alpha_k x \cos \alpha_m x \, dx = p \cos \alpha_k c \cos \alpha_m c.$$

* Communicated by the Author

† "An Extension of the Fourier method of Expansion in Sine Series," Messenger of Mathematics, vol. xxxiii. pp. 70–77 (1903). A more general discussion is given in a second paper in the same volume.

There is nothing in this method peculiar to the trigonometric functions, and we now apply it to the Bessel's expansions to obtain a generalization which is essential for the complete solution of a certain type of physical problem.

If $B_n(\mu r)$ is a particular solution of

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\left(\mu^2-\frac{n^2}{r^2}\right)u=0;$$

then, for $\mu_k \neq \mu_l$,

$$\int_a^b B_n(\mu_k r)B_n(\mu_l r)r\,dr=\frac{1}{\mu_k^2-\mu_l^2}\Big[r\{\mu_l B_n(\mu_k r)B_n'(\mu_l r)-\mu_k B_n'(\mu_k r)B_n(\mu_l r)\}\Big]_a^b, \quad \text{(i.)}$$

which is

$$=\frac{a}{\mu_k^2-\mu_l^2}\{\mu_k B_n'(\mu_k a)B_n(\mu_l a)-\mu_l B_n(\mu_k a)B_n'(\mu_l a)\},$$

if $$B_n(\mu_k b)=B_n(\mu_l b)=0;$$

and therefore

$$\int_a^b B_n(\mu_k r)B_n(\mu_l r)r\,dr=pB_n(\mu_k a)B_n(\mu_l a),$$

if the μ's are roots of

$$a\frac{B_n'(\mu a)}{B_n(\mu a)}=p\mu+q\frac{1}{\mu}. \quad \text{(ii.)}$$

Also from (i.) by approaching the limit $\mu_l=\mu_k$, we find

$$\int_a^b B_n^2(\mu_k r)r\,dr=\tfrac{1}{2}\left\{b^2B_n'^2(\mu_k b)-a^2B_n'^2(\mu_k a)-\left(a^2-\frac{n^2}{\mu_k^2}\right)B_n^2(\mu_k a)\right\}.$$

Now consider the problem of expanding a function of r, for the range between a and b, in a series of Bessel's functions of the first order

$$f(r)=\Sigma AB_0(\mu r), \quad \text{(iii.)}$$

where the μ's are determined by (ii.) and the particular solutions $B_0(\mu r)_1$ of Bessel's equation are so chosen that $B_0(\mu b)=0$. Both sides of (ii.) being odd functions of μ, the negative roots are numerically equal to the corresponding positive roots; and therefore it is sufficient to consider the positive roots alone in the summation. To determine A_k multiply (iii.) by $rB_0(\mu_k r)$ and integrate between a and b: then by the preceding results

$$\int_a^b rf(r)B_0(\mu_k r)\,dr=pB_0(\mu_k a)\Sigma AB_0(\mu a)$$
$$+A_k\tfrac{1}{2}\{b^2B_0'^2(\mu_k b)-a^2B_0'^2(\mu_k a)-(a^2+2p)B_0^2(\mu_k a)\};$$

and therefore, since there is no reason to question the validity of (iii.) in the limit when $r=a$,

$$A_k = 2\frac{\int_a^b rf(r)B_0(\mu_k r)dr - pf(a)B_0(\mu_k a)}{b^2 B_0'^2(\mu_k b) - a^2 B_0'^2(\mu_k a) - (a^2+2p)B_0^2(\mu_k a)}.$$

Through lack of this expansion the solutions of certain problems have hitherto been left incomplete. Consider, for example, the symmetrical motion of a uniformly stretched circular membrane loaded in the middle, *i. e.* of an annulus to the inner circular boundary of which a load symmetrical about the centre is attached. We have with the usual notation

$$\ddot{z} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right)$$

subject to the conditions

$$z=0, \quad \text{when} \quad r=b;$$

$$\ddot{z} = m^2\frac{\partial z}{\partial r} \quad ,, \quad r=a;$$

$$z=f(r) \quad ,, \quad t=0;$$

and $$\dot{z}=0 \quad ,, \quad t=0,$$

if the membrane starts from rest.

Hence

$$z = \sum_1^\infty A_k \cos(\mu_k ct) B_0(\mu_k r),$$

where $$B_0(\mu r) = J_0(\mu r) - \frac{J_0(\mu b)}{K_0(\mu b)} K_0(\mu r),$$

and the μ's are determined by

$$\frac{B_0'(\mu a)}{B_0(\mu a)} = -\frac{c^2}{m^2}\mu.$$

Therefore

$$A_k = 2\frac{\int_a^b rf(r)B_0(\mu_k r)dr + a\frac{c^2}{m^2}f(a)B_0(\mu_k a)}{b^2 B_0'^2(\mu_k b) - a^2 B_0'^2(\mu_k a) - \left(a^2 - 2a\frac{c^2}{m^2}\right)B_0^2(\mu_k a)}.$$

The small plane oscillations of a uniform, heavy, flexible, inelastic string hanging freely with a load attached to the end may be investigated similarly.

It may be noted that the method employed in obtaining the generalized expansion can be applied to expansions in other functions, if the necessity arises in connexion with any physical problem.

June 1907.