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## 97.46 A new mean value theorem for integrals

A. McD. Mercer

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and

$$B_2 = \int_{y=0}^{\infty} \frac{y}{1+y^2} \cdot \frac{\arctan y}{y} dy = \int_{y=0}^{\infty} \frac{\arctan y}{1+y^2} dy$$

$$= \left[ \frac{1}{2} (\arctan y)^2 \right]_{y=0}^{\infty} = \frac{1}{2} \left( \frac{\pi}{2} \right)^2 = \frac{\pi^2}{8}.$$

If in the integral  $B_1$  we put  $1+y^2 = \frac{1}{x}$  so that  $y = \sqrt{\frac{1}{x}-1}$ ,  $dy = \frac{-dx}{2x^2\sqrt{\frac{1}{x}-1}}$ , then we get

$$B_1 = \frac{1}{2} \int_{x=0}^1 \frac{-\log x}{\sqrt{\frac{1}{x}-1} \cdot \frac{1}{x}} \cdot \frac{dx}{2x^2\sqrt{\frac{1}{x}-1}} = \frac{1}{4} \int_{x=0}^1 \frac{\log x}{x-1} dx,$$

i.e.,  $B_1 = \frac{1}{4}B$  by (1). Thus the equation  $B = B_1 + B_2$  becomes  $B = \frac{1}{4}B + \frac{\pi^2}{8}$  and gives  $B = \frac{\pi^2}{6}$ .

The author is grateful to the referee for suggestions of improvements.

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IMRE PATYI

Department of Mathematics, East Carolina University, 229 Austin Building,  
E 5th St, Greenville, NC 27858-4353, USA

e-mail: [patyii@ecu.edu](mailto:patyii@ecu.edu)

## 97.46 A new mean value theorem for integrals

The *first* mean value theorem for integrals states the following.

If  $\phi(x)$  is integrable on  $(a, b)$ , is of one sign there, and  $f(x)$  is continuous on  $[a, b]$  then

$$\int_a^b f(t) \phi(t) dt = f(\xi) \int_a^b \phi(t) dt \text{ for some } \xi \in (a, b). \quad (1)$$

One form of the *second* mean value theorem for integrals states the following.

If  $\phi$  is integrable on  $(a, b)$ , and  $f(x)$  is monotone in  $[a, b]$  then there is  $x \in (a, b)$  such that

$$\int_a^b f(t) \phi(t) dt = f(a) \int_a^x \phi(t) dt + f(b) \int_x^b \phi(t) dt. \quad (2)$$

In this note it is our purpose to show that (2) can be used to give a new form of (1).

*Theorem:* Let  $\phi(x)$  be integrable on  $(a, b)$  and not identically zero there. Let

$$\int_a^x \phi(t) dt \text{ and } \int_x^b \phi(t) dt \text{ be non-negative for all } x \in [a, b] \quad (3)$$

and let

$f(x)$  be continuous and monotone on  $[a, b]$ .

Then

$$\int_a^b f(t) \phi(t) dt = f(\xi) \int_a^b \phi(t) dt \text{ for some } \xi \in (a, b).$$

*Proof:* The second mean value theorem is applicable here and, since

$$\int_a^b \phi(t) dt > 0,$$

a consequence of (2) is

$$\frac{\int_a^b f(t) \phi(t) dt}{\int_a^b \phi(t) dt} = f(a) \frac{\int_a^x \phi(t) dt}{\int_a^b \phi(t) dt} + f(b) \frac{\int_x^b \phi(t) dt}{\int_a^b \phi(t) dt}. \quad (4)$$

The two quotients on the right of (4) are non-negative and their sum is unity. Hence the right-hand side lies between  $f(a)$  and  $f(b)$  and, since  $f$  is continuous, there is  $\xi \in (a, b)$  such that

$$\frac{\int_a^b f(t) \phi(t) dt}{\int_a^b \phi(t) dt} = f(\xi).$$

That is,

$$\int_a^b f(t) \phi(t) dt = f(\xi) \int_a^b \phi(t) dt$$

as was to be shown.

This resembles the *first* mean value theorem but  $\phi(t)$  is not now required to be of one sign on  $(a, b)$ .

*An example*

Take  $a = 0$ ,  $b = 5\pi$  and  $\phi(t) = \sin t$ . Clearly  $\phi(t)$  satisfies (3), so if  $f(x)$  is continuous and monotone in  $[a, b]$  we have

$$\int_0^{5\pi} f(t) \sin t dt = f(\xi) \int_0^{5\pi} \sin t dt = 2f(\xi) \text{ for some } \xi \in (0, 5\pi).$$

This has the *form* of the first mean value theorem but  $\phi(x)$  is not of one sign on  $(0, 5\pi)$ .

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1. E. C. Titchmarsh, *The theory of functions*, (2nd. ed.), Oxford University Press (1952).

A. McD. MERCER

*Dept. of Mathematics and Statistics, Univ. of Guelph, Guelph ON, Canada*

e-mail: rex.mercer@gmail.com

## 97.47 Archimedean twin circles in the arbelos

For a point  $O$  on the segment  $AB$ , let  $\alpha$ ,  $\beta$  and  $\gamma$  be semicircles with diameters  $AO$ ,  $BO$  and  $AB$  respectively, constructed on the same side. The area surrounded by the three semicircles is called an arbelos. The perpendicular to  $AB$  passing through  $O$  divides the arbelos into two curvilinear triangles with congruent incircles (see Figure 1). Circles congruent to those circles are said to be *Archimedean*. Let  $a$  and  $b$  be the radii of  $\alpha$  and  $\beta$  respectively. The common radius of Archimedean circles is expressed by  $ab/(a+b)$  [1]. In this note we give a pair of Archimedean circles, which we hope to be new (see Figure 2).

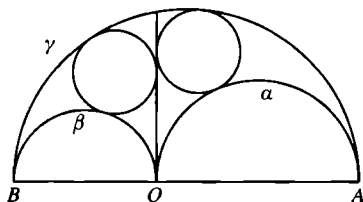


FIGURE 1

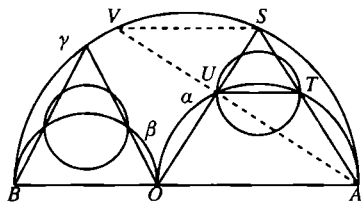


FIGURE 2

**Theorem:** Let  $S$  be the point on the semicircle  $\gamma$  such that  $|SA| = |SO|$ . If the lines  $SA$  and  $SO$  intersect  $\alpha$  at points  $T$  and  $U$  respectively, the circle with diameter  $TU$  is Archimedean.

**Proof:** An enlargement with centre  $A$  and scale factor  $(a+b)/a$  takes  $\alpha$  to  $\gamma$  and  $T$  to  $S$ . If it takes  $U$  to  $V$ ,  $|SV| = |AB| - |AO| = 2b$ . This implies that  $|TU| = 2b \times a/(a+b) = 2ab/(a+b)$ .

Similarly we can get one more Archimedean circle for the semicircle  $\beta$ .

The author would like to thank the referee for improving the proof of the theorem.

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HIROSHI OKUMURA

*251 Moo 15 Ban Kesorn Tambol Sila Amphur Muang Khonkaen 40000, Thailand*

e-mail: hiroshiokmr@gmail.com