Tutorial to evaluate measurement uncertainty as applied in 21NRM05 STASIS
Standartisation for safe implant scanning in MRI

Theoretical introduction

When evaluating measurement uncertainty, the very basic source to rely on is the document JCGM 100:2008 Evaluation of measurement data — Guide to the expression of uncertainty in measurement (GUM 1995 with minor corrections). This Guide establishes general rules for evaluating and expressing uncertainty in measurement that are intended to be applicable to a broad spectrum of measurements. It provides general rules for evaluating and expressing uncertainty in measurement.
Following are very few excerpts from the Guide, used as a basic example for evaluating the measurement data.

1. Estimate of the output quantity

In most cases, a measurand Y is not measured directly, but is determined from N input quantities X_1, X_2, ...X_N through a functional relationship

$$Y = f\left(X_1, X_2, \ldots, X_N\right) \tag{1}$$

An estimate y of the measurand Y, is obtained from Equation (1) using input estimates x_1, x_2, ..., x_N for the values of the N input quantities X_1, X_2, ..., X_N. Thus the output estimate y, which is the result of the measurement, is given by

$$y = \bar{Y} = \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{n}\sum_{k=1}^{n} f\left(X_{1,k}, X_{2,k}, \ldots, X_{N,k},\right) \tag{2}$$

That is, y is taken as the arithmetic mean or average of n independent determinations Y_k of Y, each determination having the same uncertainty and each being based on a complete set of observed values of the N input quantities X_i obtained at the same time.

2. Measurement uncertainty of the input quantities

Evaluation of the measurement uncertainty of input quantitites can be carried out in two ways:

a. Type A evaluation of measurement uncertainty,
b. Type B evaluation of measurement uncertainty.

Type A evaluation of measurement uncertainty (VIM 2.28)

It is evaluation of a component of measurement uncertainty by a statistical analysis of measured quantity values obtained under defined measurement conditions
Features:

1. Employed for input quantities which are based on replicate measurements (i.e. measurements under repeatability conditions)
2. At least 4 replicate measurements are necessary for type A evaluation
3. The uncertainty is related to the estimate of input quantity, i.e. to the arithmetic mean in this case

$$u_{\mathrm{A}} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2} \tag{3}$$

where
n number of replicate measurements

x_i the i^{th} measurement result of replicate measurements
$\bar{x}$ estimate of the overall measurement result – arithmetic mean

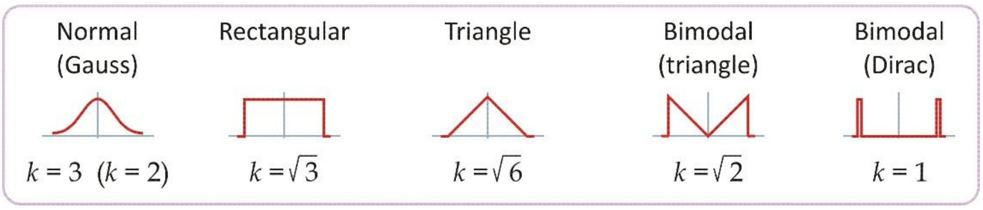
Type B evaluation of measurement uncertainty (VIM 2.29)

It is evaluation of a component of measurement uncertainty determined by means other than a Type A evaluation of measurement uncertainty.
Features:

1. employed for input quantities obtained from other sources
2. if boundaries a (sometimes designated as z_{max}) of the approximated values are given, the uncertainty evaluated by the type B method can be calculated as

$$u_B = \frac{a}{k} \tag{4}$$

where k is the value belonging to the selected approximation of the probability distribution:



EXAMPLES. Type B evaluation of measurement uncertainty is based on information:

- associated with authoritative published quantity values,
- associated with the quantity value of a certified reference material,
- obtained from a calibration certificate,
- about drift,
- obtained from the accuracy class of a verified measuring instrument,
- obtained from limits deduced through personal experience.

3. Evaluation of measurement uncertainty uy of the output quantity y

1. For non-correlated input quantities (no common influence on pairs of input quantities):

$$u_y^2 = \sum_{i=1}^{m} A_i^2 u_{x_i}^2 \tag{5}$$

where A_i (A_j respectively) are sensitivity coefficients, which can be calculated as

$$A_i = \frac{\partial f(X_1, X_2, \ldots X_m)}{\partial X_i}\bigg| X_1 = x_1, \ldots X_m = x_m \tag{6}$$

2. For correlated input quantities (common influence on pairs of input quantities exists):

$$u_y^2 = \sum_{i=1}^{m} A_i^2 u_{x_i}^2 + 2\sum_{i=2}^{m}\sum_{j<i}^{m-1} A_i A_j u_{x_{i,j}} \tag{7}$$

where $u_{x_{i,j}}$ is a covariance among estimates x_1, x_2,... x_m of correlated input quantities X_1, X_2,... X_m

3. If certain correlation between the two input quantities X_i and X_j exists, i.e. if one quantity somehow depends on the other one, their covariance must be considered as a part of the overall uncertainty of measurement. Covariance can increase or decrease the overall uncertainty of measurement.

3.a Evaluation of covariance by the type A method
If two input quantities X_i and X_j with estimates x_i and x_j are corelated, covariance evaluated by the type A method is

$$u_{\mathrm{A}\,x_{i,j}} = \frac{1}{n(n-1)}\sum_{k=1}^{n}(x_{ik} - \bar{x}_i)\left(x_{jk} - \bar{x}_j\right) \tag{8}$$

3.b Evaluation of covariance by the type B method

If two input quantities X_i and X_j with estimates x_i and x_j are correlated, the covariance evaluated by the type B method is

$$u_{\mathrm{B}\,x_{i,j}} = r_{x_{i,j}} u_{x_i} u_{x_j} \tag{9}$$

where
$r_{x_{i,j}}$ is a correlation coefficient between estimates x_i and x_j.
u_{x_i} resp. u_{x_j} are uncertainties of estimates x_i and x_j.

Finding the correlation between the power deposited into the implant and the following temperature increase

1. Scope

Determination of the relationship between the induced power and the temperature increase of the implant.

2. Technical specification ISO/TS 10974

In the *chapter 8*, the point 8.4.4.4 states that the local temperature rise ΔT or SAR at a point location in a hot spot produced by the AIMD can be related to the toral power deposition using a calibrated RF power injection method. For each AIMD hot spot, a conversion factor m between ΔT or SAR at the hot spot and injected power is experimentally determined (i.e. $\Delta T = m \cdot P_{\mathrm{inject}}$ or SAR = $m \cdot P_{\mathrm{inject}}$).

3. The task

For each implant, define a coefficient, with associated uncertainty/confidence level, that allows estimating the temperature increase after a specified time instant given the power deposited into the implant. The input data are represented by a pair – deposited power and a corresponding temperature rise.

4. Theoretical background

It is assumed that the relationship between the deposited power (denoted as X), and the temperature difference after a given time (denoted as Y), can be approximated by linear regression. The two forms of linear regression can be employed:

a. the linear regression (line) which crosses the zero point (intersection of x and y axes), i.e. $Y = \beta\, X$
b. the linear regression (line) which is shifted from the zero point (intersection of x and y axes), i.e. $Y = \alpha + \beta\, X$

For the linear regression in the form of $\boldsymbol{Y = \beta\, X}$

1. Employing the least squares method, the unknown coefficient b can be determined as follows:

$$b = \frac{1}{\sum x_i^2} \sum x_i y_i$$

where
x_i is the measured power input value (from the supplied data),
y_i is the measured temperature rise (from the supplied data).

2. The uncertainty u_b of the coefficient b can be determined as follows:

$$u_b = \frac{1}{\sqrt{\sum x_i^2}} s,$$

where s can be estimated by a formula for sample residual variance

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}[y_i - (bx_i)]^2}$$

3. The uncertainty u_y of the calculated value $y = bx$ can be determined as follows:

$$u_y = x\,u_b = x\,\frac{1}{\sqrt{\sum x_i^2}}s$$

4. Coefficient of determination R^2 is a statistical measure of how well the regression predictions approximate the real data points

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - bx_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

For the linear regression in the form of $\boldsymbol{Y = \alpha + \beta\,X}$

1. Employing the least squares method, the unknown coefficients a, and b can be determined as follows:

$$a = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

2. The uncertainties u_a, u_b of both coefficients a, and b can be determined as follows:

$$u_a^2 = \frac{\sum x_i^2}{n\sum x_i^2 - (\sum x_i)^2}s^2$$

$$u_b^2 = \frac{n}{n\sum x_i^2 - (\sum x_i)^2}s^2$$

where

$$s^2 = \frac{1}{n-2}\sum(y_i - \hat{y})^2 = \frac{1}{n-2}\sum(y_i - a - bx_i)^2$$

3. The covariance $u_{a,b}$ between the two coefficients a, and b can be determined as follows:

$$u_{a,b} = \frac{-\sum x_i}{n\sum x_i^2 - (\sum x_i)^2}s^2$$

4. The uncertainty u_y of the calculated value y can be determined as follows (the formula for u_y^2 is stated here):

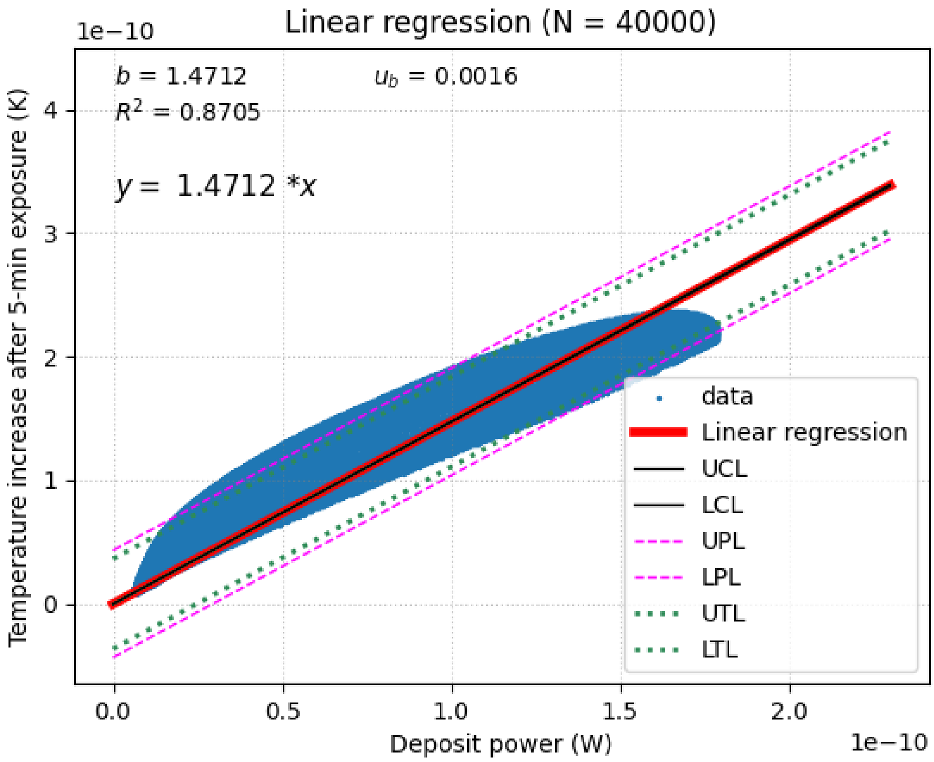
$u_y^2 = u_a^2 + x^2 \cdot u_b^2 + 2 \cdot x \cdot u_{a,b}$

5. Coefficient of determination R^2 is a statistical measure of how well the regression predictions approximate the real data points

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - a - bx_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

5. Graphic example

Ankle plate - linear regression in the form of $Y = \beta X$



Ankle plate - linear regression in the form of $Y = a + b\,X$