

## CONCLUSION.

The measurement of intensities of impulsive forces is of general interest to the scientific world and of special interest to the engineering profession.

It is suggested that in some thoroughly equipped laboratory an extension of the investigation herein described should be made.

# AN ATTEMPT AT A SYNTHETICAL DEMONSTRATION OF THE PRIMARY PROBLEMS OF THE DIFFERENTIAL CALCULUS.

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I attempt in the following paper to demonstrate the primary problems of the differential calculus by a method of direct synthetical reasoning, as distinguished from the somewhat indirect method of infinitesimals and limiting ratios. The demonstration proceeds upon the application of simple laws of proportion to variable quantities.

In the discussion the usual sign  $\int$  for a function is retained. The meaning of the new signs for the *rate of change*, the *rate-ratio*, and the *variable multiplier* will be explained in the proper place.

A *variable quantity* (or simply, *variable*) is one whose magnitude changes; a *constant quantity* (or simply, *constant*) is one whose magnitude does not change.

I designate the rapidity of the change of a quantity by the term *rate of change* (or more simply, *change-rate*), and indicate it by the letter *C* (which is the initial of "change") written before the symbol for the quantity.

Thus *C x* signifies the change-rate of *x*; *C y* the change-rate of *y*, and so on.

When the hypothesis assumes that the change-rate of a variable does not depend upon the change-rate of any other quantity, then the variable is called an *independent variable*. When the change-rate of the variable depends upon the

change-rate of some other quantity, the variable is called a *function* of that other quantity.

Properly speaking, the *change-rate* is not a quantity, although it has commensurable magnitude. It is only a property of a quantity, analogous to the velocity of a moving body, and similar to the fluxion of Sir Isaac Newton. I think the change-rate of a quantity can no more be considered a quantity, in the sense its variable is a quantity, than the color or velocity of a body can be considered the body to which it pertains.

The magnitude of the change-rate is not arbitrary in any problem. It is fixed once for all by the hypothesis of the problem itself, and can not be increased or decreased in the discussion of that problem. Its magnitude, moreover, is not dependent upon the instantaneous magnitude of its variable, but may be many times greater than the magnitude of the variable. In those cases where the magnitude of the change-rate is a constant, as hypothesis assumes it to be in the case of independent variables, the change-rate remains unchanged through all the changes of the variable, and is the same when the magnitude of the variable is zero as when it is infinitely great.

In considering the proportional effect produced upon the magnitude of a function by the changing magnitude of a variable, it becomes evident that this effect can not be due solely and entirely to the magnitude of the change-rate of the variable, but is due to the proportion which the magnitude of the change-rate bears to the simultaneous magnitude of the variable, since this proportion expresses the proportion or ratio in which the magnitude of the variable is changing under the influence of the change-rate, and that the development of this proportion must influence the development of the proportional change of the magnitude of the function.

I term this ratio of the magnitude of the change-rate to the simultaneous magnitude of its variable, the *rate-ratio*, and indicate it by the letter  $R$  written before the variable. Thus  $R x$  signifies the rate-ratio of  $x$ ;  $R y$  the rate-ratio of  $y$ , etc.

Since the rate-ratio of a variable is the proportion which the

magnitude of the change-rate bears to the simultaneous magnitude of its variable, the rate-ratio is equal to the quotient obtained by dividing the magnitude of the change-rate by the simultaneous magnitude of the variable. Or, since the symbols for the variable, the function, and the change-rate convey to the mind the notion of instantaneous magnitude, we may say more briefly and simply, the rate-ratio is the quotient of the change-rate divided by its variable. Therefore, in a similar manner we may say that the change-rate is the product of the rate-ratio multiplied by its variable, and the variable is the quotient of the change-rate divided by the rate-ratio.

Or in symbols

$$R x = \frac{C x}{x} \quad (1)$$

$$C x = (R x) x \quad (2)$$

$$x = \frac{C x}{R x} \quad (3)^*$$

Since the rate-ratio is the proportion at which, at any instant, the magnitude of a variable quantity is changing, and since this proportion is equal to the magnitude of the change-rate divided by the simultaneous magnitude of the variable, the rate-ratio of any variable quantity goes through a development which depends upon the changing proportion between the simultaneous magnitudes of the change-rate and its variable. In the case of a constant change-rate the law of the development of the rate-ratio follows the changes of the magnitude of the quotient of a constant quantity divided by a variable quantity whose magnitude is changing according to an arithmetical series. In those cases, however, when the change-rate is variable, the law of development of the rate-ratio follows the change of magnitude of a fraction whose numerator and denominator are both changing according to certain laws. Generally, however, in determining the primary laws of the calculus, we are relieved from particularly considering the law of development of the rate-ratio. We only observe the instantaneous value of the ratio, and mentally make a note of the fact that it is developing according to some law, whatever that law may be.

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\*Equation 3 is a general equation of integration.

We also observe that the expression for the magnitude of the rate-ratio,

$$\frac{C x}{x}$$

is universal, since it clearly allows for any development of the change-rate, or of the variable, or of both.

#### THE MEANING OF EQUALITY IN THE CALCULUS.

In general, when our hypothesis leads us to assume that any two quantities are equal, in reasoning on the calculus, the idea of equality has a two-fold significance: (1) that the magnitudes of the two quantities are equal at the instant of our observation; (2) that they remain equal throughout the entire development of those quantities. Thus, if we express such an hypothesis by the symbols  $x=y$ , we perceive that this equation signifies both that  $x$  and  $y$  are equal at the instant of our observation, and that they remain equal throughout their entire development. And this proposition is universally true, when comparing variables, functions, and the ratios of variables and functions. Thus, if we have the equality expressed by the symbols

$$z = \int x; y = a x; xy = z; x : y = s$$

etc., in every case the relation of equality continues throughout the entire development of the variable, or function, or ratio in question, and we perceive that the equality of constant quantities is but a special case of the general law.

In the reasoning of the calculus there is but one exception, if it be an exception, viz.: that after the analysis of the relations of the change-rate has been completed we may, in order to ascertain the momentary values of equations, make the variable equal to some constant quantity. But in this case we are no longer dealing with the laws of development of the change-rate, but only with the momentary effect produced by those laws.

Let us now examine some of the consequences of this law of equality. Suppose we are considering the case where a

variable is equal to the product of some other variable by a constant, expressed in symbols by

$$y = ax$$

This immediately gives us the proportion

$$1 : a :: x : y$$

But since the ratio  $1 : a$  is a *constant* by hypothesis, the ratio  $x : y$  must also be a *constant*, ever equal to the ratio  $1 : a$ . But this is only possible when  $x$  and  $y$  are simultaneously increasing or decreasing in the same proportion of their own simultaneous magnitudes, and when the increase or decrease of both follows the same law of development. For, in any other case, the ratio of the magnitude of  $x$  to the simultaneous magnitude of  $y$  would change during the development of the two quantities.

It therefore follows that the rate-ratio of  $y$  is the same as the rate-ratio of  $x$ , and we perceive the truth of this principle of the calculus:

I. The rate-ratio of the product of a variable by a constant is the same as the rate-ratio of the variable.

Expressed in symbols,

$$Ry = Rax = Rx = \frac{Cx}{x} \quad (4)$$

Since the change-rate of  $y$  equals the rate-ratio thereof multiplied by  $y$ , we have

$$Cy = y(Ry) = y \left( \frac{Cx}{x} \right) = ax \left( \frac{Cx}{x} \right) = aCx \quad (5)$$

and we have established this proposition:

II. The change-rate of the product of a variable by a constant is equal to the change-rate of the variable multiplied by the constant.

Evidently we may now assume that any variable is the product of a constant whose magnitude is equal to the simultaneous magnitude of its variable, multiplied by a variable whose original magnitude is unity, and whose rate-ratio is the same as the rate-ratio of the first named variable. I indicate this unity variable by the sign for unity (1) with the symbol of the variable out of which it is derived in each case written as a sub-index. Thus we may write  $x = a(I_x)$ , where  $a = x$ .

and the symbol  $(I_x)$  indicates a variable whose magnitude, at the instant of our observation, is unity, and whose rate-ratio is the same as that of  $x$ . Similarly we may write

$$y = b(I_y); z = c(I_z), \text{ etc.}$$

But the magnitude of the change-rate of  $(I_x)$  is equal to the magnitude of the rate-ratio multiplied by the simultaneous magnitude of the variable, and, therefore, the change-rate of  $(I_x)$  is equal to the rate-ratio thereof multiplied by unity. Consequently the rate-ratio and the change-rate of the unity variable are equal each to each.

In symbols

$$C(I_x) = 1 \cdot R(I_x) = \frac{C x}{x}$$

I call this unity  $(I_x)$  variable the *variable multiplier*, because it is the variable multiplier of the constant factor of the variable whose effects on a function we are considering. Certain of the conditions which arise in the discussion of the relations of the rate ratios of variables are more clearly explained by resorting to the use of it, than in any other manner.

Consider next the ratio of two quantities, and first of constants of equal magnitude. Evidently that ratio is unity and a constant. In symbols  $a : a = 1$ .

Consider next the ratio of a constant to a variable, which has momentarily the same magnitude as the constant. Evidently the momentary value of the ratio is unity, but the ratio is not now a constant.

Expressed in symbols the ratio is

$$a : x$$

We can put  $x = a(I_x)$ , and the ratio may be expressed

$$a : a(I_x)$$

Assume we started with two constants,  $a, a$ , the ratio having been

$$a : a = 1$$

Manifestly we have multiplied the consequent by the variable  $(I_x)$ , and since the antecedent remains unchanged, the ratio must be multiplied by the same variable, becoming  $1(I_x)$ , and we perceive at once that the ratio is now a variable, and has the same rate-ratio as  $x$ .

The same principle would also immediately appear upon considering the fraction

$$\frac{x}{a} \text{ or } \frac{a (I_x)}{a}$$

It is then axiomatic that the magnitude of the fraction is increasing in the same proportion of itself as the numerator is simultaneously increasing in a proportion of itself, and that the rate-ratio of the fraction must follow the same law of development as the rate-ratio of the numerator. Also, since the rate-ratio of a variable multiplied by a constant is the same as the rate-ratio of  $x$ , the rate-ratio of

$$\frac{b}{a} x$$

must be equal to the rate-ratio of  $x$ , and therefore the foregoing principles are universal, and we perceive these two propositions to be true:

III. The ratio of a constant to a variable has the same rate-ratio as the variable.

IV. The rate-ratio of a variable divided by a constant is the same as the rate-ratio of the variable.

Proposition IV also follows from proposition I. For if a variable is equal to the quotient of another variable divided by a constant, then the second variable must be equal to the product of the first variable multiplied by the constant, and this is the case of proposition I. Or in symbols, if

$$x = \frac{y}{a}$$

then must  $y = a x$ , and by proposition I,  $R y = R x$ .

But the change-rate is equal to the rate-ratio multiplied by its variable. Therefore, in symbols, for brevity, if

$$x = \frac{y}{a}, \text{ and } R x = R y$$

$$\text{then } C x = (R x) x = (R y) \frac{y}{a} = \left( \frac{C y}{y} \right) \cdot \frac{y}{a} = \frac{C y}{a} \quad (6)$$

and we have established this proposition:

V. The change-rate of the quotient of a variable divided

by a constant, is equal to the quotient obtained by dividing the change-rate by the constant.

Consider now the case of the quotient of two variables which are originally equal and have equal rate-ratios, that is, are each simultaneously increasing in the same proportion of their contemporaneous magnitudes. This quotient is manifestly unity and a constant. Or, in symbols,

$$\frac{x}{x} = 1$$

Now, suppose that, without changing its instantaneous magnitude, we increase the rate-ratio of the numerator of this fraction, by causing the numerator to increase in a greater ratio of its own magnitude than before; or, in other words of equivalent meaning, suppose that, without changing the instantaneous magnitude of the numerator, we cause it to increase more rapidly than before, by adding to its original rate-ratio some additional ratio. Evidently the numerator, with this new development of its rate-ratio, is increasing more rapidly than before, and evidently also, from the assumed conditions of its new increase, its magnitude, under the influence of the new ratio, is outstripping the simultaneous development of its old magnitude, as that proceeded under the old ratio, in exactly the ratio which was added to the old ratio. That is to say, the ratio between the simultaneous values of its old and its new magnitudes has now become a variable, instantaneously equal to unity, and increasing in the proportion of the added ratio. But the magnitude of the denominator of the fraction is, by hypothesis, equal to the original magnitude of the numerator, and the denominator also has the same rate-ratio as the numerator originally had. Therefore the magnitude of the numerator, with its new development, is outstripping the simultaneous magnitude of the denominator in exactly the same proportion as we have seen it is outstripping the development of its old magnitude as produced by its original rate-ratio. Therefore, the fraction

$$\frac{x'}{x}$$



has now become a variable; —  $x'$  representing the new development of the numerator—and we may write

$$\frac{x'}{x} = z$$

where the instantaneous magnitude of  $z$  is unity, and its rate-ratio is that which was added to the original development of the numerator. It follows, therefore, that the present rate-ratio of the numerator is equal to the original rate-ratio thereof (which is by hypothesis the same as the rate-ratio of the denominator) plus the rate-ratio of  $z$ . Or in symbols

$$R x' = R x + R z \quad (7)$$

But the equation

$$\frac{x'}{x} = z$$

gives us  $x' = z x$ , and we perceive that, in this case, the rate-ratio of  $x'$  is equal to the sum of the rate-ratios of the two variables of which  $x'$  is the product. Therefore this proposition is true:

When the magnitude of a variable is instantaneously equal to unity, the rate-ratio of the product of that variable by another variable is the sum of the rate-ratios of the variables taken separately.

But since we know, from proposition I, that the rate-ratio of any product of a variable by a constant is the same as the rate-ratio of the variable, so that we have

$$R(a x) = R x; R(b x') = R x'; R(c z) = R z$$

whatever the magnitudes of the constants may be, we perceive the foregoing principle to be universal, and we arrive at this proposition:

VI. The rate-ratio of the product of two variables is the sum of the rate-ratios of those variables taken separately.

In symbols we have, when  $z = x y$ ,

$$R z = R x + R y \quad (8)$$

Therefore

$$C z = (R z) z = (R x + R y) x y$$

and

$$C z = \left( \frac{C x}{x} + \frac{C y}{y} \right) x y = y C x + x C y \quad (9)$$

That is to say:

VII. The change-rate of the product of two variables is equal to the sum of the products of each variable by the change-rate of the other.

For the case of  $x^2$ , where  $x$  and  $y$  become equal, this resolves to

$$x C x + x C x = 2 x C x \quad (10)$$

Suppose we have to consider the rate-ratio of the product of the three variables,  $x, y, z$ .

Plainly we may let the product of  $x$  by  $y$  be equal to  $s$ , and then we have

$$R(s z) = R s + R z$$

But

$$R s = R x + R y$$

so that finally we get

$$R(x \cdot y \cdot z) = R x + R y + R z \quad (11)$$

and we perceive the general law:

VIII. The rate-ratio of the product of any number of variables is the sum of the rate-ratios of those variables taken separately.

Suppose  $s = x, y, z$

Then

$$R s = R x + R y + R z$$

and

$$C s = \left( \frac{C x}{x} + \frac{C y}{y} + \frac{C z}{z} \right) x y z = y z C x + x z C y + x y C z \quad (12)$$

That is:

IX. The change-rate of the product of any number of variables is equal to the sum of the products obtained by multiplying the change-rate of each variable by all the other variables.

This leads directly to the proposition that

$$C x^n = n x^{n-1} C x \quad (13)$$

That is,

X. The change-rate of any power of a variable is equal to the product obtained by multiplying the change-rate by the

variable raised to a power one less than its original power, and this product multiplied by the exponent of the power.

Consider next the ratio  $x : a$ .

Evidently if the antecedent is increasing in any proportion of itself, while the consequent remains constant, the ratio is decreasing in the same proportion of itself, and the proportionate change of the antecedent and of the ratio must follow the same law. But since the ratio is decreasing while the antecedent is increasing, the change of the ratio must be expressed as negative, and therefore we find in this case that the ratio is a variable quantity, changing in the same proportion as the antecedent, but negatively. Hence the rate-ratio of the ratio is  $-R x$ . But this is the same case as

$$\frac{a}{x}$$

and we perceive that—

XI. The rate-ratio of a constant quantity divided by a variable is the rate-ratio of the variable with the opposite sign.

In symbols,

$$R \left( \frac{a}{x} \right) = -R x \quad (14)$$

But,

$$C \left( \frac{a}{x} \right) = R \left( \frac{a}{x} \right) \frac{a}{x} = - \left( R x \right) \frac{a}{x} = - \frac{C x}{x} \cdot \frac{a}{x} = - \frac{a C x}{x^2} \quad (15)$$

That is,

XII. The change-rate of a constant divided by a variable at the first power is equal to the quotient obtained by dividing the product of the constant into the change-rate of the variable by the square of the variable, and this quotient having the opposite sign from that of the change-rate of the variable.

We could also deduce propositions XI and XII from proposition VI. For since  $R x y = R x + R y$ , then, when  $x y = a$ , we have  $R x + R y = 0$ , whence  $R x = -R y$ , and consequently, if  $R x$  be positive,  $R y$  must have the same expression for magnitude, but with the negative sign.

But

$$C \left( \frac{a}{x} \right) = C y = (R y) y = (-R x) y = - \frac{C x}{x} \cdot \frac{a}{x} = - \frac{a C x}{x^2} *$$

Let us next consider the case of a function which is equal to a variable plus a constant. Evidently the change-rate of such a function is solely and entirely due to the change-rate of the variable, and cannot be effected by the constant. Therefore the change-rate of such function must be the change-rate of the variable. Or, in symbols, if  $y = a + x$ , then  $C y = C x$ .

We reach the same conclusion from consideration of the rate-ratio of such a function. For, evidently, the rate-ratio thereof, that is the proportion of its own magnitude at which the function is changing, cannot now be the same as the rate-ratio of the variable, but must be to the rate-ratio of the variable in the inverse proportion of the magnitudes of the variable and the function. Or, in symbols,

$$R(a + x) : R x :: x : a + x$$

Whence we have

$$R(a + x) = \frac{(R x) x}{a + x} = \frac{C x}{a + x}$$

Consequently the change-rate of

$$a + x = R(a + x) \cdot a + x$$

which equals

$$\left( \frac{C x}{a + x} \right) (a + x) = C x \quad (16)$$

\* By a simple application of proposition VI, we shall find that the change-rate of the quotient of a constant divided by any power of a variable,

$$\left( \text{in symbols, } \frac{a}{x^n} \right)$$

is equal to the product of the change-rate of the variable into the constant, into the exponent of the power, divided by the variable raised to a power one greater than its original power, and this quotient having the minus sign. Or,

$$\left( \text{in symbols, } C \left( \frac{a}{x^n} \right) = - \frac{n a C x}{x^{n+1}} \right)$$

Similar reasoning proves that the change-rate of a variable less a constant is the same as the change-rate of the variable. Or, in symbols,  $C(x - a) = Cx$ .

Thus we have established this proposition:

XIII. The change-rate of a variable increased or diminished by a constant is the same as the change-rate of the variable.

Let us next consider the case of the sum of two variables.

When the variables are such that one may be regarded as the function of the other, we may write  $y = ax$ . So that the sum of  $x + y$  becomes  $x(1 + a)$ . But we know from previous principles that the rate-ratio of this quantity is equal to the rate-ratio of  $x$ . Let us express the relations in symbols,

$$R(x(1 + a)) = Rx$$

But the change-rate of a variable is equal to the rate-ratio thereof multiplied by the variable. That is,

$$\begin{aligned} C(x(1 + a)) &= R(x(1 + a))(x(1 + a)) = Rx(x(1 + a)) = \\ &= \frac{Cx}{x}(x(1 + a)) = Cx + aCx \end{aligned} \quad (17)$$

But since  $y = ax$ ,  $aCx$  is the change-rate of  $y$ , and we have:

XIV. The change-rate of the sum of two variables is the sum of the change-rates of the variables.

When, however, one of the variables is not a function of the other, the same law still holds true. For, in the case of the sum of two such variables, we observe we have the condition of the independent developments of the change-rates and the rate-ratios of two variables, each development unaffected by the other, and, as a necessary consequence, the total effect of the two developments must be the sum of the effects separately due to each. We thus have presented the case of a quantity made up of two independent parts, each of which is changing in some proportion of its own magnitude, unaffected by and unaffecteding the change of the other part. In such circumstances, it is evident that the proportional change of the whole, due to the proportional change of either part, must be to the proportional change of that part in the inverse ratio of the

magnitudes of the whole and the part. This gives us, as the expression for the proportional change of the whole, due to the proportional change of  $x$ ,

$$R x \left( \frac{x}{x+y} \right)$$

and similarly as to  $y$ ,

$$R y \left( \frac{y}{x+y} \right)$$

Therefore the rate-ratio of the whole is

$$R(x+y) = \frac{x(Rx) + y(Ry)}{x+y}$$

whence the change-rate of  $x+y$  is found, thus

$$C(x+y) = R(x+y)(x+y) = \left( \frac{x(Rx) + y(Ry)}{x+y} \right)(x+y) = Cx + Cy \quad (18)$$

The foregoing principles enable us to determine expressions for the change-rate of any product, quotient, power or root of a variable, or of the sum or difference, product or quotient of any combination of variables, or of the power or root of such combination.

It is now manifest that the relations of the change-rate of a variable to the change-rate of its functions are the same as those commonly said to exist between the differential of a variable and the differentials of its functions. But the method of change-rates and rate-ratios here developed has a certain logical superiority over the method of the differential, as usually treated.

In the first place, the primary definition of the change of magnitude of a variable as due to a rate of change, which is analogous to the velocity of a moving body, is a definition which corresponds to the fact. The development of a variable quantity is comparable to a ceaseless flow, but it is not comparable to a broken step-by-step change, such as the usual method of infinitesimals necessarily assumes it to be.

Again the method of rates of change is logically consistent

in defining the rate of change to be only a property of a quantity, possessing commensurable magnitude, but not itself a quantity—no more so than is the velocity of a moving body the body to which it pertains.

Finally, by insisting that the magnitude of the rate of change is not arbitrary in any problem, but is once for all fixed by the primary hypothesis of that problem, this method conforms the development of the variable through its functions to the well-known facts often presented in the concrete problems, particularly of engineering, to which the calculus is applied; in which it is logically impossible to suppose the magnitude of the rate of change to be other than the concrete conditions show it necessarily is.

The treatment of the development of a variable quantity as due to another quantity whose magnitude is arbitrary and may therefore be made infinitesimal, involves the logical fallacy that since we are unable to derive the relations of the differential from the finite magnitude assigned to it by the hypothesis of our concrete problem, we may change that magnitude to an infinitesimal magnitude, and then, having reached a limiting ratio, for the case when the differential is infinitesimal, assume, without positive proof, that ratio holds true when the differential has the assigned finite magnitude, although the operations by which we arrive at the limiting ratio cannot be performed in the case when the differential has a finite magnitude. The mind may accept the conclusion, but it objects to the logic of the method.

The method of the change-rate is a return to the conception of the great inventor of fluxions, and avoids the logical difficulties which inhere in the treatment of the differential as an arbitrary quantity.

#### THE MAXIMA AND MINIMA OF FUNCTIONS.

It has been my purpose to demonstrate the primary problems of the differential calculus, not to apply the solutions of those problems.

But the harmony of the conceptions of the change-rate and the rate-ratio with the elementary conditions of the maxima and minima of functions deserves notice.

At the instant of the maximum or minimum of a function, the differential of the function is said to be zero.

But since a differential is an infinitesimal quantity, by the usual hypothesis, and smaller than any assignable magnitude, it is not easy, upon merely logical grounds, to perceive why, at the instant of the maximum or minimum of the function, its differential need be any smaller than it always has been, nor why the limit of this quantity should not just coincide with the maximum or minimum; and the statement that the magnitude of the differential is zero at the minimum or maximum therefore becomes unnecessary and without significance.

But again, in concrete problems connected with areas of curves and like subjects, the differential is assumed to be an infinitely thin strip or lamina of the surface or solid under consideration. But if this be so, then upon no logical grounds whatever can it be shown that the division between two adjacent laminæ does not occur exactly at the minimum or maximum; but if the division between two adjacent laminæ does occur at the maximum or minimum, then there is no greater propriety in saying that the differential is zero at those points, than in saying it is zero at the indefinite number of other points where such a division occurs.

But when we once perceive that the variation of the magnitude of a function is due to a change-rate, and that if, as the function is approaching the maximum, its change-rate is positive, because it is then increasing the magnitude of the function, and if, as the function passes the maximum, its change-rate becomes negative, because it is then decreasing the magnitude of the function, it follows as a necessity of logical thinking that a positive rate can become a negative rate only by becoming zero at the instant of its change from positive to negative, and the principle becomes axiomatic that—

At the maximum or minimum of a function its change-rate is zero.

Now, the rate-ratio offers us the means of determining whether a given value for a variable, deduced by making the change-rate of the function zero, is a maximum or a minimum; for,



*When a function is passing through the maximum, the sign of its rate-ratio changes to the opposite sign from that of the function; and when the function is passing through the minimum the sign of its rate-ratio changes to the same sign as that of the function.*

This proposition is deduced from the following considerations:

At the instant of the maximum, the function has a certain magnitude (assume it to be positive). As the function passes the maximum, its magnitude cannot at once pass from positive to negative, nor can it become instantly equal to zero. For the law of the development of every variable and function is that of ceaseless, unbroken flow, which is inconsistent with either of the foregoing suppositions.

Therefore, for a certain range of development beyond the maximum, the function remains positive, but its change-rate has become negative at the instant of passing through the maximum. Therefore the rate-ratio, which is the quotient of the change-rate divided by the function, became negative as the function passed through the maximum.

For the case of the minimum, suppose the function to have a positive value, but the least possible: Upon the grounds above stated, and yet more clearly, the function remains positive as it passes the minimum. But the change-rate became positive at the instant of passing through the minimum, and therefore the sign of the rate-ratio became positive as the function passed through the minimum.

If the function is considered negative at the instant of the minimum, it will be found that the rate-ratio will become negative as the function passes through the minimum, and conversely in the case of passing through the maximum.

To apply the foregoing principles to simple cases. Suppose we are required to divide a line into two such parts that the rectangle erected on the parts of the line shall be a maximum. If the line be  $a$ , and  $x$  be one part,  $a - x$  will be the other part, and  $a x - x^2$  will be the rectangle. Since this is to be a maximum, we have

$$C(a x - x^2) = 0$$

whence

$$a C x - 2 x C x = 0$$

and

$$a = 2 x$$

whence

$$a = \frac{1}{2} x$$

Now construct the rate-ratio,

$$R(ax - x^2) = \frac{C(ax - x^2)}{ax - x^2} = \frac{Cx(a - 2x)}{ax - x^2}$$

Observation shows us:

That the magnitude of the denominator remains positive even when  $x$  becomes greater than

$$\frac{1}{2} a$$

but that the numerator becomes negative when  $x$  increases beyond

$$\frac{1}{2} a$$

because then  $a - 2x$  becomes negative. Therefore, as the function passes the value assigned to it when

$$x = \frac{1}{2} a$$

the sign of the rate-ratio becomes negative, and we know that the value

$$x = \frac{1}{2} a$$

gives a maximum value to the function.

Suppose we have a given area  $h$ , which is to be arranged into a rectangle such that the perimeter shall be a minimum.

If one side of the rectangle is  $x$ , the adjacent side is

$$\frac{h}{x}$$

and the perimeter becomes

$$2x + \frac{2h}{x}$$

But since this is to be a minimum, we have

$$C\left(2x + \frac{2h}{x}\right) = 0, \text{ whence } 2Cx - \frac{2hCx}{x^2} = 0$$

Whence,

$$2x^2 - 2h = 0, \text{ and } x = \sqrt{h}$$

Again construct the rate-ratio

$$\frac{Cx \left( \frac{2x^2 - 2h}{x^2} \right)}{\frac{2x^2 + 2h}{x}}$$

which is equal to

$$\frac{Cx(2x^2 - 2h)}{x(2x^2 + 2h)}$$

Observation shows us that when  $x$  becomes greater than  $\sqrt{h}$ , the numerator and the denominator of this fraction will both be positive, and we therefore know that the value  $x = \sqrt{h}$  is that which makes the function a minimum.

NEW YORK, MARCH 13, 1897.

## CHEMICAL SECTION.

(*Stated meeting, Tuesday, September 21, 1897.*)

DR. LEE K. FRANKEL, Vice-President, in the chair.

### THE UPPER SCHUYLKILL RIVER.

BY OSCAR C. S. CARTER,

Professor of Geology and Mineralogy, Central High School, Philadelphia.

This river, which furnishes a water supply to Philadelphia and several towns along its banks, rises in the coal regions, in the northeastern part of Schuylkill County, Pa., nearly 100 miles from Philadelphia. It flows southwest, then changes to