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A bound volume of letters from Prof. De Morgan and his son G. C. De Morgan, to A. C. Ranyard, bearing upon the foundation of The London Mathematical Society, and a letter from Mrs. De Morgan.

Tracts by Professor De Morgan:—

i. "On the Mode of using the Signs + and − in Plane Geometry."
ii. (i. *continued*) "and on the Interpretation of the Equation of a Curve."
iii. "On the word 'Αριθμός."
iv. "On a Property of Mr. Gompertz's Law of Mortality."
v. "Remark on Horner's Method of Solving Equations."
vi. "Contents of the Correspondence of Scientific Men of the Seventeenth Century."
vii. "On Ancient and Modern Usage in Reckoning."
viii. "On the Difficulty of Correct Description of Books."
ix. "On the Progress of the Doctrine of the Earth's Motion, between the times of Copernicus and Galileo."
x. "On the Early History of Infinitesimals in England."

These two volumes were left by will, by Mr. Ranyard, for the acceptance of the Council.

On Fundamental Systems for Algebraic Functions. *By* H. F. Baker. Read January 10th, 1895. Received, in abbreviated form, 18th February, 1895.

In a note which has appeared in the *Math. Annal.*, Vol. XLV., p. 118, it is verified that certain forms for Riemann's integrals, given by Herr Hensel for integrals of the first kind, and deduced by him algebraically from quite fundamental considerations, can be very briefly obtained on the basis of Riemann's theory. But a desire to dispense with the homogeneous variables used by Herr Hensel has led me to under-estimate the necessity for formally establishing one particularity implied in that representation. The assumption is made (see p. 121, bottom) that a set of fundamental integral functions, such as those of which general forms are given by Kronecker, can be taken so that in the expression of an integral function by them no redundant terms, or terms of higher infinity than that of the function, need be employed. It is obvious, of course, that not all fundamental sets have this property—for instance, when $p = 2$ and λ is great enough $(1, y+x^\lambda)$ is a fundamental set not having it (as suggested to me by Herr L. Baur). But it is easy, nevertheless, to specify such a system (having, in common with all fundamental systems, the property of not being connected by any integral relation)—as in § 2 below. This forms the main object of the present note. The following paragraphs are intended to prove, what would seem to be nearly obvious, how a fundamental system for homogeneous *forms* may be thence deduced. The actual algebraic determination of a fundamental system for an arbitrary algebraic equation, which gives such interest to Herr Hensel's papers, is not considered here.

I wish to add that the matrix Ω, occurring page 123 of my note, may not without proof be assumed to have all the elements on the right of the diagonal zero, or y_i to have the simple form there indicated—an unessential modification—nor is there need, as stated on p. 129, to take $\zeta = 0$ to be a place where $F'(\eta)$ is not zero. § 6 below is added to explain the solution given on that page (*cf.* Hensel, *Math. Annal.*, XLV., 598).

I hope I shall be allowed to supply in this connexion the reference —Christoffel, "Ueber die canonische Form der Riemannschen Integrale erster Gattung," *Annali di Mat.*, 2me serie, Tom IX., 1878–1879.

[The remarks included in square brackets were added since the paper was submitted to the referees.]

1. Suppose that for rational functions whose only infinities are at the (n or less) places $x = a$ of a Riemann's surface we can construct a set $h_1, \dots h_{n-1}$ of functions, also only infinite at these places, such that every function whose only infinities are at these places can be expressed in the form

$$F = \left(1, \frac{1}{x-a}\right)_\lambda + \left(1, \frac{1}{x-a}\right)_{\lambda_1} h_1 + \left(1, \frac{1}{x-a}\right)_{\lambda_2} h_2 + \dots,$$

in such a way that the lowest integral power of $x-a$, say $(x-a)^D$, for *which* $(x-a)^D F$ *is finite at all places* $x = a$, *is sufficient, used as a multiplier, to make every term on the right finite at* $x = a$.

Let $\tau_1+1, \dots \tau_{n-1}+1$ be the lowest integers such that

$$g_i = (x-a)^{\tau_i+1} h_i$$

is finite at all the places $x = a$. Then g_i will be an integral function on the surface, and in the sense employed in the note referred to (*Math. Annal.*, XLV., 119) will be of rank τ_i. We shall also speak of it as of dimension* τ_i+1 (see below, § 4). D is the dimension of the integral function $(x-a)^D F$ and $D-1$ its rank.

Let, now, f be any integral function of rank τ, so that $(x-a)^{-(\tau+1)} f$ is only infinite at the places $x = a$, while one at least of the values of $(x-a)^{-\tau} f$ at ∞ will be ∞. By the assumption we can write

$$(x-a)^{-(\tau+1)} f = \left(1, \frac{1}{x-a}\right)_\lambda + \left(1, \frac{1}{x-a}\right)_{\lambda_1} h_1 + \dots,$$

and the D used in stating the assumption is $\tau+1$, while

$$(x-a)^{\tau+1} \left(1, \frac{1}{x-a}\right)_{\lambda_i} h_i = (x-a)^{\tau+1} \left(1, \frac{1}{x-a}\right)_\lambda \frac{g_i}{(x-a)^{\tau_i+1}}$$

is finite at $x = a$; thus

$$\tau+1 \overline{\gtrless} \lambda_i + \tau_i + 1.$$

Hence we can write

$$f = (x-a, 1)_\lambda (x-a)^{\tau+1-\lambda} + \dots + (x-a, 1)_{\lambda_i} (x-a)^{\tau+1-\lambda_i-(\tau_i+1)} g_i + \dots,$$

namely, $(1, g_1, g_2, \dots g_{n-1})$ are a fundamental system for all integral functions f, such that in using them no terms occur on the right whose rank is greater than that of the function to be expressed.

2. Consider, then, the construction of such a system as $h_1, h_2, \dots h_{n-1}$. We may assume that the places $x = a$ are none of them branch points —that is, there are n places, and that in each sheet $x-a$ is zero of the first order—and may write

$$x-a = \frac{1}{\xi},$$

and afterwards $\xi = x$, and so are reduced to finding a fundamental

* Though the dimension thus defined will depend on the values of the coefficients in the expression of the function as well as on the algebraic form.

system for integral functions when the places $x=\infty$ are distinct. We may further assume that the derived function $F'(\eta)$ does not vanish.

The orders of infinity of all integral functions in the various sheets will therefore be all expressible by integral powers of x.

Any integral function may be represented by $(R_1, R_2, \dots R_n)$, where $R_1, \dots, R_n$, which we shall call the suffixes, are the orders of infinity in the various sheets. By subtracting a suitable polynomial in x of degree R_n, we may express the function in the form

$$(R_1, R_2, \dots R_n) = (x, 1)_{R_n} + (S_1, S_1, \dots S_{n-1}, 0) \quad \dots\dots\dots (\text{i.}).$$

Consider, then, all integral functions of which the n^{th} suffix is zero. It is possible to construct such a one with given suffixes, provided the sum of the suffixes be $p+1$; otherwise it is possible with a sum less than $p+1$ [as follows immediately from the Riemann theory]. Starting with a set of suffixes $(p+1, 0, 0, \dots 0)$, consider how far the first suffix can be reduced by increasing the $2, 3, \dots (n-1)^{\text{th}}$ suffixes. In constructing the successive functions with smaller first suffix, it will be necessary in the most general case to increase some of the other $2^{\text{nd}}, 3^{\text{rd}}, \dots (n-1)^{\text{th}}$ suffixes, and there will be a certain arbitrariness as to the way in which this shall be done. But, if we consider only those functions of which the sum of the suffixes is less than $p+2$, there will be only a finite number possible for which the first suffix has a given value. There will, therefore, only be a finite number of functions of the kind considered for which the further condition is satisfied *that the first suffix is the least possible such that it is not less than any of the others.* Let this least value be r_1, and suppose there are k_1 functions satisfying this condition. Call them the reduced functions of the first class, and in general let any function whose n^{th} suffix is zero be said to be of the first class when its first suffix is greater or not less than its other suffixes. In the same way reckon as functions of the second class all those (with n^{th} suffix zero) whose second suffix is greater than the first suffix, and greater than or equal to the following suffixes. Let the functions whose second suffix has the least value consistently with this condition be called the reduced functions of the second class, k_2 in number suppose, the value of their second suffix being r_2. Then r_2 is the least *dimension* occurring among functions of the second class. In general, reckon to the i^{th} class $(i<n)$ all those functions, with n^{th} suffix zero, whose i^{th} suffix is greater than the preceding suffixes and not less

than the succeeding suffixes; let these be k_i reduced functions of this class with i^{th} suffix equal to r_i. Clearly none of $r_1, \ldots r_{n-1}$ is zero.

Let now $(S_1 \ldots S_{i-1} R_i S_{i+1} \ldots S_{n-1}, 0)$ be any function of the i^{th} class other than a reduced function of this class,

$$(R_i > S_1, > S_2, \ldots, > S_{i-1}, \geqq S_{i+1}, \ldots \geqq S_{n-1}, R_i > r_i),$$

and let $(s_1 \ldots s_{i-1} r_i s_{i+1}, \ldots s_{n-1}, 0)$ be a selected one of the reduced functions of the i^{th} class

$$(r_1 > s_1, \ldots > s_{i-1} \geqq s_{i+1}, \ldots \geqq s_{n-1}).$$

Any one of the k_i reduced functions may be chosen. Then, by choice of a proper constant coefficient λ, we can write

$$(S_1 S_2 \ldots S_{i-1} R_i S_{i+1} \ldots S_{n-1}, 0) - \lambda x^{R_i - r_i} (s_1 s_2 \ldots s_{i-1} r_i s_{i+1} \ldots s_{n-1}, 0)$$
$$= (T_1 \ldots T_{i-1}, R'_i, T_{i+1}, \ldots T_{n-1}, R_i - r_i) \quad \ldots\ldots\ldots\ldots \text{(ii.)},$$

where $R'_i < R_i$.

T_1 may be as great as the greater of $S_1, R_i - (r_i - s_1)$, but is certainly less than R_i, and, similarly, $T_2, \ldots T_{i-1}$ are certainly less than R_i.

T_{i+1} may be as great as the greater of $S_{i+1}, R_i - (r_i - s_{i+1})$, and is therefore not greater than R_i, and, similarly, $T_{i+2}, \ldots T_{n-1}$ are certainly not greater than R_i.

Further, if $(x, 1)_{R_i - r_i}$ be a suitable polynomial of order $R_i - r_i$, we can write

$$(T_1 \ldots T_{i-1} R'_i \ldots T_{n-1}, R_i - r_i) - (x, 1)_{R_i - r_i}$$
$$= (S'_1 \ldots S'_{i-1} R''_i S'_{i+1} \ldots S'_{n-1}, 0) \ldots\ldots\ldots \text{(iii.)},$$

where R''_i may be as great as the greater of $R'_i, R_i - r_i$, and is certainly less than R_i.

S'_1 may be as great as the greater of $T_1, R_i - r_i$, and is certainly less than R_i, and so for $S'_2 \ldots S'_{i-1}$.

S'_{i+1} may be as great as the greater of $T_{i+1}, R_i - r_i$, and is certainly not greater than R_i, and so for $S'_{i+2} \ldots S'_{n-1}$

Now, there are two possibilities:

Either $(S'_1 \ldots S'_{i-1} R''_i S'_{i+1} \ldots S'_{n-1}, 0)$ is still of the i^{th} class, namely,

$$R''_i > S_1, \ldots > S'_{i-1} \geqq S'_{i+1} \ldots \geqq S'_{n-1},$$

and in this case it is of lower dimension (R''_i) than

$$(S_1 \ldots S_{i-1} R_i S_{i+1} \ldots S_{n-1}, 0).$$

Or it is a function of another class, of dimension possibly as great as before (though not greater), but such that the number of suffixes whose value is this dimension is at least one less than before.

[And, so far as these are essential to the remainder of the argument, these possibilities can be stated in the simpler forms:

(1) Either $(S'_1 \dots S'_{i-1} R''_i S'_{i+1} \dots S'_{n-1}, 0)$ is of lower dimensions than $(S_1 \dots S_{i-1} R_i S_{i+1} \dots S_{n-1}, 0)$;

(2) Or it is of the same dimension, and then belongs to a more advanced class $\{$that is to the $(i+k)^{\text{th}}$ class, where $k > 0\}$.]

In the same way, if $(t_1 \dots t_{i-1} r_i t_{i+1} \dots t_{n-1}, 0)$ be a reduced function of the i^{th} class other than $(s_1 \dots s_{i-1} r_i s_{i+1} \dots s_{n-1}, 0)$, we can, by choice of a suitable coefficient μ, write

$$(t_1 \dots t_{i-1} r_i t_{i+1} \dots t_{n-1}, 0) - \mu (s_1 \dots s_{i-1} r_i s_{i+1} \dots s_{n-1}, 0)$$
$$= (t'_1 \dots t'_{i-1} r'_i t'_{i+1} \dots t'_{n-1}, 0) \dots\dots\dots\dots (\text{iv.}),$$

where $r'_i < r_i$, $t'_1, \dots t'_{i-1}$ may be respectively as great as the greater of the pairs $(t_1, s_1) \dots (t_{i-1}, s_{i-1})$, but are each certainly less than r_i, while, similarly, none of $t'_{i+1}, \dots t'_{n-1}$ is greater than r_i.

The function $(t'_1 \dots t'_{i-1} r'_i t'_{i+1} \dots t'_{n-1}, 0)$ cannot be of the i^{th} class, since no function of the i^{th} class has its suffix less than r_i, and, whatever class it belongs to, though its dimensions may be as great as before, the number of suffixes having a value equal to this dimension is at least one less than before.

Hence, selecting $n-1$ reduced functions, one from each class, say $g_1, g_2, \dots g_{n-1}$, *any* function whatever, of dimension R_i, can be expressed as a sum of

(1) Powers of x; (2) one of $g_1 \dots g_{n-1}$ multiplied by powers of x; (3) a function which is either of lower dimension, or, if of the same dimension, has not as many suffixes reaching the dimension as the function expressed.

In none of the equations (ii.), (iii.), (iv.) does there occur any term of dimension greater than R_i; and in equation (i.) no term on the right is of higher dimension than the left.

Hence any function can be expressed in the form

$$f = (x, 1)_\lambda + (x, 1)_\mu g_1 + \dots + (x, 1)_\nu g_{n-1} + F,$$

where F is a function of lower dimension than that of f, and no terms occur of higher dimension than that of f.

Hence any function can be expressed

$$f = (x, 1)_\lambda + (x, 1)_\mu g_1 + \ldots + (x, 1)_{\nu_1} g_{n-1} + F_1,$$

where F_1 is one of the $k_1 + \ldots + k_{n-1}$ reduced functions, and hence in the form

$$(x, 1)_{\lambda_2} + (x, 1)_{\mu_2} g_1 + \ldots + (x, 1)_{\nu_2} g_{n-1} + F_2,$$

where F_2 is a reduced function of the lowest dimension occurring, and thence, since there are no functions of dimension less than those of the lowest reduced function, can be expressed in the form

$$f = (x, 1)_L + (x, 1)_M g_1 + \ldots + (x, 1)_N g_{n-1},$$

and none of the terms on the right are of higher dimension than that of f.

[It is easy to see that the resulting system of fundamental functions is practically independent of the order in which the sheets are arranged. Moreover, the theorem can be stated more generally, having regard to functions infinite in all but one of the (simple) poles of *any* algebraic function of the surface.]

3. As an example we may give (see next page) the specifications of the reduced functions for a surface of four sheets in the case in which no integral function exists of aggregate order less than $p+1$.

Taking the standard reduced functions to be those which are here first written (at random), we may exemplify the way in which others are expressible by them, in two cases:

(*a*) When $p+1 = 3M$,

$$(M, M+1, M-1, 0) - \lambda\,(M-2, M+1, M+1, 0) = \{M, M, M+1, 0\},$$

the right hand denoting a function whose infinity orders are at least not higher than those marked —, while

$$\{M, M, M+1, 0\} - \lambda_1\,(M-1, M, M+1, 0)$$
$$= \{M, M, M, 0\} = A\,(M, M, M, 0) + B$$

(since a function of $p+1$ poles has two constants, as here). Hence we have the expression

$$(M, M+1, M-1, 0) = \lambda g_2 + \lambda_1 g_1 + A g_1 + B.$$

(*b*) When $p+1 = 3P+1$, we obtain

$$(P+1, P+1, P-1, 0) = \lambda g_1 + A\,(P, P+1, P, 0) + B$$
$$= \lambda g_1 + A\,[\lambda_1 g_2 + C g_3 + D] + B.$$

	Reduced Functions of			Respectively of Rank	Whose Sum is
	First Class	Second Class	Third Class		
$p+1 = 3M$	$(M, M, M, 0)$	$(M-2, M+1, M+1, 0)$ $(M-1, M+1, M, 0)$ $(M, M+1, M-1, 0)$	$(M-1, M, M+1, 0)$	$M-1, M, M$	$3M-1 = p$
$p+1 = 3N-1$	$(N, N, N-1, 0)$ $(N, N-1, N, 0)$	$(N-1, N, N, 0)$	$(N-1, N-1, N+1, 0)$	$N-1, N-1, N$	$3N-2 = p$
$p+1 = 3P+$	$(P+1, P, P, 0)$ $(P+1, P+1, P-1, 0)$ $(P+1, P-1, P+1, 0)$	$(P-1, P+1, P+1, 0)$ $(P, P+1, P, 0)$	$(P, P, P+1, 0)$	P, P, P	$3P = p$

[The specification of all reduced forms of given class for any number of sheets is clearly an easy arithmetical problem. A set of reduced forms when

$$p+1 = (n-1)\,k-r,$$

where $r<n-1$, is clearly given by

$$g_1, \dots g_{r+1}$$

$$= (k, \dots k, k-1, \dots k-1, 0)(k-1, k, \dots k, k-1, \dots k-1, 0) \dots$$

$$\dots (k-1, \dots k-1, k, \dots k, 0),$$

$$g_{r+2}, \dots g_{n-1}$$

$$= (k-1, \dots k-1, k+1, k, \dots k, 0)(k-1, \dots k-1, k, k+1, k, \dots k, 0) \dots$$

$$\dots (k-1, \dots k-1, k, \dots k, k+1, 0),$$

wherein in the first row there are r numbers $k-1$ in each symbol, and in the second row there are $r+1$ numbers $k-1$ in each symbol; in each case $k, \dots k$ denotes a set of numbers k, and $k-1, \dots k-1$ denotes a set of numbers $k-1$. The ranks of the functions $g, \dots g_{r+1}$ are each $k-1$, and of the symbols $g_{r+2}, \dots g_{n-1}$ are each k; their sum is

$$(r+1)(k-1)+(n-r-2)\,k = (n-1)\,k-r-1 = p.]$$

4. Passing from the theory of fundamental systems, I proceed to consider the connexion between the dimension of an integral function, as hitherto defined, and the dimension of an integral *form*, as defined by Herr Hensel.

Let an integral algebraic function satisfy the equation

$$f^r + \dots + f^{r-i}(x, 1)_{\lambda_i} + \dots = 0.$$

Put $f = x^D K$, giving

$$K^r + \dots + K^{r-i}\left(\frac{1}{x}\right)^{iD-\lambda_i}\left(1, \frac{1}{x}\right)^{\lambda_i} + \dots = 0.$$

Let D be taken to be the least positive integer, such that K is finite at all the places $x=\infty$, so that $D-1$ is the rank of f. Then D is the integer actually, and just less than the greatest of the quantities $\dfrac{\lambda_i}{i}+1$.

Putting $x = \omega/\zeta$, $\zeta^D f = F$, we have the equation

$$F^\nu + \ldots + F^{\nu-1}\,\zeta^{iD+\lambda_i}\,(\omega,\ \zeta)_{\lambda_i} + \ldots = 0.$$

The effect of writing here $t\omega$, $t\zeta$ for ω, ζ is to multiply F by t^D. Hence we may speak of F as a homogeneous form of dimension D; integral in the sense that it does not become infinite for finite values of ω and ζ. It appears, then, that D is what Herr Hensel calls the dimension of the form F (see *Crelle*, CIX., pp. 7 and 9; *Math. Ann.*, LXV., 599).

5. We can show now how to form a fundamental system for the expression of homogeneous integral forms.

Let $g_1, \ldots g_{n-1}$ be integral functions for the surface

$$y^n + \ldots + y^{n-i}\,(x,\ 1)_{l_i} + \ldots = 0,$$

such that every integral function can be written in the form

$$f = (x,\ 1)_\mu + (x,\ 1)_{\mu_1} + \ldots,$$

the right hand containing no terms of higher rank than that of f. Namely, if D be the dimension of f, $\tau_i + 1$ that of g_i, $D \overset{=}{>} \tau_i + 1$.

Let F be any integral form which in Hensel's sense is of dimension D; that is, a form satisfying an equation of the form

$$F^\nu + \ldots + F^{\nu-i}\,(\omega,\ \zeta)_{iD} + \ldots = 0,$$

wherein the coefficient of $F^{\nu-i}$ is homogeneous in ω, ζ of degree iD, so that the form F is changed to $t^D F$, when ω, ζ are changed to $t\omega$, $t\zeta$. We consider how far the process of § 4 can be inverted. Let

$$(\omega,\ \zeta)_{iD} = \zeta^{iD-\lambda^i}\,(\omega,\ \zeta)_{\lambda_i},$$

the general case being when

$$\lambda_i = iD.$$

Then $y^{-D}F$, which we denote by f, satisfies an equation

$$f^\nu + \ldots + f^{\nu-i}\,(x,\ 1)_{\lambda_i} + \ldots = 0.$$

If no one of λ_i be so small as $iD - i$, the integer just less than the greatest of the quantities $\dfrac{\lambda_i}{i} + 1$ will be as great as D, and will therefore be D, since $\lambda_i \not> iD$. If, however, every one of λ_i is as small as $id - ri$, say

$$\lambda_i = i\,(D - r) - I_i,$$

so that L_i is zero or a greater integer, but every value of L_i is not as great as i, the greatest of the quantities

$$\frac{\lambda_i}{i}+1 = D-r+\left(1-\frac{L_i}{i}\right)$$

is greater than $D-r$, and less than or equal to $D-r+1$. Hence the rank of f is $D-r+1$. In this case, however, $G = \zeta^{-r}F$ satisfies an equation

$$G^r+\ldots+G^{r-i}\zeta^{L_i}(\omega,\zeta)_{\lambda_i}+\ldots = 0,$$

and is an integral form of dimension $D-r$. So that, though an integral function f of rank $D-1$ necessarily leads to an integral form of dimension D, $\zeta^D f$, an integral form F of dimension D will lead to an integral function $f = \zeta^{-D}F$, whose rank is $D-r-1$, r being the greatest integer such that $\zeta^{-r}F$ is an integral form.

Since f is of rank $D-r-1$, it can be written

$$f = (x, 1)_{\mu_0}+(x, 1)_{\mu_1}g_1+\ldots,$$

where $$D-r-(\mu_i+\tau_i+1) \geqq 0.$$

Therefore $G = \zeta^{D-r}f$ can be written

$$G = \ldots+(\omega,\zeta)_{\mu_i}\zeta^{D-r-(\mu_i+\tau_i+1)}\gamma_i+\ldots,$$

where $\gamma_i = \zeta^{\tau_i+1}g_i$ is an integral form of dimension τ_i+1.

Therefore every integral form $F\,(=\zeta^r G)$ can be written in terms of the fundamental system γ_i in the form

$$F = \ldots+(\omega,\zeta)_{k_i}\gamma_i+\ldots,$$

where $k_i = D-(\tau_i+1)$ is positive.

Conversely, every expression of this form is an integral form of dimension D.

Hence $$1,\ \zeta^{\tau_1+1}g_1,\ \zeta^{\tau_2+1}g_2,\ \ldots\ \zeta^{\tau_{n-1}+1}g_{n-1}$$

form a basis for the representation of integral forms.

6. If, for a surface

$$f(y, x) = y^n+Q_1y^{n-1}+\ldots+Q_n = 0,$$

we are given a fundamental system of integral forms $\gamma_1, \gamma_2, \ldots$ (such as found in § 5), we can determine their dimensions by expressing them in terms of the variables ω, ζ, η, where ($D-1$ being the rank of y), $y = \zeta^{-D}\eta$, $x = \omega/\zeta$; or by forming the equations satisfied by them. If, however, we do not use homogeneous variables and are given a fundamental system of integral functions $g_1, g_2, \ldots$, and require to determine the ranks of them, we may either consider their orders of infinity at $x = \infty$, or form the equation satisfied by them (using then § 4). But, in a certain case, we can more simply use the remark "When the functions $g_1, g_2, \ldots$ are of the form

$$g_i = \frac{y^i + y^{i-1} Q_1 + \ldots + Q_i}{x^{m_i} (x, 1)^{n_i}},$$

which is a common case, or, more generally, of the form

$$g_i = \frac{y^i + y^{i-1} R_1 + \ldots + R_i}{x^{m_i} (x, 1)^{n_i}},$$

wherein $R_1, \ldots R_i$ are such polynomials in x that, by the substitution $y = \zeta^{-D}\eta$, $x = 1/\zeta$, the numerator becomes changed to

$$\zeta^{-iD} [\eta^i + \eta^{i-1} S_1 + \ldots + S_i],$$

wherein $S_1, \ldots S_i$ are *integral* polynomials in ζ, and the equation is such that

$$F'(\eta) = \frac{\partial}{\partial \eta} F(\eta, \zeta) = \frac{\partial}{\partial \eta} [\zeta^{nD} f(\eta\zeta^{-D}, \zeta^{-1})]$$

does not vanish at any place $\zeta = 0$, *then* the effect of substituting in g_i for y, x respectively $y = \eta\zeta^{-D}$, $x = 1/\zeta$ is of the form

$$g_i = \zeta^{-(\tau_i + 1)} \frac{\eta^i + \eta^{i-1} \overline{Q}_1 + \ldots}{(1, \zeta)_{n_i}},$$

wherein $\overline{Q}_1, \ldots \overline{Q}_i$ are *integral* polynomials in ζ. Namely, the rule for the rank (τ_i) of an integral function g_i whose expression is given is to notice the power $\tau_i + 1$ of ζ which disengages itself under the specified substitution."

In fact, this equation shows (1) that $\zeta^{\tau_i + 1} g_i$ is finite when $\zeta = 0$ (since η is finite when $\zeta = 0$), and (2) that $\zeta^{\tau_i + 1 - \epsilon} g$ (ϵ a positive integer), which is equal to

$$\zeta^{-\epsilon} \frac{\eta^i + \eta^{i-1} \overline{Q}_i + \ldots}{(1, \zeta)_{n_i}},$$

cannot be finite at every place $\zeta = 0$, unless $\eta^i + \eta^{i-1} \overline{Q}_i + \ldots$ vanish at every such place. But, since $F'(\eta)$ does not vanish, there are n distinct values of η at $\zeta = 0$, and this polynomial, of order less than n, cannot vanish for every one of them.

When $F'(\eta)$ is zero at $\zeta = 0$, and there are only k distinct values of η there, we can still use this remark to determine the rank of $g_1, g_2, \dots g_{k-1}$.

For instance, when the surface is

$$y^3+y^2(x, 1)_2+yx(x, 1)_1+Ax^2 = 0,$$

y is of rank 1, and a fundamental set of integral functions is $1, \frac{x}{y}, \frac{x^2}{y^2}$. These are respectively replaceable by $1, \frac{y^2+yQ_1+Q_2}{x}$, $y+Q_1$, and these lead, by $y = \eta\zeta^{-2}$, $x = 1/\zeta$, to

$$g_1 = \zeta^{-2}\left[\eta+(1, \zeta)_2\right],$$

$$g_2 = \zeta^{-3}\left[\eta^2+\eta(1, \zeta)_2+\zeta^2(1, \zeta)_1\right].$$

At $\zeta = 0$ there are two distinct values of η, the rank of g_1 is correctly given, but that of g_2 incorrectly, by noticing the power of ζ which disengages itself.

It is easy to see that $\zeta g_2 = -A\zeta\frac{x}{y}$ is finite at $\zeta = 0$, and that $1, \zeta\frac{x}{y}, \zeta^2\frac{x^2}{y^2}$ form a fundamental system of integral forms. (*Cf.* Hensel, *Math. Ann.*, XLV., 599, and *Math. Ann.*, XLV., 129. Herr Hensel's fundamental system $(1, \eta, \eta^2)$ should be printed $(1, \eta_1, \eta_1^2)$.)

Electric Vibrations in Condensing Systems. *By* J. LARMOR.

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1. In forming a theory of rapid electric vibrations, the first point to settle is as to the conditions that may be taken to hold at the boundary of the dielectric medium, where it abuts on a good conductor like a metal. It is well known that when the vibrations are of such short period as free vibrations usually are, the currents in the conductor are confined to mere sheets on the surface. Inside these surface sheets the electric force is null and the magnetic force is null; for if any such forces, of alternating character, existed, there would be currents induced by them, contrary to the fact. The cir-