

Notes on Laplace's Coefficients. By J. W. L. GLAISHER, M.A.

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1. The present paper merely consists of a few detached notes having relation to Laplace's (or, more correctly, Legendre's) coefficients; the n^{th} Laplace's coefficient, P_n , being defined as the coefficient of h^n in the expansion of

$$\frac{1}{\sqrt{(1-2ha+h^2)}}$$

in ascending powers of h .

2. It is natural to examine to what results we are led by expressing $(1-2ha+h^2)^{-\frac{1}{2}}$ as a definite integral, and then expanding in powers of h . For this purpose there are three formulæ available, viz. :—

$$\frac{\pi}{\sqrt{(a^2-b^2)}} = \int_0^\pi \frac{dt}{a+b \cos t} \dots\dots\dots (1),$$

$$\frac{\sqrt{\pi}}{2\sqrt{a}} = \int_0^\infty e^{-at^2} dt \dots\dots\dots (2),$$

$$\frac{\pi}{2\sqrt{a}} = \int_0^\infty \frac{dt}{a+t^2} \dots\dots\dots (3).$$

The first of these has been discussed by Jacobi (Crelle, t. xxvi., p. 81); it leads at once* to the formula given by Laplace ("Mécanique Céleste," 1825, t. v. p. 33), viz.,—

$$\pi \cdot P_n = \int_0^\pi \{a - \cos t \sqrt{(a^2-1)}\}^n dt \dots\dots\dots (4).$$

From the second we have

$$\begin{aligned} \sqrt{\pi} \cdot P_n &= \text{coefficient of } h^n \text{ in } 2 \int_0^\infty e^{-(1-a^2)t^2 - (h-a)^2 t^2} dt \\ &= \frac{2}{n!} \int_0^\infty e^{-(1-a^2)t^2} \left(\frac{d}{dh} \right)^n e^{-(h-a)^2 t^2} dt_{(h=0)} \\ &= \frac{2}{n!} \int_0^\infty e^{-(1-a^2)t^2} \left(-\frac{d}{da} \right)^n e^{-a^2 t^2} dt, \end{aligned}$$

which we may write

$$\sqrt{\pi} \cdot P_n = \frac{1}{n!} \int_{-\infty}^\infty e^{-(1-a^2)t^2} \left(-\frac{d}{da} \right)^n e^{-a^2 t^2} dt \dots\dots\dots (5),$$

and is the form to which (2) leads.

* See Meyer, "Vorlesungen über die Theorie der bestimmten Integrale" (Leipzig, 1871), p. 407.

Formula (3) gives

$$\frac{\pi}{2\sqrt{(1-2ha+h^2)}} = \int_0^\infty \frac{dt}{(h-a)^2 + \gamma^2} \text{ if } \gamma = \sqrt{(1-a^2+t^2)}$$

$$= \int_0^\infty \frac{dt}{2\gamma i} \left(\frac{1}{h-a-\gamma i} - \frac{1}{h-a+\gamma i} \right),$$

whence

$$\pi \cdot P_n = \frac{1}{i} \int_0^\infty \frac{dt}{\sqrt{(1-a^2+t^2)}} \left\{ \frac{1}{[a-i\sqrt{(1-a^2+t^2)}]^{n+1}} - \frac{1}{[a+i\sqrt{(1-a^2+t^2)}]^{n+1}} \right\} \dots (6).$$

3. The integral (2) can be made to give Ivory's expression for P_n without difficulty. We have

$$\begin{aligned} \sqrt{\pi} \cdot P_n &= \text{coefficient of } h^n \text{ in } \int_{-\infty}^\infty e^{-(1-a^2)t^2 - (h-a)^2 t^2} dt \\ &= \text{,, ,,} \int_{-\infty}^\infty e^{-(1-a^2)t^2 - (h-at)^2} t^n dt \\ &= \text{,, ,,} e^{-h^2} \int_{-\infty}^\infty e^{-t^2 + 2hat} t^n dt \\ &= \text{,, ,,} \frac{e^{-h^2}}{2^n h^n} \left(\frac{d}{da} \right)^n \int_{-\infty}^\infty e^{-t^2 + 2hat} dt \\ &= \frac{1}{2^n} \cdot \text{coefficient of } h^{2n} \text{ in } e^{-h^2} \left(\frac{d}{da} \right)^n \cdot \sqrt{\pi} e^{a^2 h^2}; \\ &\quad \text{viz., in } \sqrt{\pi} \cdot \left(\frac{d}{da} \right)^n e^{(a^2-1)h^2}, \end{aligned}$$

whence

$$P_n = \frac{1}{2^n \cdot n!} \left(\frac{d}{da} \right)^n (a^2-1)^n,$$

which is the value of P_n given by Ivory ("Phil. Trans.," 1824, p. 93).

The above investigation is chiefly interesting because, as a rule, it is difficult to obtain by a direct process a result of the form $\left(\frac{d}{dx} \right)^n X^n$ without the aid of Lagrange's theorem or the previous expansion of X^n .

4. In order to identify (5) with the ordinary expanded value of P_n , viz.,

$$\begin{aligned} P_n &= \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{n!} \left(a^n - \frac{n \cdot n-1}{2 \cdot 2n-1} a^{n-2} \right. \\ &\quad \left. + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \cdot 4 \cdot 2n-1 \cdot 2n-3} a^{n-4} - \&c. \right) \dots \dots (7), \end{aligned}$$

it is necessary to find the expansion of $e^{ax} \left(\frac{d}{da}\right)^n e^{-ax}$. This may be obtained in two ways, viz., by means of a definite integral, or by finding the coefficient of h^n in $e^{(a+h)x}$.

Starting with $\sqrt{\pi} \cdot e^{ax} = \int_{-\infty}^{\infty} e^{-x^2+2ax} dx,$

we have $\sqrt{\pi} \cdot \left(\frac{d}{da}\right)^n e^{ax} = 2^n \int_{-\infty}^{\infty} e^{-x^2+2ax} x^n dx$
 $= 2^n e^{a^2} \int_{-\infty}^{\infty} e^{-(x-a)^2} x^n dx = 2^n e^{a^2} \int_{-\infty}^{\infty} e^{-x^2} (x+a)^n dx,$

whence

$$e^{-a^2} \left(\frac{d}{da}\right)^n e^{a^2} = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \left\{ a^n + n \cdot a^{n-1} x + \frac{n \cdot n-1}{1 \cdot 2} a^{n-2} x^2 + \&c. \right\} dx$$

$$= 2^n \left\{ a^n + \frac{1}{2} \cdot \frac{n \cdot n-1}{1 \cdot 2} a^{n-2} + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4} + \&c. \right\} \dots\dots\dots (8)$$

Or we may proceed thus,

$$\left(\frac{d}{da}\right)^n e^{ax} = n! \cdot \text{coefficient of } h^n \text{ in } e^{(a+h)x},$$

whence

$$e^{-a^2} \left(\frac{d}{da}\right)^n e^{a^2} = n! \cdot \text{coefficient of } h^n \text{ in } 1 + h(2a+h) + \frac{h^2}{1 \cdot 2} (2a+h)^2 + \&c.$$

$$= \text{coefficient of } h^n \text{ in}$$

$$h^n (2a+h)^n + n h^{n-1} (2a+h)^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} h^{n-2} (2a+h)^{n-2} + \&c.$$

$$= (2a)^n + n \cdot n-1 (2a)^{n-2} + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2} (2a)^{n-4} + \&c. \dots\dots (9),$$

which is readily identified with (8).

Writing the binomial theorem coefficients $n, \frac{n \cdot n-1}{1 \cdot 2}, \dots$ as B_1, B_2, \dots and replacing a by iat in (8), we find

$$e^{a^2 t^2} \left(-\frac{d}{da}\right)^n e^{-a^2 t^2} = 2^n t^n \left\{ a^n t^n - \frac{1}{2} B_2 a^{n-2} t^{-2} + \frac{1}{2} \cdot \frac{3}{2} B_4 a^{n-4} t^{n-4} - \&c. \right\},$$

whence

$$\int_{-\infty}^{\infty} e^{-t^2} \cdot e^{a^2 t^2} \left(-\frac{d}{da}\right)^n e^{-a^2 t^2} dt = 2^n \left\{ a^n \cdot \Gamma\left(\frac{2n+1}{2}\right) - \frac{1}{2} B_2 a^{n-2} \cdot \Gamma\left(\frac{2n-1}{2}\right) \right.$$

$$\left. + \frac{1}{2} \cdot \frac{3}{2} B_4 a^{n-4} \cdot \Gamma\left(\frac{2n-3}{2}\right) - \&c. \right\},$$

which agrees with (7), and affords a verification of (5).

5. In connexion with the integral

$$\int_{-\infty}^{\infty} e^{-ax^2 \pm 2bx} dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{b^2}{a}},$$

which has been used in the last section, it is worth noticing that it gives rise to a curious relation between the differentials of e^{a^2} and ae^{a^2} . For we have

$$\left(\frac{d}{db}\right)^{2n} \frac{e^{\frac{b^2}{a}}}{\sqrt{a}} = \left(-\frac{d}{da}\right)^n \frac{e^{\frac{b^2}{a}}}{\sqrt{a}},$$

which becomes, on replacing a by a^{-1} ,

$$\left(\frac{d}{db}\right)^{2n} e^{a^2 b^2} = \frac{1}{a} \left(2a^3 \frac{d}{da}\right)^n a e^{a^2 b^2}$$

Now, generally, $a^n \left(\frac{d}{da}\right)^n \phi(ab) = b^n \left(\frac{d}{db}\right)^n \phi(ab)$;

so that $\frac{a^{2n}}{b^{2n}} \left(\frac{d}{da}\right)^{2n} e^{a^2 b^2} = \frac{1}{a} \left(2a^3 \frac{d}{da}\right)^n a e^{a^2 b^2}$,

or, putting $b = 1$,

$$a^{2n+1} \left(\frac{d}{da}\right)^{2n} e^{a^2} = \left(2a^3 \frac{d}{da}\right)^n a e^{a^2} \dots\dots\dots (10),$$

the relation in question.

In the same manner, from the integral

$$\int_0^{\infty} e^{-ax^2 - 2bx} dx = \frac{1}{\sqrt{a}} e^{\frac{b^2}{a}} \operatorname{Erfc} \frac{b}{\sqrt{a}},$$

where

$$\operatorname{Erfc} x = \int_x^{\infty} e^{-x^2} dx,$$

we find $\frac{a^{2n}}{b^{2n}} \left(\frac{d}{da}\right)^{2n} e^{a^2 b^2} \operatorname{Erfc} ab = \frac{1}{a} \left(2a^3 \frac{d}{da}\right)^n a e^{a^2 b^2} \operatorname{Erfc} ab \dots\dots\dots (11),$

or, putting $b = 1$,

$$a^{2n+1} \left(\frac{d}{da}\right)^{2n} e^{a^2} \operatorname{Erfc} a = \left(2a^3 \frac{d}{da}\right)^n a e^{a^2} \operatorname{Erfc} a \dots\dots\dots (12).$$

The formula (10) follows from (11) by changing the sign of b and adding, since

$$\operatorname{Erfc} x + \operatorname{Erfc} (-x) = \sqrt{\pi}.$$

6. Laplace's formula,

$$\pi \cdot P_n = \int_0^{\pi} \{a \pm \cos t \sqrt{a^2 - 1}\}^n dt,$$

leads at once to a simple relation between the coefficients of different

orders, which it is just possible may not have been remarked. For, E being a symbol of operation such that $EP_n = P_{n+1}$, then

$$\pi \cdot e^{-hE} P_0 = \int_0^\pi e^{-h\alpha - h \cos t \sqrt{\alpha^2 - 1}} dt,$$

whence
$$\pi e^{-h(E-\alpha)} P_0 = \int_0^\pi e^{-h \cos t \sqrt{\alpha^2 - 1}} dt;$$

or, equating the coefficients of h^n ,

$$\pi (E-\alpha)^n P_0 = (\alpha^2 - 1)^{n/2} \int_0^\pi \cos^n t dt;$$

so that $P_n - n\alpha P_{n-1} + \frac{n \cdot n-1}{1 \cdot 2} \alpha^2 P_{n-2} \dots \pm \alpha^n P_0 = 0$, if n be uneven $\left. \begin{aligned} & \text{and} \\ & = \frac{1 \cdot 3 \cdot 5 \dots n-1}{2 \cdot 4 \cdot 6 \dots n} \text{ if } n \text{ be even} \end{aligned} \right\} (13).$

Examples: $n=2$, $\frac{1}{2} (3\alpha^2 - 1) - 2\alpha \cdot \alpha + \alpha^2 = \frac{1}{2} (\alpha^2 - 1);$

$n=3$, $\frac{1}{2} (5\alpha^2 - 3\alpha) - 3\alpha \cdot \frac{1}{2} (3\alpha^2 - 1) + 3\alpha^2 \cdot \alpha - \alpha^3 = 0.$

7. The formula (13), found in the last section, would not of course be suitable for the actual calculation of P_n , as P_n is given in terms of P_{n-1} , P_{n-2} ... P_0 . The usual formulæ which are most convenient for the calculation of P_n are

$$P_n = \alpha P_{n-1} - \frac{1-\alpha^2}{n} \frac{dP_{n-1}}{d\alpha} \dots\dots\dots (14),$$

and

$$P_n = \frac{2n-1}{n} \alpha P_{n-1} - \frac{n-1}{n} P_{n-2} \dots\dots\dots (15);$$

but a third formula, almost as convenient as either, was given by Mr. R. Kalley Miller, in his problem paper in the Mathematical Tripos, 1871 (Thursday morning), viz., that

$$P_n = \alpha P_{n-1} + (n-1) \int_0^\pi P_{n-1} d\alpha + C \dots\dots\dots (16),$$

where $C=0$ or $(-)^{1/2} \frac{n!}{2^n (\frac{1}{2}n!)^2}$, according as n is uneven or even.

Mr. Miller's formula is easily deduced from (14) by means of the equation

$$\frac{d}{d\alpha} \left\{ (1-\alpha^2) \frac{dP_{n-1}}{d\alpha} \right\} + n-1 \cdot n P_{n-1} = 0,$$

which gives
$$-\frac{1-\alpha^2}{n} \frac{dP_{n-1}}{d\alpha} = (n-1) \int_0^\pi P_{n-1} d\alpha + C,$$

C being so chosen that the two sides of this equation shall agree when $\alpha=0$, that is to say, that C shall be zero if n be uneven, but, if n be

even, shall be equal to

$$\begin{aligned}
 & -\frac{1}{n} \text{ coefficient of } h^{n-1} \text{ in } \frac{d}{da} \frac{1}{\sqrt{(1-2ha+h^2)}} \quad (\alpha=0); \\
 & = -\frac{1}{n} \text{ coefficient of } h^{n-2} \text{ in } (1+h^2)^{-\frac{1}{2}} \\
 & = \frac{1}{n} \cdot (-)^n \frac{\frac{3}{2} \cdot \frac{5}{2} \cdots \frac{n-1}{2}}{1 \cdot 2 \cdots \frac{n-2}{2}} = (-)^n \frac{n!}{2^n (\frac{1}{2}n!)^2}.
 \end{aligned}$$

8. The formula (5), viz.,

$$\sqrt{\pi} \cdot P_n = \frac{1}{n!} \int_{-\infty}^{\infty} e^{-(1-\alpha^2)t^2} \left(-\frac{d}{dt} \right)^n e^{-\alpha^2 t^2} dt,$$

admits of the following transformation:

$$\text{Since} \quad \alpha^n \left(\frac{d}{da} \right)^n e^{-\alpha^2 t^2} = t^n \left(\frac{d}{dt} \right)^n e^{-\alpha^2 t^2},$$

$$\text{we obtain} \quad \sqrt{\pi} \cdot P_n = \frac{1}{\alpha^n \cdot n!} \int_{-\infty}^{\infty} t^n e^{-(1-\alpha^2)t^2} \left(-\frac{d}{dt} \right)^n e^{-\alpha^2 t^2} dt,$$

which, by integrating by parts n times, is easily seen to give

$$\sqrt{\pi} \cdot P_n = \frac{1}{\alpha^n \cdot n!} \int_{-\infty}^{\infty} e^{-\alpha^2 t^2} \left(\frac{d}{dt} \right)^n e^{-(1-\alpha^2)t^2} t^n dt \dots\dots\dots (17).$$

This result may also be obtained as follows: In § 3 it was shown that

$$\begin{aligned}
 \sqrt{\pi} \cdot P_n &= \text{coefficient of } h^n \text{ in } \int_{-\infty}^{\infty} e^{-t^2 + 2h\alpha t - h^2} t^n dt \\
 &= \quad \quad \quad \frac{1}{\alpha^n} \int_{-\infty}^{\infty} e^{-t^2 + 2h\alpha^2 t - \alpha^2 h^2} t^n dt \\
 &= \quad \quad \quad \frac{1}{\alpha^n} \int_{-\infty}^{\infty} e^{-(1-\alpha^2)t^2 - \alpha^2(t-h)^2} t^n dt \\
 &= \quad \quad \quad \frac{1}{\alpha^n} \int_{-\infty}^{\infty} e^{-(1-\alpha^2)(t+h)^2 - \alpha^2 t^2} (t+h)^n dt \\
 &= \frac{1}{\alpha^n \cdot n!} \int_{-\infty}^{\infty} e^{-\alpha^2 t^2} \left(\frac{d}{dt} \right)^n e^{-(1-\alpha^2)t^2} t^n dt.
 \end{aligned}$$

9. It is, perhaps, worth remarking that Ivory's formula

$$P_n = \frac{1}{2^n \cdot n!} \left(\frac{d}{da} \right)^n (\alpha^2 - 1)^n$$

shows at once that the denominators of the coefficients of the powers of α in the expanded value of P_n contain only powers of 2, for, by the differentiation, we obtain as the coefficient of each power of α a product

of n factors; which must be a multiple of $n!$ Thus we can always tabulate *exactly* Laplace's coefficients for arguments which expressed as decimals do not themselves circulate, *i. e.*, $P_n(a)$ is not a circulating decimal unless a is so.

10. We can obtain an expression similar to (5) for the coefficient of h^n in

$$\frac{1}{(1-2ha+h^2)^m},$$

m being any positive quantity integral or fractional, by means of the gamma function formula

$$\frac{1}{a^m} = \frac{1}{\Gamma m} \int_0^\infty e^{-at} t^{m-1} dt,$$

$$\text{for } \frac{1}{(1-2ha+h^2)^m} = \frac{1}{\Gamma m} \int_0^\infty e^{-(1-a^2)t-(h-a)^2 t} t^{m-1} dt,$$

$$\text{and the coefficient of } h^n = \frac{1}{\Gamma m \cdot n!} \int_0^\infty t^{m-1} e^{-(1-a^2)t} \left(-\frac{d}{da}\right)^n e^{-a^2 t} dt \dots (18).$$

Of course, instead of (2), the integrals

$$\int_0^\infty \sin at^2 dt = \int_0^\infty \cos at^2 dt = \frac{\sqrt{\pi}}{2\sqrt{(2a)}}$$

might have been used, but the forms to which they lead present no point of superiority over those involving exponentials.

11. The most natural way of proving the fundamental theorem

$$\int_{-1}^1 P_m P_n da = 0 \quad \text{or} \quad \frac{2}{2n+1} \dots \dots \dots (19),$$

according as m and n are unequal or equal, seems to be by examining the coefficient of $h^m k^n$ in the integral

$$\int_{-1}^1 \frac{da}{\sqrt{\{(1-2ha+h^2)(1-2ka+k^2)\}}} \dots \dots \dots (20),$$

instead of employing the differential equation satisfied by P_n .

By means of the indefinite integral

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{\sqrt{c}} \log \{b+2cx+2\sqrt{c} \cdot \sqrt{(a+bx+cx^2)}\},$$

we find

$$\begin{aligned} (20) &= \frac{1}{2\sqrt{(hk)}} \log \frac{h+k+hk^2+h^2k-4hk-2\sqrt{(hk)}(1-h-k+hk)}{h+k+hk^2+h^2k+4hk-2\sqrt{(hk)}(1+h+k+hk)} \\ &= \frac{1}{2\sqrt{(hk)}} \log \frac{\{1+\sqrt{(hk)}\}^2 \{\sqrt{h}-\sqrt{k}\}^2}{\{1-\sqrt{(hk)}\}^2 \{\sqrt{h}-\sqrt{k}\}^2} = \frac{1}{\sqrt{(hk)}} \log \frac{1+\sqrt{(hk)}}{1-\sqrt{(hk)}}, \end{aligned}$$

which gives (19) at once. This is the method adopted by Lord Rayleigh, in his memoir in the "Philosophical Transactions" for 1870 (p. 579), for finding the value of

$$\int_0^1 P_m P_n da,$$

which is shown to be equal to the coefficient of $h^m k^n$ in

$$\frac{1}{\sqrt{hk}} \log \frac{\{1 + \sqrt{hk}\} \{\sqrt{h} - \sqrt{k}\}}{h\sqrt{(1+k^2)} - k\sqrt{(1+h^2)}} \dots\dots\dots (21).$$

In the "Proceedings of the Royal Society," t. xxiii. p. 300 (March 4, 1875), Mr. Todhunter has proved that

$$\{2m(2m+1) - (2n-1)2n\} \int_0^1 P_{2m} P_{2n-1} da \\ = (-)^{m+n} \frac{1.3.5\dots 2m-1}{2.4.6\dots 2m} \cdot \frac{1.3.5\dots 2n-1}{2.4.6\dots 2n-2},$$

so that we have the striking theorem that, in the expansion of (21), the coefficient of $h^m k^n$ is equal to zero if m and n be unequal and both even or both uneven, is equal to $\frac{1}{2n+1}$ if m and n be equal, and is equal to

$$\frac{(-)^{\frac{1}{2}(m+n+1)}}{m(m+1) - n(n+1)} \cdot \frac{1.3.5\dots m-1}{2.4.6\dots m} \cdot \frac{1.3.5\dots n}{2.4.6\dots n-1},$$

if m be even and n uneven.

12. In a memoir entitled "On the Equation of Laplace's Functions, &c." ("Philosophical Transactions," 1857, p. 43), the late Professor W. F. Donkin showed that, if $\alpha = \cos \theta$, then

$$P_n = \frac{1}{n!} (\sin \theta)^{-n} \left(\sin \theta \frac{d}{d\theta} \sin \theta \right)^n \cdot 1 \dots\dots\dots (22),$$

and in the same memoir there occur (pp. 51 and 53) the two identities

$$\left(\sin \theta \frac{d}{d\theta} \sin \theta \right)^n \left(\tan \frac{\theta}{2} \right)^n = (1.3.5\dots 2n-1) (\sin \theta)^{2n} \dots (23),$$

$$(\sin \theta)^{-n} \left(\sin \theta \frac{d}{d\theta} \sin \theta \right)^n \left(\tan \frac{\theta}{2} \right)^i \\ = (1-\alpha^2)^{\frac{1}{2}i} \alpha^{n-i} \left(\frac{d}{da} \frac{1}{\alpha} \right)^n \alpha^{n+i} (1+\alpha)^{n-i} \dots\dots (24),$$

the latter resulting from the necessary equality of two expressions for the same quantity, the one found by Boole and the other by Donkin. On (24) Donkin remarks: "This equivalence and that expressed by equation (27), art. 14 [viz. (23)], are instances of theorems by no means obvious or easy to verify directly." In a postscript he states, "Mr. Cayley has been kind enough to communicate to me direct verifications of the equation (27), art. 14, and of the identity referred to in

art. 16 [viz. (24)]. Assuming a formula established in Mr. Cayley's paper 'On certain Formulæ for Differentiation, &c.' ['Cambridge and Dublin Journal,' vol. ii., p. 124, equation (2)], the former of the two theorems just mentioned is easily obtained, the latter not without a good deal of trouble." As Professor Cayley's solutions have not, I believe, been published, I conclude these notes with verifications of the three formulæ (22), (23), (24).

13. Putting $\cot \theta = t$, we have

$$-\frac{d}{dt} = \sin^2 \theta \frac{d}{d\theta},$$

and equation (22) becomes

$$P_n = \frac{1}{n!} (1+t^2)^{\frac{1}{2}(n+1)} \left(-\frac{d}{dt}\right)^n \frac{1}{\sqrt{1+t^2}}.$$

This value of P_n can be easily verified, for, since $\alpha = \cos \theta$ and $t = \cot \theta$,

$$\begin{aligned} P_n &= \text{coefficient of } h^n \text{ in } \frac{1}{\sqrt{\left\{1 - \frac{2ht}{\sqrt{1+t^2}} + h^2\right\}}} \\ &= \text{ " " } \frac{\sqrt{1+t^2}}{\sqrt{\{1+t^2 - 2ht\sqrt{1+t^2} + h^2(1+t^2)\}}} \\ &= (1+t^2)^{\frac{1}{2}n} \text{ coefficient of } h^n \text{ in } \frac{\sqrt{1+t^2}}{\sqrt{1+t^2 - 2ht + h^2}} \\ &= (1+t^2)^{\frac{1}{2}(n+1)} \text{ coefficient of } h^n \text{ in } \frac{1}{\sqrt{\{1 + (h-t)^2\}}} \\ &= \frac{1}{n!} (1+t^2)^{\frac{1}{2}(n+1)} \left(-\frac{d}{dt}\right)^n \frac{1}{\sqrt{1+t^2}}. \end{aligned}$$

14. Putting, as before, $\cot \theta = t$, (23) takes the form

$$\left(-\frac{d}{dt}\right)^n \frac{\{\sqrt{1+t^2} - t\}^n}{\sqrt{1+t^2}} = \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{(1+t^2)^{\frac{1}{2}(2n+1)}} \dots \dots \dots (25),$$

and, to verify this, we notice that, by Lagrange's theorem,

if $x = t + hfx$,

then $\frac{1}{n!} \left(\frac{d}{dt}\right)^n (f^n t \cdot R^n t) = \text{coefficient of } h^n \text{ in the expansion of } R^n x \cdot \frac{dx}{dt}.$

In our case, $x = t + h \{\sqrt{1+x^2} - x\} \dots \dots \dots (26),$

and $R^n x = \frac{1}{\sqrt{1+x^2}};$

therefore $R^n x \cdot \frac{dx}{dt} = \frac{1}{\sqrt{1+x^2} - h \{x - \sqrt{1+x^2}\}} = \frac{1}{\sqrt{1+x^2} + x - t}.$

Writing, for the moment, $\sqrt{1+x^2} + x = p$, we have

$$\frac{1}{p} = \sqrt{(1+x^2)} - x,$$

whence $1+x^2 = \left(x + \frac{1}{p}\right)^2$ and $1+2px-p^2=0$.

Thus $x = \frac{p^2-1}{2p}$, and (26) becomes

$$\frac{p^2-1}{2p} = t + \frac{h}{p},$$

viz.,

$$(p-t)^2 = t^2 + 2h + 1,$$

so that

$$\frac{1}{\sqrt{(1+x^2)}+x-t} = \frac{1}{p-t} = \frac{1}{\sqrt{1+2h+t^2}},$$

in the expansion of which quantity the coefficient of h^n is

$$\frac{1}{n!} \cdot (-)^n \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{(1+t^2)^{\frac{1}{2}(2n+1)}}$$

thus proving (25).

15. To verify (24), we observe that

$$\begin{aligned} & \frac{1}{n!} (\sin \theta)^{-n-1} \left(\sin^2 \theta \frac{d}{d\theta} \right)^n \sin \theta \left(\tan \frac{\theta}{2} \right)^i \\ &= \frac{1}{n!} (1+t^2)^{\frac{1}{2}(n+1)} \left(-\frac{d}{dt} \right)^n \left\{ \frac{\sqrt{(1+t^2)}-t}{\sqrt{(1+t^2)}} \right\}^i \\ &= (1+t^2)^{\frac{1}{2}(n+1)} \text{coefficient of } h^n \text{ in } \frac{\{\sqrt{(1+t^2-2th+h^2)}-t+h\}^i}{\sqrt{(1+t^2-2th+h^2)}} \\ &= (1+t^2)^{\frac{1}{2}} \text{coefficient of } h^n \text{ in } \\ & \quad \frac{\{\sqrt{[1+t^2-2t\sqrt{(1+t^2)}h+h^2(1+t^2)]}-t+h\sqrt{(1+t^2)}\}^i}{\sqrt{\{1+t^2-2t\sqrt{(1+t^2)}h+h^2(1+t^2)\}}} \\ &= \text{coefficient of } h^n \text{ in } (\operatorname{cosec} \theta)^i \frac{\{\sqrt{(1-2h \cos \theta + h^2)} - \cos \theta + h\}^i}{\sqrt{(1-2h \cos \theta + h^2)}} \dots (27). \end{aligned}$$

Now, the right-hand side of (24) is

$$\frac{(1-\alpha^2)^{\frac{1}{2}i}}{\alpha^{i-1}} \cdot \alpha^n \left(\frac{d}{a da} \right)^n \frac{1}{a} \cdot \alpha^n (1+\alpha)^n \cdot \left(\frac{\alpha}{1+\alpha} \right)^i;$$

and, taking $\alpha^2 = \beta$,

$$\begin{aligned} & \frac{\alpha^n}{n!} \left(\frac{d}{a da} \right)^n \left\{ \alpha^n (1+\alpha)^n \cdot \frac{1}{a} \left(\frac{\alpha}{1+\alpha} \right)^i \right\} \\ &= \frac{\beta^{\frac{1}{2}n}}{n!} 2^n \left(\frac{d}{d\beta} \right)^n \left\{ (\sqrt{\beta} + \beta)^n \cdot \frac{1}{\sqrt{\beta}} \cdot \left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}} \right)^i \right\} \\ &= \text{coefficient of } h^n \text{ in } \frac{1}{\sqrt{x}} \left(\frac{\sqrt{x}}{1+\sqrt{x}} \right)^i \frac{dx}{d\beta}, \end{aligned}$$

where

$$x = \beta + 2h\sqrt{\beta}(\sqrt{x}+x) \dots \dots \dots (28);$$

and, in the formation of $\frac{dx}{d\beta}$, the $\sqrt{\beta}$ that multiplies h is to be treated as

constant. Writing for β its value, viz. $\cos^2 \theta$, (28) gives

$$x(1 - 2h \cos \theta) - 2h \cos \theta \sqrt{x} = \cos^2 \theta;$$

whence
$$\sqrt{x} = \frac{h + \sqrt{(1 - 2h \cos \theta + h^2)}}{1 - 2h \cos \theta} \cos \theta,$$

and we find that

$$\frac{\sqrt{x}}{1 + \sqrt{x}} = \frac{\sqrt{(1 - 2h \cos \theta + h^2)} - \cos \theta + h}{\sin^2 \theta} \cos \theta;$$

also, on the understanding mentioned above, with regard to the factor $\sqrt{\beta}$,

$$\frac{1}{\sqrt{x}} \frac{dx}{d\beta} = \frac{1}{\cos \theta \sqrt{(1 - 2h \cos \theta + h^2)}};$$

so that the coefficient of h^n in $\frac{1}{\sqrt{x}} \left(\frac{\sqrt{x}}{1 + \sqrt{x}} \right)^i \frac{dx}{d\beta}$

$$= \text{coefficient of } h^n \text{ in } \frac{\{\sqrt{(1 - 2h \cos \theta + h^2)} - \cos \theta + h\}^i}{(\sin \theta)^{2i} \sqrt{(1 - 2h \cos \theta + h^2)}} (\cos \theta)^{i-1};$$

and this, on multiplication by

$$\frac{(1 - a^2)^{ii}}{a^{i-1}}, \text{ that is, by } \frac{(\sin \theta)^i}{(\cos \theta)^{i-1}},$$

becomes identical with (27).

On the Mechanical Description of a Sphero-Conic. By MR. HART.

[Read May 13th, 1875.]

Let P be any point on a small circle of a sphere whose centre is B and radius subtends angle $2b$ at centre of sphere.

Let A be a fixed point, $AB = 2a$, O the middle point of AB.

Then the locus of Q, the middle

point of AP, is a sphero-conic. For if $OQ = \rho$, $QOB = \theta$, $AQ = QP = x$,

$$\cos x = \cos a \cos \rho - \sin a \sin \rho \cos \theta \dots\dots\dots (1),$$

$$\frac{\cos 2b - \cos 2a \cos 2x}{\sin 2a \sin 2x} = \frac{\cos \rho - \cos a \cos x}{\sin a \sin x} \dots\dots\dots (2);$$

whence, eliminating x ,

$$\tan^2 \rho = \frac{\sin^2 b}{(\cos^2 a - \sin^2 b) \sin^2 \theta + \cos^2 b \cos^2 \theta},$$

which is the polar equation to a sphero-conic whose centre is O, minor axis coincides with AB and $= 2b$, major axis perpendicular to AB

