

## The Whirling and Transverse Vibrations of Rotating Shafts

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*XV. The Whirling and Transverse Vibrations of Rotating Shafts. By C. CHREE, Sc.D., LL.D., F.R.S.\**

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*Preliminary Discussion.*

§ 1. LATERAL vibrations are those executed by a bar when bent to one side and then released. They are connected with the "whirling" of rotating shafts, but in a way which has not, I think, hitherto been clearly recognized. The subject of "whirling" has been treated by Professor Greenhill† in a well-known paper dealing with an unloaded shaft rotating under various terminal conditions. More recently the subject has been treated in an elaborate and important paper by Prof. Dunkerley‡. He employed two ways of calculating the critical speeds of rotation. The first makes use of the ordinary elastic solid equations applicable to thin rods acted on by "centrifugal force"; this is the method followed by Greenhill. On attempting to apply this method to loaded shafts, Dunkerley reached results which he considered hopelessly complicated. In his second method, which is due apparently to Prof. Osborne Reynolds, Dunkerley calculated a critical speed for the loaded shaft, in which the mass of the shaft itself was neglected. Calling the frequency thus obtained  $N_2$ , and that found for the unloaded shaft by the

\* Read March 11, 1904.

† Proc. Institution of Mechanical Engineers, 1883, p. 182.

‡ Phil. Trans. A, 1894, p. 279.

first method  $N_1$ , he deduced a final value  $N$  for the frequency from the equation

$$1/N^2 = 1/N_1^2 + 1/N_2^2. \quad (1)$$

By speed or frequency Dunkerley means the number of revolutions per *minute*, i. e.

$$N = 30\omega/\pi, \quad (2)$$

where  $\omega$  is the angle through which the shaft rotates in one second.

When more than one load exists, Dunkerley calculated critical speeds  $N_2, N_3, \dots$  for each separately, and deduced a critical speed for the whole system from the formula

$$1/N^2 = 1/N_1^2 + 1/N_2^2 + 1/N_3^2 + \dots \quad (3)$$

He made a number of experiments with a miniature shaft, loaded with one or both of two pulleys, and in many cases the speeds at which whirling commenced agreed remarkably with those calculated. In some cases better agreement was obtained with a formula of the type

$$1/N^2 = 1/N_2^2 + a/N_1^2,$$

in which  $a$  is an experimental constant.

Dunkerley (*l. c.* p. 281) noticed that there was a connexion between the speed of whirling and the frequency of the lateral vibrations of the shaft when not rotating; but I see no indication in his paper that he had grasped the real nature of the connexion.

Dunkerley's method (*l. c.* p. 358) of arriving at his working equation (3)—which he characterizes himself as “empirical”—is not convincing mathematically. In fact, the result does not appear to be in general strictly true; and there seems nothing in Dunkerley's work which serves to bring out the limitations. As we shall see later in particular cases, the agreement between theory and observation is not by itself sufficient evidence of the general applicability of formula (1). Again, the applications by Dunkerley of the Euler-Bernoulli elastic theory are cumbersome mathematically, and the formulæ to which they lead, and which were employed by Dunkerley, often admit of great simplification, without appreciable diminution of accuracy, under his experimental conditions. Also,

as already explained, the true nature of the relationship to lateral vibrations is not brought out.

For these several reasons I have thought it worth while to examine the whole question afresh from a variety of points of view, making liberal use, however, of Dunkerley's experimental results, and referring to his formulæ in the several cases, so that the present work is in many respects supplementary to his. To go fully into the mathematical investigations in each case would occupy an undue amount of space, and as the same methods are employed in the different cases, I have deemed it sufficient to give one or two illustrations in the Appendix at the end of the paper.

§ 2. It is unnecessary to describe the method employed by Greenhill and Dunkerley for the unloaded shaft, as it is simply the approximate Bernoulli-Eulerian method described in mathematical textbooks treating of "thin" rods. Dunkerley's second method, in so far as it relates to finding the critical speed for a loaded but massless shaft, is really analogous to a method illustrated by Lord Rayleigh\* in obtaining approximate frequencies of vibration. Both assume the displacement of the bar to be of a simple algebraic type; but Dunkerley applies ordinary statical equations, whilst Rayleigh applies dynamical equations deduced by means of the principle of energy. Rayleigh, however, advances in justification of his method a result based on very general reasoning, viz. that a considerable departure from the true type of vibration leads to only a small error in the estimate of the frequency.

I am not prepared to say that Rayleigh's general theory is impervious to criticism. A general theorem may pass muster even with acute critics, simply from failing to suggest points of view which decline to be left out of account in actual practice. Again, a theorem may be practically satisfactory within certain limits, and yet those limits may be so difficult to recognize that applications may be fraught with peril to any but one of the very few men who combine profound physical insight with first-rate mathematical ability. Still, taking all these things into account, I think it will be generally recognized that, in view of the empirical nature of Dunkerley's

\* 'Theory of Sound,' vol. i. Arts. 182, 183, &c.

second method, the application of Rayleigh's method to the problem of whirling is, if practicable, highly desirable. Numerous applications of it will be made here, and there is an illustration of the mathematical details in the Appendix.

§ 3. Before treating individual cases, I shall describe in unmathematical language the true nature of the connexion between lateral vibrations and the phenomenon of whirling. Ordinarily, when a shaft held at one or both ends is acted on by forces tending to bend it, on the removal of these forces it tends to return to its original straight position; in doing so it overshoots the mark and vibrates to and fro laterally. The velocity of its approach to the equilibrium position, and the frequency of the vibrations subsequently executed, are greater the larger the elastic stresses produced in the bar by a given lateral displacement. When the bar is rotating round its longitudinal axis, and is displaced laterally, the elastic stresses tend as before to bring it back to the undisturbed position; but the "centrifugal forces" have exactly the opposite tendency; they thus reduce the righting forces, and so diminish the frequency of vibration. If we take the simplest case where there are no complications from the mass of the shaft itself, and where only the mass (not the moment of inertia) of the load requires to be taken into account, it may be shown that if  $k/2\pi$  be the frequency of the vibrations which the shaft executes when displaced laterally at a time when it is rotating with uniform angular velocity  $\omega$ , and  $K/2\pi$  be the corresponding frequency in the absence of rotation, then

$$k^2 = K^2 - \omega^2. \quad (4)$$

As  $\omega$  is increased, the frequency of vibration and so the stability of the bar diminish, until eventually when

$$\omega = K \quad (5)$$

the frequency becomes *nil*, i. e. the period becomes infinite, or the righting power vanishes. In fact, the position is similar to that of a ship whose C.G. has come to coincide with the metacentre. The case is not one in which forced vibrations are set up with a frequency equal to one of the natural periods. What leads to whirling is the indirect action of the rotation in reducing to zero the righting forces which naturally

act on the shaft when displaced laterally. The case of a loaded but massless shaft is of course an extreme one ; but all the other cases which I have examined present similar features. The case selected for mathematical treatment in the Appendix is that of a shaft supported at both ends ; this admits of a variety of sub-cases, illustrative of various points.

[*March 15.*—To prevent misconception, it seems desirable to state explicitly and prove—as was done when the paper was read—that the formula of § 3,

$$k^2 + \omega^2 = K^2,$$

applies exactly to all *unloaded* shafts, to the degree of accuracy possessed by ordinary equations for thin rods. The elastic bodily equation has the following forms :

$$\begin{aligned} d^4y/dx^4 &= \Omega^2(\sigma\rho y/EI) \text{ for rotation with whirling velocity } \Omega, \\ &= K^2(\sigma\rho y/EI) \text{ for vibration without rotation,} \\ &= (k^2 + \omega^2)(\sigma\rho y/EI) \text{ for vibration when velocity of} \\ &\hspace{15em} \text{rotation is } \omega. \end{aligned}$$

The value of  $\mu$  in the typical equation

$$d^4y/dx^4 = \mu^4 y$$

depends only on the terminal conditions. Thus for any, the same, system of supports we have

$$k^2 + \omega^2 = K^2 = \Omega^2.]$$

§ 4. As stated above, whirling is not really a case of coincidence of period between a vibrating system and disturbing forces. A rotating shaft may, however,—like any other shaft—be acted on by periodic forces which tend directly to set up lateral vibrations. In considering the effect of any such forces, it must be borne in view that what one has to look to is the frequency of the lateral vibrations of the shaft *as reduced by the rotation*. The possibility of forced vibrations of this kind is an additional reason for considering the effects of rotation on the period.

§ 5. In the main I shall follow Dunkerley's classification of the principal cases of whirling shafts, but shall not number separately cases where the shaft is with, and without, a load.

The cases are determined by the number and nature of the supports.

If  $x$  be taken parallel, and  $y$  perpendicular, to the undisturbed position of the axis of the shaft, the bending being supposed to occur in the plane  $xy$ , clearly at any support

$$y=0.$$

If the end of a shaft simply rests on a support, then on the Euler-Bernoulli theory, as the terminal section must be free from a couple,

$$d^2y/dx^2=0.$$

At such an end the shaft is said to be "supported." If, on the other hand, the shaft be constrained to retain a fixed direction at an end, the second terminal condition is

$$dy/dx=0.$$

If a shaft is "supported" at any intermediate point, then clearly  $y$  must vanish there, while  $dy/dx$  and  $d^2y/dx^2$  must be continuous. A sudden change of  $dy/dx$  would imply fracture, while a sudden change of  $d^2y/dx^2$  would imply the action of a couple at the supported section.

When an end is quite free, resting on no support, both stress and couple vanish, and so

$$d^2y/dx^2=d^3y/dx^3=0.$$

*Notation used.*

$E$  = Young's modulus for shaft, assumed homogeneous and isotropic.

$\rho$  = density of material of shaft, supposed uniform.

$\sigma$  = cross section (and so  $\sigma\rho$  = mass per unit length).

$M$  = mass of load, when there is one.

$I$  = moment of inertia of  $\sigma$  about diameter perpendicular to plane of bending.

$I'$  = moment of inertia of  $M$  about an axis through its C.G. perpendicular to plane of bending.

$\omega$  = angular velocity of rotation.

$k/2\pi$  = frequency of lateral vibration, taking rotation into account.

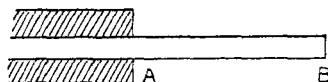
$T$  = variable part of kinetic energy of system.

$V$  = " " potential " "

$\mu = \omega^4(\rho\sigma/EI)^{\frac{1}{2}}.$

## CASE 1 (Dunkerley's Cases I., VII., and IX.).

§ 6. Overhanging shaft fixed in direction at end A.  
Length  $AB=l$ .



The terminal conditions are :

$$\begin{aligned} \text{At A (or } x=0), \quad y &= dy/dx = 0; \\ \text{,, B (or } x=l), \quad d^2y/dx^2 &= d^3y/dx^3 = 0. \end{aligned}$$

(a) Unloaded bar : Euler-Bernoulli solution (*cf.* Dunkerley, *l. c.* p. 288).

The displacement is of the type

$$y = \alpha \{ (\cosh \mu x - \cos \mu x) (\sinh \mu l + \sin \mu l) - (\sinh \mu x - \sin \mu x) (\cosh \mu l + \cos \mu l) \}, \dots \quad (1)$$

where  $\alpha$  is a constant.

The equation for  $\mu$ —and so for  $\omega$ —(see Dunkerley's equation (A) p. 289) is

$$1 + \cosh \mu l \cos \mu l = 0. \quad \dots \quad (2)$$

The smallest root (see Rayleigh's 'Sound,' art. 174) is

$$\mu l = 1.8751.$$

But by definition of  $\mu$ ,

$$\omega^2 = (EI/\sigma \rho l^4)(\mu l)^4,$$

$$\text{i. e.} \quad \omega^2 = 12.36 (EI/\sigma \rho l^4), \quad \dots \quad (3)$$

$$\text{or } \omega = 3.516 (EI/\sigma \rho l^4)^{\frac{1}{2}}. \quad \dots \quad (4)$$

Usually I shall record only the value of  $\omega^2$ , as often more convenient than that of  $\omega$ . In general  $\omega^2$  will equal  $(EI/\sigma \rho l^4)$  multiplied by some numerical quantity.

The above method leads to a somewhat complicated expression for the displacement, and throws no direct light on the relationship of whirling to lateral vibrations.

§ 7. (b) Supposing the bar still unloaded, replace the Euler-Bernoulli expression (1) by the much simpler one

$$y = \eta (x^4 - 4lx^3 + 6l^2x^2). \quad \dots \quad (5)$$

If for  $\eta$  we substitute  $g\rho\sigma/24EI$ , we have the displacement which the shaft would experience if bending under its own



weight. For the present purpose we assume

$$\eta \propto \cos kt,$$

where  $k/2\pi$  is the frequency of the lateral vibrations.

Neglecting the inertia of the motion of cross-sections relative to their centres of gravity, we find from (5)

$$\left. \begin{aligned} T &= \frac{1}{2}(\eta^2 + \omega^2 \eta^2) \frac{104}{45} \sigma \rho l^3, \\ V &= \frac{1}{2} \eta^2 \frac{144}{5} EI l^3; \end{aligned} \right\} \dots \dots (6)$$

whence, by Lagrange's equation,

$$k^2 + \omega^2 = 12.46 (EI/\sigma \rho l^3). \dots \dots (7)$$

This gives the frequency of vibration for any assigned value of  $\omega$ .

The critical angular velocity answering to whirling is that for which  $k$  vanishes, or the motion becomes unstable; it is thus given by

$$\omega^2 = 12.46 (EI/\sigma \rho l^3). \dots \dots (8)$$

The values given by (3) and (7) for the critical value of  $\omega$  differ by less than  $\frac{1}{2}$  per cent.

§ 8. (c) Mass  $M$ , inertia  $I'$ , at end of massless shaft.

Assume (*cf.* Rayleigh's 'Sound,' art. 183)

$$y = (3z - l\theta)(x/l)^2 + (l\theta - 2z)(x/l)^3, \dots \dots (9)$$

where  $z$  and  $\theta$  are the values of  $y$  and  $dy/dx$  (which may be regarded as the inclination of the shaft to its undisturbed direction) at the point of attachment of the load.

By Lagrange's equations, or otherwise, we obtain for the frequency

$$\{(k^2 + \omega^2)M - 12EI/l^3\} \{(k^2 - \omega^2)I' - 4EI/l\} = 36(EI/l^2)^2. (10)$$

For any assigned value of  $\omega$ , (10) gives two values of  $k^2$ , answering to two different types of vibration. Only one of these—which answers normally to the smaller value of  $k^2$ —is properly speaking of the lateral type.

For the critical angular velocity answering to whirling, we put  $k=0$  and obtain a quadratic equation for  $\omega^2$ , identical with that obtained otherwise by Dunkerley (*l. c.* p. 304).

One of these values of  $\omega^2$  is negative, and has no application to the present problem.

If, as in Dunkerley's experiments,  $I'$  has but little effect, a first approximation to the desired value of  $\omega^2$ —obtained by omitting  $I'$  altogether—is

$$\omega^2 = 3EI/Ml^3. \quad (11)$$

As a second approximation, neglecting  $(I')^2$  we find

$$\omega^2 = (3EI/Ml^3)(1 + 9I'/4Ml^2). \quad (12)$$

§ 9. (*d*) If we know, to start with, that the effect of  $I'$  is small, we can simplify the work by taking in place of (9)

$$y = \eta(3lx^2 - x^3)/2l^3. \quad (13)$$

If we substituted  $Mgl^3/3EI$  for  $\eta$  we should have the displacement produced in the shaft by a weight  $Mg$  at the end.

Assuming  $\eta$  in (13) proportional to  $\cos kt$ , and still neglecting the mass of the shaft, we have

$$\left. \begin{aligned} T &= \frac{1}{2}M(\dot{\eta}^2 + \omega^2\eta^2) + \frac{1}{2}I'(\dot{\eta}^2 - \omega^2\eta^2)(9/4l^2), \\ V &= \frac{1}{2}EI \cdot 3\eta^2/l^3 \end{aligned} \right\} \quad (14)$$

Thence we have for the frequency equation

$$(k^2 + \omega^2)M + (k^2 - \omega^2)(9I'/4l^2) = 3EI/l^3, \quad (15)$$

and for the critical angular velocity

$$\omega^2 = (3EI/Ml^3)(1 - 9I'/4Ml^2)^{-1}. \quad (16)$$

Omitting  $I'$  altogether we deduce (11); while retaining  $I'$ , but omitting  $(I')^2$ , we have (12).

§ 10. (*e*) Loaded shaft of appreciable mass.

Dunkerley's hypothesis (see § 1) supplies as the equation for the critical angular velocity

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2, \quad (17)$$

where  $\omega_1^2$  and  $\omega_2^2$  are given by (3) and (10) respectively. Supposing  $I'$  so small that (12) is applicable, we deduce

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{12 \cdot 36 EI} + \frac{Ml^3}{3EI} - \frac{3}{4} \frac{I'l}{EI}. \quad (18)$$

§ 11. (*f*) Instead of assuming the truth of (17) we may, following Rayleigh, assume (13) as the type of displacement, no longer neglecting the mass of the shaft. This adds to the value of  $T$  in (14) the term

$$\frac{1}{2}(\dot{\eta}^2 + \omega^2\eta^2) \frac{33}{140} \sigma \rho l,$$

and so leads to the frequency equation

$$(k^2 + \omega^2) \left( M + \frac{33}{140} \sigma \rho l \right) + (k^2 - \omega^2) 9I' / 4l^2 = 3EI / l^3. \quad (19)$$

For the critical angular velocity, noticing that  $140/11 = 12.73$ , we have

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{12.73 EI} + \frac{M l^3}{3 EI} - \frac{3}{4} \frac{I'}{EI}, \quad \dots \quad (20)$$

which is certainly in close agreement with the result (18) obtained by Dunkerley's hypothesis.

(g) If instead of (13) we assume the type (5), but regard  $I'$  as small, we obtain in place of (20)

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{12.46 EI} + \frac{M l^3}{3.2 EI} - \frac{5}{9} \frac{I'}{EI} \dots \quad (21)$$

## CASE 2 (Dunkerley's Cases II., VIII., and X.).

Shaft *supported* at both ends.

§ 12. At each end we have

$$y = d^2 y / dx^2 = 0.$$

(a) Unloaded shaft: Euler-Bernoulli solution.

The displacement is of the type

$$y = \alpha \sin \mu x \dots \dots \dots (1)$$

where  $\alpha$  is a constant, and the equation for  $\mu$  is

$$\mu l = \pi, \dots \dots \dots (2)$$

where  $l$  is the total length of the shaft.

From this we have

$$\omega^2 = \pi^4 (EI / \sigma \rho l^4) = 97.41 (EI / \sigma \rho l^4) \dots \dots \dots (3)$$

(b) Instead of the Euler-Bernoulli method for the unloaded shaft, we may assume

$$y = \eta x (\bar{l}^3 - 2lx^2 + x^3) \dots \dots \dots (4)$$

If  $\eta$  were replaced by  $g\sigma\rho/24EI$  this would represent the bending of the shaft under its own weight.

Assuming  $\eta \propto \cos kt$ , we have

$$\left. \begin{aligned} T &= \frac{1}{2} (\dot{\eta}^2 + \omega^2 \eta^2) \frac{31}{630} \sigma \rho l^3, \\ V &= \frac{1}{2} \eta^2 (24/5) EI l^3 \end{aligned} \right\}; \dots \dots \dots (5)$$

whence we have for the frequency equation

$$k^2 + \omega^2 = (3024/31)(EI/\sigma \rho l^4), \dots \dots (6)$$

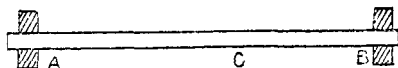
and for the critical velocity

$$\omega^2 = 97.55 EI/\sigma \rho l^4, \dots \dots \dots (7)$$

which presents an exceedingly close agreement with (3).

§ 13. (c) Mass (M, I') on massless shaft at a point C

$$(AC = a, BC = b, a + b = l).$$



Assuming Rayleigh type formulæ, and employing Lagrange's equations (see Appendix, § 41) we find for the frequency equation

$$\{M(k^2 + \omega^2) - 3EI(a^{-3} + b^{-3})\} \{I'(k^2 - \omega^2) - 3EI(a^{-1} + b^{-1})\} \\ = 9(EI)^2(b^{-2} - a^{-2})^2. \quad (8)$$

From this, putting  $k=0$ , we obtain a quadratic equation for the critical angular velocity, identical with that found otherwise by Dunkerley (*l.c.* p. 307).

If in (8) we absolutely neglect I', we have the simple result

$$k^2 + \omega^2 = 3EI/Ma^2b^2; \dots \dots \dots (9)$$

whilst, retaining I' but neglecting I'^2, we have for the critical angular velocity

$$\omega^2 = (3EI/Ma^2b^2) \left\{ 1 + \frac{I'}{M} \left( \frac{a-b}{ab} \right)^2 \right\}. \quad (10)$$

(d) If in the last sub-case we start with the assumption that I' is small, we may take

$$\left. \begin{array}{l} \text{for AC, } y = \eta bx (l^2 - b^2 - x^2), \\ \text{for BC, } y' = \eta ax' (l^2 - a^2 - x'^2) \end{array} \right\} \dots \dots \dots (11)$$

If  $\eta$  were replaced by  $gM/6EI$  this would represent the bending of the bar under the weight of M.

Assuming  $\eta \propto \cos kt$  we find by Lagrange's equations

$$(k^2 + \omega^2)Ma^2b^2 + (k^2 - \omega^2)I'(a-b)^2 = 3EI. \quad (12)$$

For the critical angular velocity, putting  $k=0$ , we have

$$\omega^2 = (3EI/Ma^2b^2) \{ 1 - (I'/M)(a-b)^2/a^2b^2 \}^{-1}. \quad (13)$$

When  $(I'/M)^2$  is neglected this is identical with (10).

## § 14. (e) Loaded shaft of appreciable mass.

On Dunkerley's hypothesis we have for the critical velocity

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2,$$

where  $\omega_1^2$  is given by (3), and  $\omega_2^2$  by (8) with  $k$  put  $=0$ . Thus when  $I'$  is so small that (10) is applicable, we have

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{97.41EI} + \frac{Ma^2b^2}{3EI l} - \frac{I'(a-b)^2}{3EI l} \quad \dots \quad (14)$$

(f) If we employ the same Rayleigh formulæ as in (c), but do not neglect the mass of the shaft, we replace (8) by

$$\begin{aligned} & \left\{ (k^2 + \omega^2) \left( M + \frac{17}{35} \sigma \rho l \right) - 3EI(a^{-3} + b^{-3}) \right\} \times \\ & \quad \left\{ (k^2 - \omega^2)I' + (k^2 + \omega^2) \frac{2}{105} \sigma \rho (a^3 + b^3) - 3EI(a^{-1} + b^{-1}) \right\} \\ & = (a^2 - b^2)^2 \left\{ 3EIa^{-2}b^{-2} + \frac{3}{35} (k^2 + \omega^2) \sigma \rho \right\}^2 \dots \dots \quad (15) \end{aligned}$$

This may be expected to give best results when the load is at or near the centre of the span, as the assumed type of displacement, which answers to the load only, is then nearest to that natural to a massive but unloaded shaft.

If the load is at the exact centre (15) gives

$$k^2 + \omega^2 = 48EI \div \left\{ Ml^3 + \frac{17}{35} \sigma \rho l^4 \right\} \dots \dots \quad (16)$$

When  $M$  is neglected this gives

$$k^2 + \omega^2 = 98.8EI \sigma \rho l^4 \dots \dots \quad (17)$$

Putting  $k=0$  in (16) and (17) we have of course the corresponding critical angular velocities. The value obtained from (17) is a very fair approximation for the case of an unloaded shaft (cf. (3)).

§ 15. (g) When  $I'$  is small, but the load  $M$  is not near the centre of the span, and is of the same order as the mass  $m$  of the shaft, better results are obtained from the following displacement types:—

$$\left. \begin{aligned} & \text{for AC, } y = \eta x \{ m(l-x)(l^2 + lx - x^2) + 4Mb(l^2 - l^2 - x^2) \}, \\ & \text{for BC, } y' = \eta x' \{ m(l-x')(l^2 + lx' - x'^2) + 4Ma(l^2 - a^2 - x'^2) \}. \end{aligned} \right\} \quad (18)$$

If  $\eta$  were replaced by  $g/24EI$  this would give the bending of the shaft under its own weight and that of the load

combined. Assuming  $\eta \propto \cos kt$  we obtain the frequency equation

$$(k^2 + \omega^2)A + (k^2 - \omega^2)I'B = C, \quad . \quad . \quad . \quad (19)$$

giving for the critical angular velocity

$$\omega^2 = C \div (A - I'B); \quad . \quad . \quad . \quad . \quad (20)$$

where for brevity

$$\begin{aligned} A &= 1 + \frac{18M}{31m} \left( \frac{ab}{l^2} \right) \left\{ 17 + 52 \frac{ab}{l^2} + 76 \left( \frac{ab}{l^2} \right)^2 + 36 \left( \frac{ab}{l^2} \right)^3 \right\} \\ &+ \frac{96}{31} \left( \frac{M}{m} \right)^2 \left( \frac{ab}{l^2} \right)^2 \left\{ 8 + 121 \frac{ab}{l^2} + 117 \left( \frac{ab}{l^2} \right)^2 \right\} + \frac{40320}{31} \left( \frac{M}{m} \right)^3 \left( \frac{ab}{l^2} \right)^3, \\ B &= \frac{630}{31} \frac{(a-b)^2}{ml^4} \left( 1 + 2 \frac{ab}{l^2} + \frac{8M}{m} \frac{ab}{l^2} \right), \\ C &= \frac{3024EI}{31ml^3} \left\{ 1 + 10 \frac{Mab}{ml^2} \left( 1 + \frac{ab}{l^2} \right) + 40 \left( \frac{M}{m} \right)^2 \left( \frac{ab}{l^2} \right)^2 \right\}. \end{aligned}$$

§ 16. Prof. Dunkerley carried out a number of experiments under the conditions of Case 2. He employed a shaft unloaded, or loaded with one or other of two pulleys of different sizes. As a preliminary to comparing his theory with observation, he had to calculate critical speeds for the pulleys alone, neglecting the mass of the shaft. Table I. compares the number of revolutions per minute which he calculated from a formula equivalent to (8), with  $k$  omitted, with the corresponding numbers which are given by the much simpler formulæ (9) and (13). The results apply of course only to the particular shaft and pulleys employed by Dunkerley.

TABLE I.— $N_2$  (Number of revolutions per minute for whirling).

Pulley.	Formula.	$b/l =$	1/2.	1/3.	1/6.	1/11.	1/32.
I.	(8) [Dunkerley's]...		1495	1683	2705	4621	13537
	(9)		1495	1682	2690	4532	12343
	(13)		1495	...	2710	4636	(16364)
II.	(8) [Dunkerley's] ...		997	1122	1808	3116	10355
	(9)		997	1121	1794	3015	8231
	(13)		997	...	1808	3119	(13716)

§ 17. The results from the formula (13) for the smallest value of  $b/l$  are put in brackets because the assumption on which the formula is based—viz., that the contribution from

$I'$  is small—is then far from being satisfied, so that the result is *a priori* unsatisfactory. The closeness with which the simplest formula (9) approaches to the results from the complicated formula (8) is rather surprising.

In Dunkerley's case the value given by (3) for  $N_1$ —the critical number of revolutions per minute for the shaft when unloaded—is 1122. Thus for the value  $1/32$  of  $b/l$  we have

$$\begin{aligned}\omega_2/\omega_1 = N_2/N_1 &= 13537/1122 \text{ for pulley I. (or 12 roughly),} \\ &= 10355/1122 \quad \text{,,} \quad \text{II. (or 9 roughly).}\end{aligned}$$

Thus the contribution from  $\omega_2$ , or  $N_2$ , to the critical speed for the loaded shaft on Dunkerley's hypothesis (1) § 1 is only about one-eightieth of that from  $\omega_1$ , even for the case of the heavier pulley II. Under such circumstances an error of even 100 per cent. in the value of  $\omega_2$ , or  $N_2$ , would exert but little influence. It is thus clear that under the conditions of Dunkerley's experiments the simple formula (9) would for all practical purposes be as satisfactory as (8).

§ 18. Table II. compares the speeds at which Dunkerley observed whirling to commence with those which he calculated from (3) and (8) on his hypothesis (1) § 1, and with those which are given by the formulæ (15) and (20) singly.

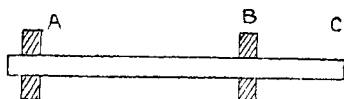
TABLE II.—Critical revolutions per minute.

Pulley I.					Pulley II.			
$b/l$	Observed.	(3) & (8) Dunkerley.	(15)	(20)	Observed.	(3) & (8) Dunkerley.	(15)	(20)
1/2	921	897	901	899	769	745	747	769
1/3	952	933	951	940	803	793	...	799
1/6	1044	1036	1113	1050	942	953	...	974
1/8	1101	1069	1179	1079	1007	1013	...	1081
1/11	1123	1089	1241	1100	1046	1055	...	1074
1/32	1150	1117	1362	1121	1130	1115	...	1121

Formula (15) is derived from a type of displacement which treats the shaft as massless, so its failure to agree well with experiment when the load is near one of the ends was to be anticipated. The simple formula (14) would give results in close agreement with those calculated by Dunkerley.

## CASE 3 (Dunkerley's Cases III. and XI.).

§ 19. Shaft supported at end A and at a second point B,



with BC overhanging ( $AB=l$ ,  $BC=c$ ). Taking B as origin of coordinates  $x$ ,  $y$  for BC, and  $x'$ ,  $y'$  for BA, we have:

At  $x=0$ ,  $y=0$  with continuity in  $dy/dx$  and  $d^2y/dx^2$ ,  
 „  $x=c$ ,  $d^2y/dx^2=d^3y/dx^3=0$  (when there is no load),  
 „  $x'=l$ ,  $y'=d^2y'/dx'^2=0$ .

(a) Unloaded shaft: Euler-Bernoulli solution. The displacements are cumbersome to record. The equation determining  $\mu$  (cf. Dunkerley, *l.c.* equation (A) p. 291) is

$$(\cosh \mu l \sin \mu l - \sinh \mu l \cos \mu l)(\cosh \mu c \sin \mu c - \sinh \mu c \cos \mu c) - 2 \sinh \mu l \sin \mu l (1 + \cosh \mu c \cos \mu c) = 0. \quad (1)$$

Except for special values of  $c/l$  this is somewhat intractable. On his p. 292 Dunkerley specifies 3.08 as the limiting value to which  $\mu l$  approaches when  $c/l$  is indefinitely reduced. This is not quite correct; the true limiting value is  $\pi$ , exactly as in (a) Case 2.

Treating  $c/l$  as small though not negligible, I find for the next approximation

$$\mu l = \pi (1 - \frac{1}{6} \pi^2 c^3 / l^3). \quad (2)$$

Whence, substituting its numerical value for  $\pi$ , we have for the critical angular velocity

$$\omega^2 = 97.41 (EI / \sigma \rho l^4) (1 - 6.6 c^3 / l^3). \quad (3)$$

This is satisfactory so long as  $(c/l)^3 (\pi^3/6) \coth \pi$  is small compared with unity.

A way of treating this problem by an assumed type of vibration will be found below in (f).

§ 20 (b) Mass (M, I') on overhanging part of massless shaft at distance  $c$  from the nearest support.

Formulae of the Rayleigh type are

$$\left. \begin{aligned} \text{for BC, } y &= \frac{3z - c\theta}{c(3c + 4l)} (2lx + 3x^2) + \frac{\theta c(2l + 3c) - 2z(l + 3c)}{3c + 4l} \left(\frac{x}{c}\right)^3, \\ \text{„ BA, } y' &= \frac{c\theta - 3z}{cl(3c + 4l)} (2l^2x' - 3lx'^2 + x'^3). \end{aligned} \right\}$$



By Lagrange's equations we find for the frequency equation

$$\left\{ M(k^2 + \omega^2) - 12EI \frac{3c+l}{c^3(3c+4l)} \right\} \left\{ I'(k^2 - \omega^2) - 12EI \frac{c+l}{c(3c+4l)} \right\} \\ = \frac{36(EI)^2(3c+2l)^2}{c^4(3c+4l)^2}. \quad (5)$$

Putting  $k=0$  we obtain an equation for the critical angular velocity which is identical with Dunkerley's (*l. c.* p. 314).

If in (5) we neglect  $I'$  altogether, we find

$$k^2 + \omega^2 = 3EI \div Mc^2(c+l). \quad (6)$$

Proceeding to a second approximation, retaining only the lowest power of  $I'$ , we find for the critical angular velocity, putting  $k^2=0$ ,

$$\omega^2 = \frac{3EI}{Mc^2(c+l)} \left\{ 1 + \frac{1}{4} \frac{I'}{M} \frac{(3c+2l)^2}{c^2(c+l)^2} \right\}. \quad (7)$$

This last result will not be satisfactory when  $c/l$  is very small, or the load very close to B; but under these conditions  $1/\omega^2$  is very small, so that on Dunkerley's hypothesis the load has but little real influence on the critical velocity.

(c) If in the problem treated under (b) we assume the effect of  $I'$  small to start with, we may employ the simpler type

$$\text{for BC, } y = \eta(2clx + 3cx^2 - x^3), \\ \text{,, BA, } y' = -\eta(c/l)(2l^2x' - 3lx'^2 + x'^3). \quad (8)$$

If  $\eta$  were replaced by  $gM/6EI$  this would represent the bending of the shaft under the weight of  $M$ .

Treating  $I'$  as small, we find by assuming  $\eta \propto \cos kt$

$$(k^2 + \omega^2)4Mc^2(c+l)^2 + (k^2 - \omega^2)I'(3c+2l)^2 = 12EI(c+l). \quad (9)$$

When  $I'$  is wholly neglected, this agrees with (6).

For the critical angular velocity, we have from (9)

$$\omega^2 = \frac{3EI}{Mc^2(c+l)} \left\{ 1 - \frac{I'}{M} \frac{(3c+2l)^2}{4c^2(c+l)^2} \right\}^{-1}, \quad (10)$$

which agrees with (7) when  $(I')^2$  is neglected.

§ 21. (d) Loaded shaft of appreciable mass.

On Dunkerley's hypothesis the critical angular velocity is given by

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2,$$

where  $\omega_1$  is given by (1) and  $\omega_2$  by (5) with  $k$  omitted. So

long as the effect of  $I'$  is small, and  $c/l$  does not exceed  $1/4$ , the following approximation is deducible from (3) and (7)

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{97 \cdot 41 EI} \left( 1 - 6 \cdot 6 \frac{c^3}{l^3} \right)^{-1} + \frac{Mc^2(c+l)}{3EI} - \frac{I'(3c+2l)^2}{12EI(c+l)}. \quad (11)$$

(e) If, while assuming the displacements (8), we allow for the mass of the rod, we replace (9) by

$$(k^2 + \omega^2) \left[ M \cdot 4c^2(c+l)^2 + \sigma \rho \left\{ \frac{4}{3} l^2(l^3 + c^3) + \frac{33}{35} (l^5 + c^5) - \frac{11}{5} l(l^4 - c^4) \right\} \right] + (k^2 - \omega^2) I'(3c+2l)^2 = 12EI(c+l). \quad (12)$$

Putting  $k=0$  we have a form of the critical velocity equation in which allowance is made for the inertia of both shaft and load.

(f) As an alternative to (c) we may assume a type of displacement answering to the form taken by the shaft when bending under its own weight, viz. R≡

for BC,

$$y = \eta \{ lx(4c^2 - l^2) + 6c^2x^2 - 4cx^3 + x^4 \};$$

for BA,

$$y' = \eta \{ -3l^3x' + 6l^2x'^2 - 4lx'^3 + x'^4 + (2/l)(l^2 - c^2)(2l^2x' - 3lx'^2 + x'^3) \}; \quad \left. \begin{array}{l} Q \equiv \\ S \equiv \end{array} \right\}$$

In the statical problem  $\eta \equiv g\sigma\rho/24EI$ . In the kinetic problem, assuming  $\eta \propto \cos kt$ , we find

$$\begin{aligned} (k^2 + \omega^2) & \left[ 9Mc^2(c+l)^2(3c^2 + cl - l^2)^2 + \sigma \rho \left\{ l^9 - \frac{33}{10} l^7(l^2 - c^2) \right. \right. \\ & \left. \left. + \frac{96}{35} l^5(l^2 - c^2)^2 + 3c^3l^2(4c^2 - l^2)^2 + \frac{78}{5} c^6l(4c^2 - l^2) + \frac{104}{5} c^9 \right\} \right] \\ & + (k^2 - \omega^2) I' \times 9 \{ 4c^2(l+c) - l^3 \}^2 = \frac{216}{5} EI(l^5 - 5l^3c^2 + 10lc^4 + 6c^5). \end{aligned}$$

Putting  $k=0$ , we have a second formula for the critical velocity, in which allowance is made for both shaft and load.

If in (14) we omit  $M$  and  $I'$  we have a result appropriate to the unloaded bar in (a). It is not, however, very satisfactory unless  $c/l$  is small. When powers of  $(c/l)$  above the fourth are neglected it gives for the critical angular velocity

$$\omega^2 = 97 \cdot 55 (EI/\sigma \rho l^4) \{ 1 - 0 \cdot 06 (c/l)^2 - 6 \cdot 8 (c/l)^3 + 3 \cdot 5 (c/l)^4 \}, \quad (15)$$

a result very similar to (3).

(g) An assumption which appears more natural at first sight than that made in either (e) or (f) is that the type of displacement answers to the bending of the shaft under its own weight and that of the load combined. This gives

$$\left. \begin{aligned} \text{for CB, } y &= \eta \left\{ \frac{1}{6} M(\xi^3 - c^2 \xi) + \frac{1}{24} \sigma \rho (\xi^4 - c^3 \xi) + (c - \xi) H \right\}, \\ \text{,, AB, } y' &= \eta \left\{ \frac{1}{6} P(\xi'^3 - l^2 \xi') + \frac{1}{24} \sigma \rho (\xi'^4 - l^3 \xi') \right\}, \end{aligned} \right\} \quad (16)$$

$$\left. \begin{aligned} \text{where } \xi &\equiv c - x, & Pl &\equiv Mc - \frac{1}{2} \sigma \rho (l^2 - c^2), \\ \xi' &\equiv l - x', & H &\equiv \frac{1}{3} (Pl^2 + Mc^2) + \frac{1}{8} \sigma \rho (l^3 + c^3). \end{aligned} \right\} \quad (17)$$

Writing for shortness

$$\left. \begin{aligned} R &\equiv Mc^2 H^2 + \sigma \rho \left\{ \frac{2}{945} (P^2 l^7 + M^2 c^7) + \frac{11}{8640} \sigma \rho (Pl^8 + Mc^8) + \frac{(\sigma \rho)^2}{7 \cdot 2^2} (l^9 + c^9) \right. \\ &\quad \left. + \frac{1}{3} c^3 H^2 - \frac{7}{180} c^5 M H - \frac{1}{90} c^8 \sigma \rho H \right\}, \\ Q &\equiv (H + \frac{1}{6} c^2 M + \frac{1}{24} c^3 \sigma \rho)^2, \\ S &\equiv EI \left\{ \frac{1}{3} (P^2 l^3 + M^2 c^3) + \frac{1}{4} \sigma \rho (Pl^4 + Mc^4) + \frac{1}{20} (\sigma \rho)^2 (l^5 + c^5) \right\}, \end{aligned} \right\} \quad (18)$$

we find for the frequency equation

$$(k^2 + \omega^2) R + (k^2 - \omega^2) I' Q = S, \quad \dots \quad (19)$$

and so for the critical angular velocity

$$\omega^2 = S \div (R - Q I'). \quad \dots \quad (20)$$

Obviously in general the evaluation of R and S is laborious.

§ 22. (h) It will be found that none of the types (e), (f), or (g) gives results which invariably accord well with experiment. It is clear that the most natural way of bending may be such that portions of the bar on opposite sides of the support at B have "centrifugal forces" acting on them in opposite directions. This suggests the use of a type of displacement answering to an imaginary gravitational force oppositely directed on opposite sides of B. Such a type is

$$\left. \begin{aligned} \text{for BC, } y &= \eta \{ x^4 - 4cx^3 + 6c^2x^2 + 2A(3cx^2 - x^3) - Bx \}, \\ \text{,, BA, } y' &= -\eta \{ x'^4 - 4lx'^3 + 6l^2x'^2 + 2A'(3lx'^2 - x'^3) - Bx' \}, \end{aligned} \right\} \quad (21)$$

where

$$\left. \begin{aligned} A &= 2M/\sigma\rho, \quad A' = 2P/\sigma\rho, \\ B &= (8Pl^2 + 3\sigma\rho l^3)/\sigma\rho, \\ Pl &= -Mc - \frac{1}{2}\sigma\rho(l^2 + c^2). \end{aligned} \right\} \quad (22)$$

Putting for shortness

$$\left. \begin{aligned} R' &= Mc^2 \{ 3(l^3 - c^3) + 8(Pl^2 - Mc^2)/\sigma\rho \}^2 + \frac{104}{45} \sigma\rho(l^3 + c^3) \\ &+ \frac{59}{5} (Pl^3 + Mc^3) - \frac{26}{15} (8Pl^2 + 3\sigma\rho l^3)(l^6 + c^6) + \frac{528}{35\sigma\rho} (P^2l^7 + M^2c^7) \\ &+ \frac{1}{3\sigma\rho} (l^3 + c^3)(8Pl^2 + 3\sigma\rho l^3)^2 - \frac{22}{5\sigma\rho} (8Pl^2 + 3\sigma\rho l^3)(Pl^5 + Mc^5), \\ S' &= 144EI \left\{ \frac{1}{5}(l^5 + c^5) + \{Pl^4 + Mc^4 + \frac{4}{3\sigma\rho}(P^2l^3 + M^2c^3)\}/\sigma\rho \right\}, \end{aligned} \right\} \quad (23)$$

we find for the frequency equation

$$k^2 + \omega^2 = S'/R', \quad (24)$$

and for the critical angular velocity

$$\omega^2 = S'/R'. \quad (25)$$

§ 23. Table III. compares the results obtainable from Dunkerley's formula, corresponding to (5) with  $k=0$ , with those given by the simple formulæ (6) and (10) for the case when the mass of the shaft is neglected.

TABLE III.

$c/l$ .....	$\frac{1.00}{30.66}$		$\frac{2.57}{29.10}$		$\frac{3.66}{28.0}$		$\frac{4.99}{26.66}$		$\frac{7.66}{24.00}$		$\frac{10.32}{21.33}$	
Pulley Formula	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.
(5) .....	16390	13816	4808	3353	3318	2277	2428	1643	1572	1056	1162	782
(6) .....	12014	8020	4603	3157	3256	2209	2393	1617	1562	1051	1161	779
(10) .....	15214	14733	4747	3350	3307	2275						

The results from (5) are taken from Dunkerley's paper. The values of  $c/l$  really differed slightly for the two pulleys; the values given in the headings to the Table are the means for the two cases.

It will be seen that except in the first instance, where the pulley was only an inch from a support, the simple formula (6) differs but little from (5). The difference for the larger values of  $c/l$  is so small that the improvement obtained when (10) is substituted for (6) is hardly worth considering.

§ 24. Table IV. gives particulars of the critical number of revolutions per minute of an unloaded shaft as observed and calculated by Dunkerley, and as calculated from several of the other equations advanced above.

TABLE IV.—Unloaded overhanging Shaft, critical speeds.

$c/l$ .	Observed value.	Calculated values from				
		(1) by Dunkerley.	(3)	(14)(M=0).	(15)	(25)(M=0).
1/10 ...	1309	1301	1351			
1/7 ...	1435	1397	1450			
1/5 ...	1472	1516	1571		1574	
1/3 ...	1606	1704		2013		1747
1/2 ...	1558	1606		3325		1627
1 ...	1002	1031		1058		1046

The values calculated by Dunkerley for small values of  $c/l$  are apparently affected by the error referred to in (a) ; if this were corrected his values should practically coincide with those obtained from (3). The large difference between the observed values answering to the values  $1/3$  and  $1/2$  of  $c/l$  and those calculated from (14) is to be ascribed to the fact that the type of displacement assumed in (f) answers more nearly to a higher harmonic than to the fundamental vibration. This serves to illustrate a contingency which must never be lost sight of when applying Rayleigh's method. We know from his general theory \* that the value so calculated for  $k$  can never be too low—excluding of course errors of calculation—so that when results have been obtained from more than one assumed type of vibration we need never be at loss which to prefer.

§ 25. Table V. compares the results observed by Dunkerley in a shaft carrying a pulley with those variously calculated.

\* 'Theory of Sound,' vol. i. Art. 89.

TABLE V.—Loaded overhanging Shaft, critical speeds.

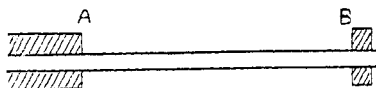
Pulley.	$l$ inches.	$c$ inches.	Observed speed.	Calculated speeds from				
				(1) & (5) by Dunkerley.	(e) (12).	(f) (14).	(g) (20).	(h) (25).
I.	30.70	1.00	1223	1170			1215	
	29.10	2.61	1329	1256			1342	
	28.00	3.69	1384	(1288)	[1678]	1385		
	26.66	5.02	1407	(1286)	(1578)	1454	[1839]	1394
	24.00	7.69	1224	1156	1267			1229
	21.33	10.35	968	941	971		1073	
II.	30.63	1.00	1227	1173			1220	
	29.10	2.54	1276	1213			(1416)	
	28.00	3.63	1281	(1191)	[1473]	1311	[1927]	
	26.66	4.96	1215	(1114)	1280	(1326)		1200
	24.00	7.63	928	898	947			944
	21.33	10.29	712	703	711			

Calculated results differing widely from the observed are put in [ ] brackets; those whose divergence is less but still conspicuous are put in ( ) brackets. When  $c/l$  is small, all the formulæ necessarily supply results which approach closely to those for an unloaded shaft.

When there is a conspicuous difference between observed and calculated values, the latter, in accordance with Rayleigh's principle, are invariably the larger, except in the case of Dunkerley's own calculations.

#### CASE 4 (Dunkerley's Cases IV. and XII.).

§ 26. Shaft fixed in direction at one end (A,  $x=0$ ), and supported at the other (B,  $x=l$ ).



(a) Unloaded shaft: Euler-Bernoulli solution.

The displacement is given by

$$y = \alpha \{ (\cosh \mu x - \cos \mu x)(\sinh \mu l + \sin \mu l) - (\sinh \mu x - \sin \mu x)(\cosh \mu l + \cos \mu l) \}, \quad (1)$$

where  $\alpha$  is a constant.

The equation determining  $\mu$  (*cf.* Dunkerley, *l. c.* p. 294) is

$$\coth \mu l = \cot \mu l. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The least root of this (Rayleigh's 'Sound,' arts. 180 and 174) is

$$\mu l = 3.9266. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Answering to which we have for the critical angular velocity

$$\omega^2 = 237.7 (EI/\sigma \rho l^4). \quad . \quad . \quad . \quad . \quad (4)$$

(b) Instead of the Euler-Bernoulli method, we may assume for the unloaded shaft

$$y = \eta x^2(l-x)(3l-2x). \quad . \quad . \quad . \quad . \quad (5)$$

If  $\eta$  were replaced by  $g\sigma\rho/48EI$ , this would give the bending of the shaft under its own weight.

For the dynamical problem we find

$$\left. \begin{aligned} T &= \frac{1}{2}\sigma\rho(\dot{\eta}^2 + \omega^2\eta^2)(19/630)l^3, \\ V &= \frac{1}{2}\eta^2(36/5)EI l^5; \end{aligned} \right\} \quad . \quad . \quad . \quad (6)$$

whence we have for the frequency equation

$$k^2 + \omega^2 = (4536/19)(EI/\sigma\rho l^4), \quad . \quad . \quad . \quad (7)$$

and for the critical angular velocity

$$\omega^2 = 238.7 EI/\sigma\rho l^4. \quad . \quad . \quad . \quad . \quad (8)$$

The value of  $\omega$  given by (8) is only 0.2 per cent. in excess of that given by the exact equation (4).

(c) Mass ( $M, I'$ ) on massless shaft.

Supposing the mass at C ( $AC=a$ ,  $BC=b$ ), and measuring  $x$  from A to C, and  $x'$  from B to C, we have for the Rayleigh type of displacement

$$\left. \begin{aligned} \text{for AC, } y &= (3z-a\theta)(x^2/a^2) + (a\theta-2z)(x^3/a^3), \\ \text{for BC, } y' &= \frac{1}{2}(3z+b\theta)(x'/b) - \frac{1}{2}(b\theta+z)(x'^3/b^3). \end{aligned} \right\} \quad . \quad (9)$$

By Lagrange's equations we find

$$\left\{ M(k^2 + \omega^2) - 3EI \frac{a^3 + 4b^3}{a^3b^3} \right\} \left\{ I'(k^2 - \omega^2) - EI \frac{3a + 4b}{ab} \right\} \\ = (3EI)^2 \left( \frac{a^2 - 2b^2}{a^2b^2} \right)^2. \quad . \quad (10)$$

When  $k$  is omitted this agrees with Dunkerley's equation (*l. c.* p. 321). Equation (10) splits into two factors when

$$a/b = \sqrt{2} = 1.414.$$

In this position of the load the frequencies of what may be called the transverse and the oscillational vibrations are respectively given by

$$\left. \begin{aligned} k^2 &= -\omega^2 + (51 + 36\sqrt{2})(EI/MI^3) \equiv -\omega^2 + 101.9(EI/MI^3), \\ k^2 &= \omega^2 + (7 + 5\sqrt{2})(EI/I'l) \equiv \omega^2 + 14.14(EI/I'l). \end{aligned} \right\} \quad (11)$$

If we wholly neglect  $I'$  in (10), we have for the critical velocity

$$\omega^2 = 12EI l^3 \div \{Ma^3b^2(3a+4b)\}; \quad . \quad . \quad (12)$$

while retaining  $I'$ , but neglecting  $(I')^2$ , we have

$$\omega^2 = \frac{12EI l^3}{Ma^3b^2(3a+4b)} \left\{ 1 + \frac{9I'(a^2-2b^2)^2}{Ma^2b^2(3a+4b)^2} \right\}. \quad . \quad (13)$$

(d) If we assume  $I'$  small to commence with, still neglecting the mass of the shaft, we may take

$$\left. \begin{aligned} \text{in AC, } y &= \eta \{ 2l^3(3ax^2 - x^3) - a^2(2a+3b)(3lx^2 - x^3) \}, \\ \text{in BC, } y' &= \eta \{ 3a^2bl^2x' - a^2(2a+3b)x'^3 \}. \end{aligned} \right\} \quad (14)$$

When  $\eta$  is replaced by  $Mg/12EI l^3$  we have the bending of the shaft under the weight of  $M$ .

For the kinetic problem, we find for the frequency equation

$$\begin{aligned} (k^2 + \omega^2)Ma^3b^2(3a+4b)^2 + (k^2 - \omega^2)9I'a(a^2-2b^2)^2 \\ = 12EI l^3(3a+4b), \quad . \quad (15) \end{aligned}$$

and for the critical angular velocity

$$\omega^2 = \frac{12EI l^3}{Ma^3b^2(3a+4b)} \left\{ 1 - \frac{9I'(a^2-2b^2)^2}{Ma^2b^2(3a+4b)^2} \right\}^{-1}. \quad . \quad (16)$$

This agrees with (12) when  $I'$  is neglected, with (13) when  $(I')^2$  only is neglected.

§ 27. (e) Load  $(M, I')$  on massive shaft.

On Dunkerley's hypothesis we have

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2,$$

where  $\omega_1$  is given by (4), and  $\omega_2$  by (10), with  $k$  omitted.

If the effect of  $(I')$  is small we should get from (4) and (13)

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{237.7 EI} + \frac{Ma^3b^2(3a+4b)}{12EI l^3} - \frac{3 I'a(a^2-2b^2)^2}{4 EI l^3(3a+4b)}. \quad (17)$$

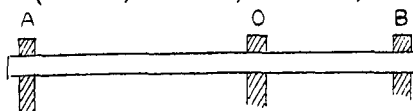
In this case I have not worked out an independent dynamical method taking both shaft and load into account; but the difficulties would be less than in case (3).



CASE 5 (Dunkerley's Cases V. and XIII.).

§ 28. Shaft supported at ends A and B, and at intermediate point O

$$(OA=a, \quad OB=b, \quad a+b=l).$$



(a) Unloaded shaft: Euler-Bernoulli solution.

As pointed out by Dunkerley, the mathematical conditions are all satisfied if the two relations (*cf.* § 12)

$$\left. \begin{aligned} \mu a &= i\pi, \\ \mu b &= j\pi \end{aligned} \right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where  $i$  and  $j$  are integers, can exist simultaneously.

To have a real application to the practical problem,  $i$  and  $j$  must be small integers, so that (1) is of very limited scope. If, however, the spans are equal, or if the longer, say  $a$ , is a multiple of  $b$ , we have obviously

$$\mu b = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Answering to this, we obtain for the critical velocity

$$\omega^2 = \pi^4 EI / \sigma \rho b^4 = 97.41 EI / \sigma \rho b^4. \quad . \quad . \quad . \quad (3)$$

This is the same result (*cf.* (a) Case 2) as for a shaft of length  $b$  supported at both ends.

Excluding the above special cases, the general equation obtained by Dunkerley (*l. c.* p. 296) is

$$\coth \mu a - \cot \mu a + \coth \mu b - \cot \mu b = 0. \quad . \quad . \quad . \quad (4)$$

If  $b/a$  be very small, a first approximation is

$$\coth \mu a - \cot \mu a = 0,$$

the least root of which (*cf.* (a) case 4) is

$$\mu a = 3.9266.$$

This reduction to (a) case 4 only implies, what is physically obvious, that when the support O is close to an end it serves to fix the terminal direction of the shaft. If  $b/a$  in (4) is treated as small, and  $(b/a)^2$  as negligible, we deduce

$$\mu a = 3.9266 (1 - b/3a), \quad . \quad . \quad . \quad . \quad (5)$$

whence

$$\omega^2 = 237.7 (1 - 4b/3a) (EI / \sigma \rho a^4). \quad . \quad . \quad (6)$$

(b) As (4) is somewhat unmanageable except for special cases, I have tried several algebraic types of displacement, amongst them

$$\left. \begin{array}{l} \text{for AO, } y = \eta \{x^4 - 4a^3x - C(x^3 - 3a^2x) - Dx\}, \\ \text{for BO, } y' = \eta \{x'^4 - 4b^3x' - C'(x'^3 - 3b^2x') + Dx'\}. \end{array} \right\} \quad (7)$$

Here  $x$  is measured from A towards O, and  $x'$  from B towards O, and

$$C \equiv (3a^2 + ab - b^2)/2a, \quad C' \equiv (3b^2 + ab - a^2)/2b, \\ D \equiv ab(a - b).$$

Replacing  $\eta$  in (7) by  $g\sigma\rho/24EI$  we should have the bending of the shaft under its own weight.

From the kinetic method I find

$$k^2 + \omega^2 = (3/5)(EI/\sigma\rho) \{3(a^5 + b^5) + 5ab(a^3 + b^3) - 5a^2b^2(a + b)\} \div \left\{ \frac{37}{9}(a^5 + b^5) \right. \\ \left. + \frac{68}{35}(Ca^7 + C'b^7) + \frac{1}{3}D^2(a^3 + b^3) - \frac{113}{20}(Ca^5 + C'b^5) + \frac{7}{3}D(a^5 - b^5) - \frac{8}{5}D(Ca^5 - C'b^5) \right\}$$

Putting  $k^2 = 0$  we have the critical angular velocity. The evaluation, though perfectly straightforward, is in general tedious.

For  $b/a$  small, however, we easily find

$$k^2 + \omega^2 = (4536/19)(1 - 4b/3a)(EI/\sigma\rho a^4), \quad \dots \quad (9)$$

and thence for the critical velocity

$$\omega^2 = 238.7(1 - 4b/3a)(EI/\sigma\rho a^4), \quad \dots \quad (10)$$

a result of course in close agreement with (6) (*cf.* also (8) and (4) of case 4, § 26).

Again, if  $b = a$  we obtain from (8) for the period and critical velocity

$$k^2 + \omega^2 = (4536/19)(EI/\sigma\rho a^4), \quad \dots \quad (11)$$

$$\omega^2 = 238.7(EI/\sigma\rho a^4). \quad \dots \quad (12)$$

(c) As an alternative type to (7) let us take

$$\left. \begin{array}{l} \text{for AO, } y = -\eta \{x^4 - 4a^3x - C(x^3 - 3a^2x) - Dx\}, \\ \text{for BO, } y' = \eta \{x'^4 - 4b^3x' - C'(x'^3 - 3b^2x') - Dx'\}, \end{array} \right\} \quad (13)$$

where now

$$C \equiv (3a^3 + 4a^2b + b^3) \div 2a(a + b), \quad C' \equiv (a^3 + 4ab^2 + 3b^3) \div 2b(a + b) \\ D \equiv ab(a^2 + b^2)/(a + b).$$

The above displacements answer to an imaginary gravitational force oppositely directed on the two sides of O.

By Lagrange's equations we deduce

$$= (3/5)(EI/\rho pl)(3l^3 - 10abl^4 - 5a^2b^2l^2 + 20a^3b^3) \div \left\{ \frac{37}{9}(a^3 + b^3) + \frac{68}{35}(C^2a^7 + C'^2b^7) \right. \\ \left. + \frac{1}{3}D^2(a^3 + b^3) - \frac{113}{20}(Ca^5 + C'b^5) + \frac{7}{3}D(a^5 + b^5) - \frac{8}{5}D(Ca^5 + C'b^5) \right\}. \quad (14)$$

Omitting  $k^2$ , we have the critical angular velocity.

When  $b/a$  is small (14) agrees with (8) in giving (9) and (10). For  $b=a$ , however, it gives the widely different result

$$k^2 + \omega^2 = (48 \times 63/31)(EI/\sigma \rho a^4), \quad (15)$$

whence for the critical velocity

$$\omega^2 = 97.55(EI/\sigma \rho a^4). \quad (16)$$

This is identical with (7) of case (2), which applies to a single span of length  $a$ , and is in close agreement—as it should be—with (3), when  $b$  is replaced by  $a$ . The divergence of (12) really means (*cf.* § 24 and Table IV.) that its assumed type of vibration answers not to the fundamental note but to an harmonic.

§ 29. Mass (M, I') on massless shaft.

(d) Supposing the load at C, between O and A, at a distance  $c$  from O, Rayleigh type displacements are :

$$\left. \begin{aligned} \text{for OC, } y &= \frac{2b(3z - c\theta)}{c(4b + 3c)} \left( x + \frac{3x^2}{2b} \right) + \frac{\theta c(2b + 3c) - 2z(b + 3c)x^3}{4b + 3c} \\ \text{for AC, } y' &= \frac{3z + (a - c)\theta}{2(a - c)} x' - \frac{(a - c)\theta + z}{2(a - c)^3} x'^3, \\ \text{for OB, } y'' &= \frac{2b(c\theta - 3z)}{c(4b + 3c)} \left( x'' - \frac{3}{2} \frac{x''^2}{b} + \frac{1}{2} \frac{x''^3}{b^2} \right), \end{aligned} \right\} \quad (17)$$

where  $x$  and  $x''$  are measured from O, and  $x'$  from A.

Lagrange's equations lead to

$$\left[ M(k^2 + \omega^2) - 3EI \left\{ \frac{1}{(a - c)^3} + \frac{4(b + 3c)}{c^3(4b + 3c)} \right\} \right] \\ \times \left[ I'(k^2 - \omega^2) - 3EI \left\{ \frac{1}{a - c} + \frac{4(b + c)}{c(4b + 3c)} \right\} \right] \\ - 9(EI)^2 \left\{ \frac{1}{(a - c)^2} - \frac{2(2b + 3c)}{c^2(4b + 3c)} \right\}^2 = 0. \quad (18)$$

Omitting  $k^2$ , we obtain an equation for the critical angular velocity which—allowing for a misprint—agrees with Dunkerley's equation A (*l. c.* p. 326). It is worth noticing that the terms independent of  $k$  or  $\omega$  in (18) reduce to

$$36(EI)^2 a^2(a+b) \div \{c^3(a-c)^3(4b+3c)\}.$$

The quadratic (18) splits into two factors, representing pure transverse and oscillational vibrations, when

$$\begin{aligned} c^2/(a-c)^2 &= (4b+6c)/(4b+3c), \\ &= 1 \text{ when } b/c \text{ is very big (cf. (c) case 2),} \\ &= 2 \text{ when } b/c \text{ is very small (cf. (c) case 4).} \end{aligned}$$

If in (18) we altogether neglect  $I'$  we find for the critical velocity

$$\omega^2 = 12EIa^2(a+b) \div [Mc^2(a-c)^2\{4a(b+c)-c^2\}]. \quad (19)$$

(e) If we assume  $I'$  small to begin with, we may replace the displacements in (*d*) by the simpler type

$$\left. \begin{aligned} \text{in OC, } y &= \eta[-2abc(3ac-2a^2-c^2)x + 2a^2(a+b)(3cx^2-x^3) \\ &\quad - c(3ac+2ab-c^2)(3ax^2-x^3)], \\ \text{in CA, } y &= \eta[-2c^3a^2(a+b) + 2ac(c^2b+3a^2c+2a^2b)x \\ &\quad - c(3ac+2ab-c^2)(3ax^2-x^3)], \\ \text{in OB, } y' &= \eta[2abc(3ac-2a^2-c^2)x' \\ &\quad - (ac/b)(3ac-2a^2-c^2)(3bx'^2-x'^3)]. \end{aligned} \right\}$$

In the above  $x$  is measured from O to A, and  $x'$  from O to B.

Replacing  $\eta$  by  $(gM/EI)/12a^2(a+b)$ , we should have the bending of the shaft due to the weight of M at C.

From Lagrange's equations I find for the frequency equation

$$\begin{aligned} &(k^2 + \omega^2)Mc^2(a-c)^2(4ab+4ac-c^2)^2 \\ &+ (k^2 - \omega^2)I'\{4ab(a-2c) + 3c(2a^2-4ac+c^2)\}^2 \\ &= 12EIa^2(a+b)(4ab+4ac-c^2), \quad \dots \dots (21) \end{aligned}$$

and for the critical velocity

$$\begin{aligned} \omega^2 &= \frac{12EIa^2(a+b)}{Mc^2(a-c)^2(4ab+4ac-c^2)} \times \\ &\left[ 1 - \frac{I'\{4ab(a-2c) + 3c(2a^2-4ac+c^2)\}^2}{Mc^2(a-c)^2(4ab+4ac-c^2)^2} \right]^{-1}. \quad (22) \end{aligned}$$

Neglecting  $I'$  altogether, we obtain (19).

In the case of two equal spans, or  $b=a$ , omitting higher powers of  $I'$  in (22), we find

$$\omega^2 = \frac{24EIa^3}{Mc^2(a-c)^2(4a^2+4ac-c^2)} \times \left[ 1 + \frac{I'(4a^3-2a^2c-12ac^2+3c^3)^2}{Mc^2(a-c)^2(4a^2+4ac-c^2)^2} \right]. \quad (23)$$

An identical result is deducible—but not so easily—from (18).

§ 30. Load ( $M, I'$ ) on massive shaft.

(f) On Dunkerley's hypothesis the critical velocity is given by

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2,$$

where  $\omega_1$  is given by (3) or (4), and  $\omega_2$  by (18) with  $k$  omitted.

In the case of equal spans, supposing the effect of  $I'$  small, we thus find

$$\frac{1}{\omega^2} = \frac{\sigma \rho a^4}{97.41EI} + \frac{Mc^2(a-c)^2(4a^2+4ac-c^2)}{24EIa^3} - \frac{I'(4a^3-2a^2c-12ac^2+3c^3)^2}{24EIa^3(4a^2+4ac-c^2)^2}. \quad (24)$$

(g) The best algebraic type of displacement would probably answer to the bending of the shaft under a gravitational force supposed to act on both shaft and load, but oppositely directed on the two spans. I have only worked out results from the simpler type (13), which neglects the influence of the load on the displacement. This leads to the frequency equation

$$(k^2 + \omega^2)R + (k^2 - \omega^2)QI' = S, \quad (25)$$

where

$$\left. \begin{aligned} R &= M[(a-c)^4 - 4a^3(a-c) - C\{(a-c)^3 - 3a^2(a-c)\} - D(a-c)]^2 \\ &\quad + \sigma \rho \left[ \frac{37}{9}(a^9 + b^9) + \frac{68}{35}(C^2a^7 + C'^2b^7) + \frac{1}{3}D^2(a^3 + b^3) \right. \\ &\quad \left. - \frac{113}{20}(Ca^8 + C'b^8) + \frac{7}{3}D(a^6 + b^6) - \frac{8}{5}D(Ca^5 + C'b^5) \right], \\ Q &= [4\{a^3 - (a-c)^3\} - 3\{a^2 - (a-c)^2\}C + D]^2, \\ S &= \frac{3EI}{5l} \{3l^6 - 10abl^4 - 5a^2b^2l^2 + 20a^3b^3\}, \end{aligned} \right\} \quad (26)$$

the notation being the same as in (c).

Omitting  $k$  in (25) we obtain the critical angular velocity. If, for instance, the spans are equal, and  $I'$  is negligible, we obtain

$$\omega^2 = \frac{97.55EI}{\sigma \rho a^4} + \left[ 1 + \frac{315}{31} \frac{M}{\sigma \rho a} \left\{ \frac{c(a-c)}{a^2} \right\}^2 \left\{ 1 + \frac{c(a-c)}{a^2} \right\}^2 \right] \quad (27)$$

Numerical calculations are here simplified by noticing that the coefficient of  $M$  involves  $c$  and  $a-c$  symmetrically; this implies that the critical velocity is unaltered when the distances of the load from the central and terminal piers are interchanged.

As (27) is based on a displacement which omits the effect of the load, we may expect it to prove less exact the larger the load.

§ 31. Table VI. compares the results calculated by Dunkerley from (18), with  $k$  omitted, for the critical number of revolutions per minute in his shaft, supposed massless but carrying one of his pulleys, with corresponding results from the simpler formulæ (19) and (23). The spans are supposed equal.

TABLE VI.

$c/a$ .....	1/16.		1/4.		1/2.		3/4.		15/16.	
Pulley ...	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.
Dunkerley from (18)...	31664	24173	8114	4842	4987	3325	6318	4288	24550	18816
„ (19)...	24760	16510	7175	4784	4987	3325	6284	4190	19464	12978
„ (23)...	32676	27219	...	...	...	...	...	...	24976	20096

When the pulley is close to a support it exerts—*cf.* the analogous case in § 17—an exceedingly small effect on the critical velocity in the practical case of shaft and pulley combined. Except when the pulley is close to a support, we see in Table VI. a close agreement between even the simplest formula (19) and (18). The more complicated formula (23) agrees pretty closely with (18) even for the values 1/16 and 15/16 of  $c/a$ ; for the other values the two would be in practical agreement.

§ 32. Table VII. compares the critical number of revolutions actually observed by Dunkerley for the case of equal spans, with those which he calculated from (2) and (18), and the corresponding results from (27), which altogether neglects  $I'$ .

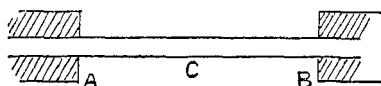
TABLE VII.

$c/a$ .....	1/16.		1/4.		1/2.		3/4.		15/16.	
Pulley ...	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.
Observed Dunkerley	4430	4524	3930	3213	3420	2600	3846	3056	4402	4220
from (2) & (18)	4440	4411	3925	3286	3334	2671	3657	3100	4411	4362
" (27)...	4438	4381	3967	3516	3604	3000	3967	3516	4438	4381

As theory led us to expect, the agreement of (27) with observation is not so good for the heavier pulley II. as for the lighter.

#### CASE 6 (Dunkerley's Cases VI. and XIV.).

§ 33. Shaft fixed in direction at both ends.



(a) Unloaded shaft: Euler-Bernoulli method (Dunkerley, *l. c.* p. 298).

Measuring  $x$  from an end A, and denoting AB by  $l$ , we have

$$y = \alpha \{ (\cosh \mu x - \cos \mu x)(\sinh \mu l - \sin \mu l) - (\sinh \mu x - \sin \mu x)(\cosh \mu l - \cos \mu l) \}, \quad \dots (1)$$

where  $\alpha$  is a constant, and the value of  $\mu$  is given by

$$\cosh \mu l \cos \mu l - 1 = 0. \quad \dots (2)$$

The least root of (2) (see Rayleigh's Sound, art. 174) is

$$\mu l = 4.73004, \quad \dots (3)$$

which gives for the critical velocity

$$\omega^2 = 500.6 (EI / \sigma \rho l^4). \quad \dots (4)$$

(b) In the above case of no carried load, an approximate frequency equation is deducible from the simple type

$$y = \eta x^2 (l - x)^2. \quad \dots \quad (5)$$

Replacing  $\eta$  by  $g\sigma\rho/24EI$  we should have the bending of the shaft under its own weight.

From the kinetic treatment we easily find

$$k^2 + \omega^2 = 504(EI/\sigma\rho l^4), \quad \dots \quad (6)$$

giving for the critical velocity

$$\omega^2 = 504(EI/\sigma\rho l^4). \quad \dots \quad (7)$$

The values given by (4) and (7) for  $\omega$  differ by only about 0.3 per cent.

§ 34. Load (M, I') on massless shaft.

(c) If the load be at C (AC =  $a$ , BC =  $b$ ) we have for Rayleigh type displacements, measuring  $x$  from A and  $x'$  from B,

$$\begin{aligned} \text{in AC, } y &= (3z - a\theta)(x/a)^2 + (a\theta - 2z)(x/a)^3, \\ \text{in BC, } y' &= (3z + b\theta)(x'/b)^2 - (b\theta + 2z)(x'/b)^3 \end{aligned} \quad \dots \quad (8)$$

Applying Lagrange's equations we have

$$\begin{aligned} \{M(k^2 + \omega^2) - 12EI(a^{-3} + b^{-3})\} \{I'(k^2 - \omega^2) - 4EI(a^{-1} + b^{-1})\} \\ = 36(EI)^2(b^{-2} - a^{-2})^2 \quad \dots \quad (9) \end{aligned}$$

Omitting  $k^2$ , we obtain for the critical velocity a result agreeing with Dunkerley's (*l. c.* p. 337).

(9) splits into factors, representing pure transverse and oscillational vibrations, when  $b = a$ . In this case we have for the respective frequencies (*cf.* (11) of § 26, and (13) and (14) of § 42)

$$\begin{aligned} k^2 &= -\omega^2 + 192EI/MI^3, \\ k^2 &= \omega^2 + 16EI/I'l. \end{aligned} \quad \dots \quad (10)$$

If we omit  $I'$  altogether in (9) we obtain for the critical velocity

$$\omega^2 = 3EI^3/Ma^3b^3. \quad \dots \quad (11)$$

As a second approximation, when  $I'$  is small, we have

$$\omega^2 = (3EI^3/Ma^3b^3) \{1 + 9I'(a - b)^2/4Ma^2b^2\}. \quad \dots \quad (12)$$



(d) If we can assume  $I'$  small to begin with, we may replace (8) by

$$\left. \begin{array}{l} \text{in AC, } y = \eta \{ ab^2(a+b)x^2 - \frac{1}{3}l^2(3a+b)x^3 \}, \\ \text{in BC, } y' = \eta \{ a^2b(a+b)x'^2 - \frac{1}{3}a^2(a+3b)x'^3 \} \end{array} \right\} \quad (13)$$

Applying Lagrange's equations we find

$$(k^2 + \omega^2) \frac{4}{3} Ma^3b^3 + (k^2 - \omega^2) I' ab(a-b)^2 = \frac{4}{3} EI(a+b)^3. \quad (14)$$

For the critical angular velocity, writing  $l$  for  $a+b$ , we have

$$\omega^2 = (3EI^3/Ma^3b^3) \{ 1 - 9I'(a-b)^2/4Ma^2b^2 \}^{-1}. \quad (15)$$

Neglecting  $I'$  this agrees with (11), neglecting  $(I')^2$  with (12).

§ 35. (c) Load ( $M$ ,  $I'$ ) on massive shaft.

On Dunkerley's hypothesis we have

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2,$$

where  $\omega_1$  is given by (4) and  $\omega_2$  by (9) with  $k$  omitted.

When  $I'$  is small we thus obtain

$$\frac{1}{\omega^2} = \frac{\sigma \rho l^4}{500 \cdot 6 EI} + \frac{Ma^3b^3}{3EI^3} - \frac{3}{4} \frac{I' ab(a-b)^2}{EI^3}. \quad (16)$$

I have not worked out a frequency equation based on an assumed type of displacement, but it would present no difficulty. The displacements would combine terms of the types (5) and (13) according to the relative masses of the shaft and load.

#### GENERAL CONCLUSIONS.

§ 36. In every case here treated when the effect of the moment of inertia of the load has been small—as was true invariably in Dunkerley's experiments, and probably often is in practice—the frequency equation has proved to be of the type

$$k^2 = K^2 - \nu \omega^2, \quad . . . . . (1)$$

where  $K/2\pi$  is the frequency of the fundamental transverse vibration of the system when not rotating. There would thus seem grounds for supposing that a formula of type (1) will often prove a close approximation to the truth. When this is the case, we can arrive at a close approximation to the velocity answering to whirling without endangering the

shaft by actually pushing the velocity to this point. All that is necessary is to determine the frequency of the lowest natural transverse vibration in the shaft when not rotating, and when rotating with any convenient velocity  $\omega_1$ .

If these frequencies be  $K/2\pi$  and  $k_1/2\pi$  respectively, then it is easily found from (1) that the frequency answering to any arbitrary value of  $\omega$  is given by

$$k^2 = K^2 - (K^2 - k_1^2)\omega^2/\omega_1^2. \quad \dots \quad (2)$$

Thus the critical angular velocity,  $\Omega$  say, being that for which  $k$  vanishes, is given by

$$\Omega^2 = \omega_1^2 K^2 / (K^2 - k_1^2) \quad \dots \quad (3)$$

As a check on the applicability of (2), it would in general be advisable to determine the frequency  $k_2/2\pi$  answering to a second angular velocity  $\omega_2$ . If (2) is strictly true, we should obviously have

$$(K^2 - k_2^2)/\omega_2^2 = (K^2 - k_1^2)/\omega_1^2, \quad \dots \quad (4)$$

the quantity on either side of the equation being a value for  $K^2/\Omega^2$ .

In all the cases solved, the quantity  $\nu$  in (1) has approached unity as a limiting value when the moment of inertia of the load has been indefinitely diminished; *i. e.* the angular velocity answering to whirling has approached the limiting value  $2\pi n$ , where  $n$  is the number of transverse vibrations of the fundamental type executed by the system when not rotating in unit of time.

The feasibility of determining frequencies of vibration in actual shaft systems, or models, is a question which I must leave to those experienced in Acoustics and practical Engineering.

§ 37. When a shaft is carried on more than two supports, it is not easy to lay down a suitable basis for the comparison of the critical velocities answering to different terminal conditions. A comparison is, however, easily instituted in Cases 1, 2, 4, and 6, when the shaft is unloaded. In all four cases, suppose the total length  $l$ , the mass  $m$  ( $\equiv \sigma \rho l$ ), the stiffness  $EI$ , and let  $\omega$  be the critical angular velocity,  $N$  ( $= 30\omega/\pi$ ) the corresponding number of revolutions per minute. Then we have the results given in Table VIII.

TABLE VIII.

Case.	State of ends.	Value of $\omega^2 \div (EI/ml^3)$ .	Value of $N \div (EI/ml^3)^{\frac{1}{2}}$ .
1.	One fixed in direction, other free. ....	12.36	33.58
2.	Both supported .....	97.41	94.25
4.	One fixed in direction, other supported...	237.7	147.2
6.	Both fixed in direction .....	500.6	213.7

The smaller  $\omega$ , or  $N$ , the more easily is the shaft caused to whirl. Table VIII. thus serves to bring out the great reduction in the tendency to whirl caused by fixing the direction at the ends of the shaft.

§ 38. A comparison is also readily made for a loaded but massless shaft under the conditions of cases 1, 2, 4, and 6, when the moment  $I'$  of the load is negligible. This is done in Table IX. The load  $M$  is supposed at the end in case 1; in the other cases its distances from the ends A and B are  $a$  and  $b$ . The so-called "minimum values" of  $\omega^2$  and  $N$  answer to that position of  $M$  which gives the smallest critical velocity. The rest of the notation has the same significance as in Table VIII.

TABLE IX.

Case.	End A.	End B.	General value of $\omega \div (EI/Ml^3)$ .	Value of $b/a$ giving minimum value of $\omega$ .	Minimum Values of	
					$\omega^2 \div (EI/Ml^3)$ .	$N \div (EI/Ml^3)^{\frac{1}{2}}$ .
1. ...	Direction fixed.	Free.	3	.....	3	16.54
2. ...	Supported.	Supported.	$3l^4/a^2b^2$	1	48	66.16
4. ...	Direction fixed.	Supported.	$12 \frac{l^6}{a^3b^2(3a+4b)}$	$(1/2)^{\frac{1}{2}}$	101.9	96.4
6. ...	Direction fixed.	Direction fixed.	$3l^6/a^3b^3$	1	192.0	132.3

In cases 2, 4, and 6, the position of the load which supplies the minimum value is precisely that for which Dunkerley's equations and mine agree in making the effect of  $I'$  vanish. Again, in every case where I have treated  $I'$  as small, the

corrective term in  $I'$  has increased  $\omega$ . There are thus reasons for regarding the above minimum values of  $N$  as pretty safe measures of the lowest speed at which whirling is likely to arise when a load is attached anywhere to a shaft of much smaller mass.

If we compare the results for  $\omega^2$  or  $N$  in Table VIII. with the corresponding minimum values in Table IX., we obtain the following as the ratios of  $m$  to  $M$  for which the critical velocities in the two cases are equal :—

Case ....	1.	2.	4.	6.
$m/M$ .....	4.12	2.03	2.33	2.61

It follows that when a load is near its most effective position, it must be taken into account as well as the shaft if we aim at a close approximation to the critical velocity.

The general values of  $\omega^2$ , however, in Table IX. show how very rapidly the tendency in  $M$  to produce whirling falls off as its position approaches a support, especially when the support is such as to fix the direction of the shaft.

§ 39. In considering liability to whirl, we must not lose sight of the possibility of the elastic strains and stresses exceeding the limit of safety before whirling is reached. As an example, let us suppose the shafts circular (solid, or hollow), of external radius  $a$  and perimeter  $p$  ( $p \equiv 2\pi a$ ), of a material for which

$E = 20 \times 10^8$  grammes weight per sq. cm. (1270 tons per  
 $\eta = 1/4$ , sq. inch),  
 $\rho = 7.8$  times the density of water.

Suppose the shafts unloaded, and denote the maximum stress-difference answering to the critical angular velocity of whirling by  $\bar{S}$ , then the following results may be proved :—

TABLE X.

	Solid Shaft, both ends		Very thin-walled hollow Shaft, both ends	
	Supported.	Fixed.	Supported.	Fixed.
$\bar{S}$ (in tons per sq. inch) ...	$6.61(p/l)^4$	$34.0(p/l)^4$	$39.7(p/l)^4$	$204.0(p/l)^4$
$\bar{S} = 2$ tons ..... { $l/p =$	1.35	2.03	2.11	3.18
{ $l/a =$	8.5	12.8	13.3	20.0

The value of  $\bar{S}$  for the very thin-walled shaft is really 6 times that for the solid shaft, for the assumed value  $1/4$  of Poisson's ratio. Since the maximum stress-difference answering to the critical velocity varies as the fourth power of the ratio borne by the perimeter (or radius) of the shaft to its length, it increases with great rapidity as the length is reduced, the section remaining unaltered. The physical properties assumed answer fairly to steel; but the results, it may be remarked, depend only on the elasticity, not on the density of the material. Under statical conditions, a stress-difference of 2 tons on the square inch is but a trifle compared to what good steel will stand; but in a rotating shaft, where there are ordinarily rapid alternations of stresses from various sources, it is probably at least as large a contribution from "centrifugal forces" as a cautious engineer will care to see.

When the ratio of the length to the circumference, or radius, is less than the values recorded in the two last lines of Table X., the stress-difference will exceed 2 tons on the square inch before the shaft whirls.

#### MATHEMATICAL APPENDIX.

§ 40. The kinetic energy of a body rotating about an axis through its C.G. is given by

$$T_r = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2),$$

where  $I_1, I_2, I_3$  are the principal moments of inertia, and  $\omega_1, \omega_2, \omega_3$  the component angular velocities about the three principal axes. Supposing the body one of revolution, and that it rotates with angular velocity  $\omega$  about a fixed direction with which its axis of figure makes a small angle  $\theta$ , then

$$\text{and} \quad \omega_1 = \omega \cos \theta, \quad \omega_2 = \omega \sin \theta, \quad \omega_3 = \dot{\theta},$$

$$T_r = \frac{1}{2}\{\omega^2 I_1 - (I_1 - I_2)\omega^2 \sin^2 \theta + I_2 \dot{\theta}^2\}. \quad . \quad . \quad . \quad (1)$$

If the body be a flat disk  $I_1 = 2I_2$ ; and if  $\theta$  be very small  $\sin \theta$  may be replaced by  $\theta$ , thus leading to

$$T_r = \frac{1}{2}\omega^2 I_1 + \frac{1}{2}I_2(\dot{\theta}^2 - \omega^2 \theta^2). \quad . \quad . \quad . \quad (2)$$

This result is applicable to a plate-shaped pulley carried

on a rotating shaft, at a place where the tangent to the axis of the shaft is inclined at an angle  $\theta$  to its undisturbed position. It is of course additional to the energy of the mass supposed collected at its centre of gravity. Unless the thickness of the pulley is small, variations of  $\theta$  throughout it may not be negligible; and unless it closely resembles a cylinder of revolution, it may be necessary to allow for  $I_1$  not being double  $I_2$ .

In the text, and subsequently in the Appendix,  $I'$  is used for  $I_2$ .

§ 41. For illustrative purposes I shall take the case of a massless shaft of length  $l$ , supported at its two ends A and B ( $AB=l$ ), carrying a load of mass  $M$ , and inertia  $I'$ , at an intermediate point C ( $AC=a$ ,  $BC=b$ ).

For AC we measure  $x$  from A, and for BC we measure  $x'$  from B. At any time  $t$  suppose C to be at a distance  $z$  from AB, in the  $xy$  plane, and let the tangent at C to the axis of the shaft make an angle  $\theta$  with AB. Then we must have

$$\left. \begin{aligned} \text{at } x=0, \quad y &= d^2y/dx^2=0, \\ x=a, \quad y &=z, \quad dy/dx=\theta, \\ x'=0, \quad y' &= d^2y'/dx'^2=0, \\ x'=b, \quad y' &=z, \quad dy'/dx'=-\theta. \end{aligned} \right\} \quad \cdot \quad \cdot \quad (3)$$

These conditions are easily seen to be satisfied by

$$\left. \begin{aligned} y &= (\beta z - a\theta)(x/2a) + (a\theta - z)(x^3/2a^3), \\ y' &= (\beta z + b\theta)(x'/2b) - (b\theta + z)(x'^3/2b^3), \end{aligned} \right\} \quad \cdot \quad \cdot \quad (4)$$

when we treat  $z$  and  $\theta$  as constants. Under the same assumption these expressions satisfy

$$d^4y/dx^4=0, \quad d^4y'/dx'^4=0,$$

equations answering to the absence of external forces on the shaft itself.

The components of velocity at the C.G. C of the load are  $\dot{z}$  perpendicular to AB in the plane of bending, and  $\omega z$  perpendicular to the plane of bending. Thus, treating  $\theta$  as small, we have for the kinetic energy of the system

$$T = \frac{1}{2}M(\dot{z}^2 + \omega^2 z^2) + \frac{1}{2}I_1\omega^2 + \frac{1}{2}I'(\dot{\theta}^2 - \omega^2\theta^2). \quad \cdot \quad \cdot \quad (5)$$

The shaft being supposed massless, contributes nothing to  $T$ . It is, however, the seat of the potential energy,  $V$ , which is given on the ordinary Euler-Bernoulli theory by

$$V = \frac{1}{2}EI \left\{ \int_0^a (d^2y/dx^2)^2 dx + \int_0^b (d^2y'/dx'^2)^2 dx' \right\} \dots (6)$$

Substituting the values of  $d^2y/dx^2$  and  $d^2y'/dx'^2$  from (4), and carrying out the integrations, we easily find

$$V = \frac{1}{2}EI \{ 3(a\theta - z)^2/a^3 + 3(b\theta + z)^2/b^3 \} \dots (7)$$

Employing these values of  $T$  and  $V$  in the two Lagrangian equations

$$\frac{d}{dt} \left( \frac{dT}{dz} \right) - \frac{dT}{dz} + \frac{dV}{dz} = 0, \dots (8)$$

$$\frac{d}{dt} \left( \frac{dT}{d\theta} \right) - \frac{dT}{d\theta} + \frac{dV}{d\theta} = 0, \dots (9)$$

we have

$$M(\ddot{z} - \omega^2 z) + 3EIz(a^{-3} + b^{-3}) + 3EI\theta(b^{-2} - a^{-2}) = 0, \dots (10)$$

$$I'(\ddot{\theta} + \omega^2 \theta) + 3EIz(b^{-2} - a^{-2}) + 3EI\theta(b^{-1} + a^{-1}) = 0. \dots (11)$$

For a vibration of frequency  $k/2\pi$  we have

$$\ddot{z} = -k^2 z, \quad \ddot{\theta} = -k^2 \theta.$$

Substituting for  $\ddot{z}$  and  $\ddot{\theta}$  in (10) and (11), and eliminating  $z$  and  $\theta$  between these two equations, we have, as in (8) of § 13,

$$\begin{aligned} \{ M(k^2 + \omega^2) - 3EI(a^{-3} + b^{-3}) \} \{ I'(k^2 - \omega^2) - 3EI(a^{-1} + b^{-1}) \} \\ = 9(EI)^2(b^{-2} - a^{-2})^2 \dots (12) \end{aligned}$$

The term  $\frac{1}{2}I_1\omega^2$  contributes nothing to Lagrange's equations and is for this reason omitted in the text.

Assigning any arbitrary value to  $\omega$ , we obtain two values of  $k^2$ —one of which may be imaginary—answering to two different types of vibration.

§ 42. In general each of the equations (10) and (11) contains both  $z$  and  $\theta$ , and each root of  $k^2$  given by (12) depends on both  $M$  and  $I'$ . There is obviously, however, a complete separation of the transverse and oscillatory movements when the load is at the centre of the span. For putting  $b=a$ , we have  $z$  only in (10) and  $\theta$  only in (11); while the

right-hand side of (12) vanishes, and we have for the transverse vibration

$$k_1^2 = 48(EI/Pl^3) - \omega^2, \quad . \quad . \quad . \quad (13)$$

for the oscillatory vibration

$$k_2^2 = 12(EI/l^3) + \omega^2. \quad . \quad . \quad . \quad (14)$$

Answering to  $\omega = 0$  we have

$$\begin{aligned} k_2^2/k_1^2 &= Pl^2/4I', \\ &= l^2/r^2, \end{aligned}$$

if the load be a thin circular disk of radius  $r$ .

Thus even when  $\omega = 0$ ,  $k_2$  will exceed  $k_1$  unless the radius of the disk be equal to the span.

As  $\omega$  increases,  $k_2$  increases while  $k_1$  diminishes; thus under ordinary circumstances the frequency of the transverse vibration is much the less of the two.

In the above special case it is obvious that  $k_2$  cannot vanish, and that it is only the transverse vibration in connexion with which instability can arise. The critical angular velocity, answering to  $k_1$  becoming nil, is given by

$$\omega^2 = 48(EI/Pl^3). \quad . \quad . \quad . \quad (15)$$

Even in the general case it is easily shown that one only of the two values of  $k^2$  supplied by (12) can possibly vanish. For assuming  $k$  zero, we find the equation to reduce to

$$\begin{aligned} \omega^4 Pl' + \omega^2 \cdot 3EI\{M(a^{-1} + b^{-1}) - I'(a^{-3} + b^{-3})\} \\ - 9(EI)^2(a+b)^2/a^3b^3 = 0, \end{aligned} \quad (16)$$

a quadratic in  $\omega^2$  whose roots are of opposite sign. As a negative value of  $\omega^2$  supplies an imaginary value of  $\omega$ , there is only one real value of  $\omega$  for which  $k$  can vanish. And as (12), regarded as an equation in  $k^2$ , cannot have equal roots, unless  $b=a$ , only one of the two values of  $k^2$  can be made to vanish.

§ 43. When there is no load, and an algebraic type of vibration is assumed, the application of the Lagrangian equations is even simpler. Taking, for example, the case of a shaft supported at both ends, we have for the displacement (*cf.* (b) case 2)

$$y = \eta x(l^3 - 2lx^2 + x^3), \quad . \quad . \quad . \quad (17)$$



whence

$$T = \frac{1}{2} \sigma \rho \int_0^l (\dot{y}^2 + \omega^2 y^2) dx = \frac{1}{2} \sigma \rho (\dot{\eta}^2 + \omega^2 \eta^2) \int_0^l x^2 (l^3 - 2lx^2 + x^3)^2 dx \\ = \frac{1}{2} (\dot{\eta}^2 + \omega^2 \eta^2) (31/630) \sigma \rho l^9, \quad \dots \quad (18)$$

$$V = \frac{1}{2} EI \int_0^l (d^2 y / dx^2)^2 dx = \frac{1}{2} EI \eta^2 \int_0^l 144 x^2 (x-l)^2 dx \\ = \frac{1}{2} \eta^2 (24/5) EI l^5. \quad \dots \quad (19)$$

Lagrange's equation

$$\frac{d}{dt} \frac{dT}{d\dot{\eta}} - \frac{dT}{d\eta} + \frac{dV}{d\eta} = 0$$

gives

$$(\ddot{\eta} - \omega^2 \eta) (31/630) \sigma \rho l^9 + (24/5) \eta EI l^5 = 0.$$

Assuming  $\eta \propto \cos kt$ , and so  $\ddot{\eta}/\eta = -k^2$ , we have (*cf.* (6), § 12)

$$k^2 + \omega^2 = \frac{24 \times 126}{31} \frac{EI}{\sigma \rho l^4} = \frac{3024}{31} (EI / \sigma \rho l^4). \quad \dots \quad (20)$$

§ 44. Under certain circumstances an equation of type (1), § 1, may be shown to be true for vibration frequencies. The ordinary differential equation for a frictionless simple harmonic motion is

$$M d^2 x / dt^2 + F x = 0, \quad \dots \quad (21)$$

where  $M$  is a quantity of the nature of a mass, and  $F$  a force of restitution, such as is exerted by a spring. The frequency  $k/2\pi$  of the corresponding vibration is given by

$$1/k^2 = M/F. \quad \dots \quad (22)$$

Suppose, now, that the force of restitution remains the same whether we apply one or a series of loads,  $M_1, M_2, \&c.$  When the loads are put on one at a time, the corresponding frequency equations are

$$1/k_1^2 = M_1/F, \quad 1/k_2^2 = M_2/F; \quad \dots \quad (23)$$

when put on all together we have for the frequency equation

$$1/k^2 = (M_1 + M_2 + \dots) / F \\ = 1/k_1^2 + 1/k_2^2 + \dots \quad \dots \quad (24)$$

This is analogous of course to Dunkerley's hypothesis, but it is far from amounting to a proof. Even if we assumed that

what is true of transverse vibration frequencies is true of whirling velocities, we should have to prove that the addition of pulleys at different parts of a shaft is equivalent to varying the load without affecting the forces of restitution.

§ 45. The following investigation would seem to show that the result is not in general strictly true, though it may be, and not improbably often is, a close approximation to the truth.

Suppose that a massless shaft of length  $l$ , supported at its ends A and B, carries a mass  $M_1$  at C ( $AC=a$ ), and a second mass  $M_2$  at D ( $CD=c$ ,  $BD=b$ ), the effect of the moment of inertia being negligible in either case.

Measuring  $x$  from A, and  $x'$  from B, we may assume the following types of displacement—derived by considering the bending of the shaft under the weight of the two loads:—

$$\left. \begin{aligned} \text{from A to C, } y &= \eta [M_1(b+c)x\{l^2 - (b+c)^2 - x^2\} \\ &\quad + M_2bx(l^2 - b^2 - x^2)], \\ \text{,, C to D, } y &= \eta [M_1ax'(l^2 - a^2 - x'^2) \\ &\quad + M_2bx(l^2 - b^2 - x^2)], \\ \text{,, D to B, } y &= \eta [M_1ax'(l^2 - a^2 - x'^2) \\ &\quad + M_2(a+c)x'\{l^2 - (a+c)^2 - x'^2\}]. \end{aligned} \right\} \quad (25)$$

Taking  $\omega$  as usual for the angular velocity, and applying Lagrange's equations, we find after algebraic manipulation  $1/(k^2 + \omega^2) = M_1\{a^2(b+c)^2/3EI\} + M_2\{b^2(a+c)^2/3EI\} - R$ , (26) where

$$\begin{aligned} R = & M_1M_2(M_1 + M_2)a^2b^2c^2(4ab + 4ac + 4bc + 3c^2) \\ & \div 12EI\{M_1^2a^2(b+c)^2 + M_2^2b^2(a+c)^2 \\ & + M_1M_2ab(2ab + 2ac + 2bc + c^2)\} \quad (27) \end{aligned}$$

For the critical angular velocity answering to whirling we put  $k=0$ , and find

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2 - R, \quad (28)$$

where

$$\omega_1^2 = 3EI/a^2(b+c)^2, \quad \omega_2^2 = 3EI/b^2(a+c)^2.$$

Referring to (9) or (10) §13, we see that  $\omega_1$  and  $\omega_2$  are the critical velocities for the shaft when loaded with the mass  $M_1$  and when loaded with the mass  $M_2$ .

In order that (28) should agree with Dunkerley's hypothesis  $R$  should vanish. It is, however, obvious that  $R$  is positive for all values of  $M_2/M_1$ , and for all values of  $a$ ,  $b$ , and  $c$ . It vanishes, it is true, when  $c$  vanishes, but the two loads then coincide in position. As  $R$  is positive, the application of Dunkerley's hypothesis gives a larger value for  $1/\omega^2$ , and so a smaller value for  $\omega^2$ , than does (28).

This does not entirely disprove Dunkerley's hypothesis, because we are not entitled to assume that (25) accords absolutely with the true type of displacement, and we know from Rayleigh's general theorem that, unless this is the case, the value given by (26) for  $k^2$  must be somewhat in excess, and consequently the value (28) for  $1/\omega^2$  somewhat too low. We may however expect, in accordance with Rayleigh's general reasoning, that (26) is a very close approach to the truth; and whilst  $R$  is usually much smaller than  $1/\omega_1^2 + 1/\omega_2^2$ , it is by no means negligible, unless one of the loads be much less than the other, or one of the three lengths,  $a$ ,  $b$ , and  $c$  be small.

Considering, however, the various sources of uncertainty, it must be allowed that in the present instance Dunkerley's hypothesis gives at least a fair first approximation. Taking, for example, the fairly representative case presented when the two loads are equal, and  $a$ ,  $b$ ,  $c$  all equal, we find

$$1/\omega^2 = (15/16)(1/\omega_1^2 + 1/\omega_2^2).$$

§ 46. In carrying out investigations in cases where there are two, three, or more loads, the physical significance of the processes is more easily seen by adopting a generalized notation. In the above case, for instance, it will be found that the displacements at the points where the loads occur are really of the types

$$\eta'(M_1 y_{11} + M_2 y_{12}) \text{ and } \eta'(M_1 y_{12} + M_2 y_{22}),$$

where  $y_{11}$  and  $y_{12}$  are the displacements at the point where  $M_1$  occurs, due respectively to unit loads at this point and at the point where  $M_2$  occurs. (By a well-known general theorem  $y_{12}$  and  $y_{21}$  are equal.) The kinetic and the potential energies

vary respectively as

$$M_1(M_1y_{11} + M_2y_{12})^2 + M_2(M_1y_{12} + M_2y_{22})^2 \text{ and} \\ (M_1^2y_{11} + 2M_1M_2y_{12} + M_2^2y_{22}),$$

and the function which appears in the expression for  $1/\omega^2$  really varies as

$$M_1y_{11} + M_2y_{22} - \\ (y_{11}y_{22} - y_{12}^2)M_1M_2(M_1 + M_2) \div \{M_1^2y_{11} + 2M_1M_2y_{12} + M_2^2y_{22}\}.$$

The sign of  $R$  in (26) and (28) really turns on the sign of  $(y_{11}y_{22} - y_{12}^2)$ .

#### DISCUSSION.

Dr. GLAZEBROOK said there was one point he would like to bring out in connexion with Dr. Chree's paper. The author had referred to a theorem due to Lord Rayleigh: a theorem which he knew well in its application to sound, although he had never seen it employed in connexion with rotating shafts. The theorem is that in any vibrating system the period of natural vibrations is a stationary period, so that the application of any small external force produces only a second-order change in the period. By use of this theorem it is possible to assume some definite law and calculate periods with sufficient accuracy by working with expressions much less complicated than would otherwise be the case.

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