The experiments here described are merely a preliminary to large scale experiments in actual closed circuit telegraphy the writer hopes to be able to try later on.

One drawback to the magnetic induction form of telegraphy is the rapid rate at which the effect falls off with the distance. In the case of true radiation at long distances the forces vary inversely as the distance, but a more rapid rate of decay, something between the inverse cube and inverse square, holds good for the inductive effect at least at short distances. Hence the use of magnetic oscillators as transmitters is never likely for this reason alone to rival the electric or open oscillator, but there may be circumstances under which it is possible to use them with advantage. In conclusion the author desires to mention that the actual measurements recorded in this paper were taken by his assistant, Mr. G. B. Dyke, B.Sc., with the kind help of Mr. K. W. McMillan, and to these gentlemen is due an acknowledgement of their share in the work, in making these observations with much intelligence and care.

LXIX. *The Asymptotic Expansion of Bessel Functions of High Order.* By J. W. Nicholson, *D.Sc., B.A., Isaac Newton Student in the University of Cambridge* *.

In certain investigations in the theory of diffraction by large obstacles the author recently found it necessary to obtain some approximate formulæ for the Bessel functions whose order is half an odd integer. The results can be applied to a large number of physical problems, and in fact supply the key to the solution of the majority of problems connected with the bending of waves round large spheres, with which little progress has hitherto been made. The Bessel functions are of several types, determined by the relation between their order and argument. The attention of investigators has been mainly confined to the types in which the order is small in comparison with the argument, which may be of any magnitude. In this paper, expressions will be obtained for functions of large argument, and of order comparable with, but less than that argument. This special problem has received little attention, and a memoir by Lorenz† appears to furnish the only contribution yet made to the

* Communicated by the Author.

† *Œuvres Scientifiques,* i. p. 405 *et seq.*

subject. The results obtained by Lorenz may be summarized as follows:—

Writing
$$J_{n+\frac{1}{2}}(z)=\left(\frac{2R}{\pi z}\right)^{\frac{1}{2}}\sin\phi \quad . \quad . \quad . \quad (1)$$
$$J_{-n-\frac{1}{2}}(z)=(-)^n\left(\frac{2R}{\pi z}\right)^{\frac{1}{2}}\cos\phi, \quad . \quad . \quad . \quad (2)$$
in all cases in which $n+\frac{1}{2}$ is less than z, which is large; then, if $z-n-\frac{1}{2}$ is of higher order than $z^{\frac{1}{3}}$,
$$R=\left(\frac{z}{z^2-\overline{n+\frac{1}{2}}^2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$\phi=(z^2-\overline{n+\tfrac{1}{2}}^2)^{\frac{1}{2}}-\frac{n\pi}{2}+(n+\tfrac{1}{2})\sin^{-1}\frac{n+\frac{1}{2}}{z}. \quad . \quad . \quad (4)$$

The formulæ deduced by Lorenz for higher values of n, more closely equal to, or greater than z, are not relevant to the present purpose.

When z and n are only moderately large, these forms cease to be good approximations, and in this paper it is proposed to generalize them, and to carry the calculation to higher orders.

Making, with Lorenz, the substitutions (1) and (2) in the relation
$$J'_+J_- - J'_-J_+ = (-)^n\cdot\frac{2}{\pi z},$$
it appears, after some reduction, that
$$\frac{d\phi}{dz}=\frac{1}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But for extremely great values of z, in comparison with n, Lommel's* ordinary formula yields
$$\phi=z-\frac{n\pi}{2}.$$

Thus
$$\phi=z-\frac{n\pi}{2}-\int_z^\infty\left(\frac{1}{R}-1\right)dz \quad . \quad . \quad . \quad . \quad (6)$$
in the more general case when n is of order z.

The differential equation for the functions J must therefore be satisfied by the form
$$f=\left(\frac{R}{z}\right)^{\frac{1}{2}}\exp.\,\iota\int^z\frac{dz}{R}\cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

* *E. g.*, *vide* Whittaker, Modern Analysis, 1902, p. 294.

Substituting this expression in the equation

$$\frac{d^2f}{dz^2}+\frac{2}{z}\frac{df}{dz}+\left(1-\frac{n\,.\,n+1}{z^2}\right)f=0,$$

it appears that

$$RR''-\tfrac{1}{2}R'^2+\left(1-\frac{n\,.\,n+1}{z^2}\right).2R^2-2=0,$$

where the accent indicates differentiation with respect to z. If this be again differentiated it becomes linear, and yields

$$R'''+4\left(1-\frac{n\,.\,n+1}{z^2}\right)R'+4\frac{n\,.\,n+1}{z^3}R=0, \quad . \quad (8)$$

which may be integrated in series.

By reference again to Lommel's formula*, it appears that when z is very great in comparison with n, R takes the value unity.

The series solution of (8) satisfying this condition is

$$R=1+\frac{n\,.\,n+1}{z^2}.\frac{1}{2}+\frac{n-1\,.\,n\,.\,n+1\,.\,n+2}{z^4}.\frac{1.3}{2.4}+\dots \quad (9)$$

When n is of order z, and $z-n$ is not small, this leads at once to

$$R=\frac{z}{\sqrt{z^2-\overline{n+\tfrac{1}{2}}^2}},$$

as proved by Lorenz. The value of ϕ in (4) follows by (6). We proceed to obtain a definite integral for the function R.

Writing $m=2n+1$, so that m is an odd integer,

$$R=1+\frac{m^2-1^2}{4z^2}.\frac{1}{2}+\frac{m^2-1^2\,.\,m^2-3^2}{(4z^2)^2}.\frac{1.3}{2.4}+\dots$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}S(\theta)\,d\theta,$$

where

$$S(\theta)\equiv 1+\frac{m^2-1^2}{1}.\frac{\cos^2\theta}{4z^2}+\frac{m^2-1^2\,.\,m^2-3^2}{1}.\left(\frac{\cos^2\theta}{4z^2}\right)^2+\dots \quad (10)$$

But by a well-known result†, since m is odd,

$$\frac{\sinh mt}{m}=\sinh t+\frac{m^2-1^2}{3\,!}\sinh^3 t+\frac{m^2-1^2\,.\,m^2-3^2}{5\,!}\sinh^5 t+\dots$$

* *Loc. cit.* † *Vide* Chrystal's Algebra, Part II. p. 180.

Thus, if
$$\sinh t = \frac{u\cos\theta}{2z},$$
then
$$\frac{2z\sinh mt}{m\cos\theta} = u + \frac{m^2-1^2}{3!}u^3\left(\frac{\cos\theta}{2z}\right)^2 + \ldots\ldots$$

But when s is an integer,
$$\int_0^\infty e^{-u}u^{s+1}\,du = s\,!$$

Hence
$$S(\theta) = \frac{2z}{m\cos\theta}\int_0^\infty e^{-u}\sinh mt\,du\,;$$
or, since $u\cos\theta = 2z\sinh t$,
$$S(\theta) = \frac{4z^2}{m\cos\theta}\int_0^\infty e^{-2z\sinh t/\cos\theta}\sinh mt\cosh t\,dt,$$
and
$$R = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} S(\theta)d\theta.$$

The usual rules for the change in the order of integration are satisfied by the presence of the exponential factor, and
$$R = \frac{8z^2}{m\pi}\int_0^\infty \sinh mt\cosh t\,dt\int_0^{\frac{\pi}{2}}\sec^2\theta\,.\,d\theta\,.\,e^{-\lambda\sec\theta}, \qquad (11)$$
where $\lambda = 2z\sinh t$.

Let $K_0(\lambda)$ be the second solution of Bessel's equation of order zero, with independent variable λ, defined by
$$\pi K_0(\lambda) = \int_0^\infty e^{-\lambda\cosh u}\,du. \quad . \quad . \quad . \quad . \qquad (12)$$

Then writing $\cos\theta = \operatorname{sech} u$,
$$\int_0^{\frac{\pi}{2}}\sec^2\theta\,.\,d\theta\,e^{-\lambda\sec\theta} = \int_0^\infty \cosh u\,.\,du\,.\,e^{-\lambda\cosh u}$$
$$= -\pi K_0'(\lambda).$$

Thus
$$R = -\frac{8z^2}{m}\int_0^\infty \sinh mt\cosh t\,K_0'(2z\sinh t)\,dt$$
$$= -\frac{4z}{m}\int_0^\infty \sinh mt\frac{d}{dt}\{K_0(2z\sinh t)\}dt,$$
or
$$R = 4z\int_0^\infty K_0(2z\sinh t)\cosh mt\,dt, \quad . \quad . \qquad (13)$$
after integration by parts, m still denoting $2n+1$.

As a double integral of reversible type,

$$\mathrm{R}=\frac{4z}{\pi}\int_0^\infty\int_0^\infty e^{-2z\sinh t\cosh\psi}\cosh mt\,dt\,d\psi. \quad . \quad (14)$$

First approximation to R.

The most significant portion of the integral, with the assumed magnitudes of m and z, occurs near $t=0$. Thus writing $\sinh t=t$, and integrating over a small range which is itself capable of being regarded as infinite owing to the large argument of the exponentials,

$$\mathrm{R}=\frac{4z}{\pi}\int_0^\infty d\psi\int_0^\infty e^{-2zt\cosh\psi}\cosh mt\,dt,$$

to the first order of approximation.

This leads to

$$\begin{aligned}\mathrm{R}&=\frac{2z}{\pi}\int_0^\infty d\psi\left\{\frac{1}{2z\cosh\psi-m}+\frac{1}{2z\cosh\psi+m}\right\}\\&=\frac{8z^2}{\pi}\int_0^\infty\frac{d(\sinh\psi)}{4z^2-m^2+4z^2\sinh^2\psi}\\&=\frac{2z}{(4z^2-m^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)\end{aligned}$$

in accordance with (3).

Approximation of any order to R.

We proceed to obtain an expansion of the integral

$$\mathrm{I}=\int_0^\infty e^{-\lambda v}\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where λ is large, and v' or $\frac{dv}{dt}$ is never zero in the range of integration, v also being everywhere positive. By an integration by parts,

$$\mathrm{I}=\left[-\frac{1}{\lambda v'}e^{-\lambda v}\right]_0^\infty+\frac{1}{\lambda}\int_0^\infty\frac{d}{dt}\left(\frac{1}{v'}\right)e^{-\lambda v}\,dt.$$

Under the conditions supposed, the second term is of lower order in λ than the first. Continuing the process, an asymptotic expansion is obtained, each term being of a higher order in $\frac{1}{\lambda}$ than that immediately preceding. Although not in general convergent, this series may be used for calculating

R to a high order, as in the usual theory of asymptotic expansions. The series is evanescent at the upper limit on account of the exponential, and thus, if $v_0=0$,

$$I \doteq \left(\frac{1}{\lambda v'} + \frac{1}{\lambda^2 v'}\frac{d}{dt}\cdot\frac{1}{v'} + \dots\right)_{v=0}.$$

If E denote the operation $\frac{d}{dv} \equiv \frac{1}{v'}\frac{d}{dt}$,

$$I = \frac{1}{\lambda}\left(1+\frac{E_0}{\lambda}+\frac{E_0^2}{\lambda^2}+\dots\right)\frac{1}{v_0'}, \quad . \quad . \quad . \quad (17)$$

which may be symbolically written

$$I = (\lambda - E_0)^{-1}\cdot(v_0')^{-1}, \quad . \quad . \quad . \quad . \quad (18)$$

where the sign of equality denotes asymptotic equivalence.

Now
$$R = \frac{2z}{\pi}\int_0^{i\infty}(I_1+I_2)\,d\psi, \quad . \quad . \quad . \quad . \quad (19)$$

where
$$(I_1, I_2) = \int_0^{\infty} e^{-2z\sinh t\cosh\psi \pm mt}\,dt.$$

If m is of the same order as z, and such that $z-m$ is not of low order, these integrals are of the form treated above. Writing $\lambda = 2z\cosh\psi$, $m=\lambda\mu$, $\mu<1$, and denoting I_1 or I_2 by I,

$$I = \int_0^{\infty} e^{-\lambda(\sinh t \mp \mu t)}\,dt. \quad . \quad . \quad . \quad . \quad (20)$$

Thus in the above notation,

$$v = \sinh t \mp \mu t, \qquad v_0 = 0, \qquad v_0' = 1 \mp \mu,$$
$$v_0^{(2n)} = 0, \qquad v_0^{(2n+1)} = 1.$$

By help of these results it is readily proved that, if $w=v_0'$,

$$\left.\begin{aligned} &E_0\left(\frac{1}{w}\right) = E_0^3\left(\frac{1}{w}\right) = \dots = 0 \\ &E_0^2\left(\frac{1}{w}\right) = -\frac{1}{w^4} \\ &E_0^4\left(\frac{1}{w}\right) = -\frac{1}{w^6}+\frac{10}{w^7} \\ &E_0^6\left(\frac{1}{w}\right) = -\frac{1}{w^8}+\frac{56}{w^9}-\frac{280}{w^{10}} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (21)$$

and so on, where $w=1\mp\mu$, $\lambda w = 2z\cosh\psi \mp m$.

Accordingly,

$$I=\frac{1}{\lambda w}-\frac{1}{\lambda^3 w^2}-\frac{1}{\lambda^5 w^6}+\frac{10}{\lambda^5 w^7}-\frac{1}{\lambda^7 w^8}+\frac{56}{\lambda^7 w^9}-\frac{280}{\lambda^7 w^{10}}+\ldots,$$

which may obviously be expressed in the form

$$I=\left\{1+\frac{z}{3!}\frac{\partial^3}{\partial z\partial m^2}+\frac{z}{5!}\frac{\partial^5}{\partial z\partial m^5}+\frac{10z^2}{6!}\frac{\partial^6}{\partial z^2\partial m^4}\right.$$
$$\left.+\frac{z}{7!}\frac{\partial^7}{\partial z\partial m^6}+\frac{56z^2}{8!}\frac{\partial^8}{\partial z^2\partial m^6}+\frac{280z^3}{9!}\frac{\partial^9}{\partial z^3\partial m^6}+\ldots\right\}\frac{1}{\lambda w}$$
$$=D\cdot\frac{1}{\lambda w}, \qquad . \quad . \quad . \quad (22)$$

where D denotes the operation in brackets.

Accordingly by (19),

$$R=\frac{2z}{\pi}\int_0^\infty (I_1+I_2)d\psi$$
$$=\frac{2z}{\pi}\int_0^\infty D\left\{\frac{1}{2z\cosh\psi-m}+\frac{1}{2z\cosh\psi+m}\right\}d\psi$$
$$=\frac{2z}{\pi}.D\int_0^\infty\left\{\frac{1}{2z\cosh\psi-m}+\frac{1}{2z\cosh\psi+m}\right\}d\psi,$$

since the integral obviously satisfies all necessary conditions for differentiation.

The value of the integral is $\frac{\pi}{(4z^2-m^2)^{\frac{1}{2}}}$ as in (15), and thus

$$R=2z\left\{1+\frac{z}{3!}D_1D_2^2+\frac{z}{5!}D_1D_2^4+\frac{10z^2}{6!}D_1^2D_2^4+\frac{z}{7!}D_1D_2^6\right.$$
$$\left.+\frac{56z^2}{8!}D_1^2D_2^6+\frac{280z^3}{9!}D_1^3D_2^6+\ldots\right\}\frac{1}{\sqrt{4z^2-m^2}}, \quad (23)$$

where $D_1\equiv\frac{\partial}{\partial z}$, $D_2\equiv\frac{\partial}{\partial m}$.

If $\frac{m}{2z}\equiv\frac{n+\frac{1}{2}}{z}=\sin\alpha$, the second approximation becomes

$$R=\sec\alpha-\frac{1}{8z^2}(\cos^2\alpha+5\sin^2\alpha)\sec^7\alpha. \quad . \quad (24)$$

Second form for the function R.

The formula just proved suggests the existence of an asymptotic series of the form

$$\frac{R}{2z}=\frac{\lambda_1}{x}+\frac{\lambda_3}{x^3}+\frac{\lambda_5}{x^5}+\ldots\ldots, \quad . \quad . \quad . \quad (25)$$

where $x=(4z^2-m^2)^{\frac{1}{2}}$ and $\lambda_1=1$.

Such a series may be obtained directly from the differential equation. Writing $R=2zy$ in (8), it is found that

$$zy'''+3y''+4zy'\left(1-\frac{m^2-1}{4z^2}\right)+4y=0, \quad . \quad . \quad (26)$$

where $4n\,.\,n+1$ has been written as $4m^2-1$. With a new independent variable $x=(4z^2-m^2)^{\frac{1}{2}}$, this may be reduced to

$$x^2(x^2+m^2)^2y'''+3x(x^4-m^4)y''+3m^4y'+x^4(1+x^2)y'+x^5y=0$$
$$. \quad . \quad . \quad . \quad (27)$$

Writing
$$y=\frac{\lambda_1}{x}+\frac{\lambda_2}{x^2}+\ldots\ldots$$

the relation between successive coefficients (s odd) becomes

$$(s+3)\lambda_{s+4}+(s+2)^3\lambda_{s+2}+2m^2s\,.\,s+1\,.\,s+2\,.\,\lambda_s$$
$$+m^4s\,.\,s-2\,.\,s+2\,.\,\lambda_{s-2}=0, \quad . \quad . \quad (28)$$

and corresponding to $\lambda_1=1$,

$$\lambda_3=-\tfrac{1}{2}, \quad \lambda_5=\tfrac{1}{8}(27-24m^2)$$
$$\lambda_7=\tfrac{1}{16}(1160m^2-1125-40m^4). \quad . \quad . \quad (29)$$

Any succeeding coefficient may be at once found by (28). Every third term in the resulting series is two orders (in m or x, and therefore in n or z) smaller than that immediately preceding. Thus the second is two orders smaller than the first, the fifth two smaller than the fourth, and so on.

Finally,

$$\frac{R}{2z}=\frac{1}{(4z^2-m^2)^{\frac{1}{2}}}-\frac{1}{2\,.\,(4z^2-m^2)^{\frac{3}{2}}}+\frac{27-24m^2}{8\,.\,(4z^2-m^2)^{\frac{5}{2}}}$$
$$+\frac{1160m^2-1125-40m^4}{16(4z^2-m^2)^{\frac{7}{2}}}+\ldots\ldots \quad . \quad . \quad (30)$$

The terms here written give the value of R correctly to four places of decimals when z is only 10, and $n=8$, a case in which n and z are nearly equal.

The function ϕ.

By (5) $$\frac{\partial\phi}{\partial z}=\frac{1}{\mathrm{R}}=\frac{x}{2z}\left(1+\frac{\lambda_3}{x^2}+\frac{\lambda_5}{x^4}+\ldots\right)^{-1},$$

where $x^2=4z^2-m^2$.

Thus $$\frac{\partial\phi}{\partial z}=\frac{x}{2z}\left(1+\frac{\mu_1}{x^2}+\frac{\mu_2}{x^4}+\ldots\right), \quad . \quad . \quad . \quad (31)$$

where $$\left.\begin{aligned}\mu_1&=-\lambda_3=\tfrac{1}{2},\\ \mu_2&=-\lambda_5+\lambda_3^2=\tfrac{1}{8}(24m^2-25),\\ \mu_3&=-\lambda_7+2\lambda_3\lambda_5-\lambda_3^3=\tfrac{1}{16}\,(40m^4-1112m^2-1073),\end{aligned}\right\}(32)$$

and so on.

Thus by (6)

$$\phi-z-\frac{n\pi}{2}=\int_z^\infty \frac{x}{2z}dz\left\{\frac{2z}{x}-1-\frac{\mu_1}{x^2}-\frac{\mu_2}{x^4}\ldots\right\}$$

$$=\int_0^\alpha \frac{m}{2}\cot^2\theta\left(\frac{1-\cos\theta}{\cos\theta}-\frac{\mu_1}{m^2}\tan^2\theta-\frac{\mu_2}{m^4}\tan^4\theta\ldots\right)d\theta,$$

where $$\sin\alpha=\frac{m}{2z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

or $$\phi+\frac{n\pi}{2}-\frac{m}{2}\sin^{-1}\frac{m}{2z}-z\left(1-\frac{m^2}{4z^2}\right)^{\frac{1}{2}}$$

$$=-\frac{1}{2m}\int_0\left(\mu_1+\frac{\mu_2}{m^2}\tan^2\theta+\frac{\mu_3}{m^4}\tan^4\theta+\ldots\right)d\theta$$

and finally

$$\phi=\frac{\pi}{4}+z\left(\cos\alpha-\overline{\frac{\pi}{2}-\alpha}\,.\sin\alpha\right)-\frac{\alpha}{2m}\left(\mu_1-\frac{\mu_2}{m^2}+\frac{\mu_3}{m^4}-\ldots\right)$$

$$-\frac{1}{2m}\left\{\frac{\mu_2}{m^2}\tan\alpha-\frac{\mu_3}{m^4}\left(\tan\alpha-\frac{1}{3}\tan^3\alpha\right)+\frac{\mu_4}{m^6}\left(\tan\alpha-\right.\right.$$

$$\left.\left.\frac{1}{3}\tan^3\alpha+\frac{1}{5}\tan^5\alpha\right)-\ldots\right\}. \quad . \quad . \quad (34)$$

Final Results.

Collecting the results, it appears that when n and z are both large, and $z-n$ is of order not too small in comparison with z, and if

$$\left.\begin{aligned} J_{n+\frac{1}{2}}(z) &= \sqrt{\frac{2R}{\pi z}}\sin\phi \\ J_{-n-\frac{1}{2}}(z) &= (-)^n\sqrt{\frac{2R}{\pi z}}\cos\phi \\ m = 2n+1 &= 2z\sin\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Then
$$R = \sec\alpha + \frac{\lambda_3}{(2z)^2}\cdot\sec^3\alpha + \frac{\lambda_5}{(2z)^4}\cdot\sec^5\alpha + \ldots, \qquad (36)$$

where

$$(s+3)\lambda_{s+4} + (s+2)^3\lambda_{s+2} + 2m^2 s\,.\,s+1\,.\,s+2\,.\,\lambda_s + m^4 s\,.\,s^2-4\,.\,\lambda_{s-2} = 0$$
$$\lambda_3 = -\tfrac{1}{2},\quad \lambda_5 = \tfrac{1}{8}(27-24m^2),\quad \lambda_7 = \tfrac{1}{16}(1160m^2-1125-40m^4) \quad . \quad . \quad . \quad (37)$$

and every third term, commenceing with the second, is two orders smaller than the one preceding. Moreover,

$$\phi = \frac{\pi}{4} + z\left(\cos\alpha - \overline{\frac{\pi}{2} - \alpha}\sin\alpha\right) - \frac{\alpha}{2m}\left(\mu_1 - \frac{\mu_2}{m^2} + \frac{\mu_3}{m^4}\ldots\right)$$
$$-\frac{1}{2m}\left\{\frac{\mu_2}{m^2}\tan\alpha - \frac{\mu_3}{m^4}\left(\tan\alpha - \frac{1}{3}\tan^3\alpha\right) + \frac{\mu_5}{m^6}\left(\tan\alpha - \right.\right.$$
$$\left.\left.\frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha\right) - \ldots\right\}, \quad . \quad . \quad (38)$$

where the coefficients μ and λ are connected by

$$\left(1 + \frac{\mu_1}{x} + \frac{\mu_2}{x^2} + \ldots\right) = \left(1 + \frac{\lambda_3}{x} + \frac{\lambda_5}{x^2} + \ldots\right)^{-1}, \quad . \quad (39)$$

and every third term of the brackets in ϕ, commencing with the second, is two orders below the one preceding.

In calculating results to a definite order, the coefficients may be simplified. For example, if R is required to order z^{-2} inclusive, it is sufficient to take

$$\lambda_3 = -\tfrac{1}{2},\quad \lambda_5 = -3m^2,\quad \lambda_7 = -\tfrac{5}{2}m^4,\quad \lambda_9 = 0.$$

For most physical problems connected with the intensity of shadow behind large spheres, the approximation to order z^{-2} is sufficient.

If
$$\psi = \frac{\pi}{4} + z\left(\cos\alpha - \overline{\frac{\pi}{2} - \alpha}\sin\alpha\right), \quad . \quad . \quad . \quad (40)$$

it is found that to this order, on reduction after substitution of the values of R and ϕ,

$$J_{n+\frac{1}{2}}(z)=\sqrt{\frac{2\sec\alpha}{\pi z}}\left(1-\frac{\sec^6\alpha}{16z^2}\cdot\overline{1+4\sin^2\alpha}\right)\left\{\left(1-\frac{c^2}{2z^2}\right)\sin\psi-\frac{c}{z}\cos\psi\right\}$$

$$J_{-n-\frac{1}{2}}(z)=(-)^n\sqrt{\frac{2\sec\alpha}{\pi z}}\left(1-\frac{\sec^6\alpha}{16z^2}\cdot\overline{1+4\sin^2\alpha}\right)\left\{\left(1-\frac{c^2}{2z^2}\right)\cos\psi+\frac{c}{z}\sin\psi\right\}, \quad (41)$$

where $c=\frac{1}{8}\sec\alpha\,(1+\frac{5}{3}\tan^2\alpha),\quad 2n+1=2z\sin\alpha$. . (42)

When n and z differ so little that α is approximately 90°, the character of the expansion changes, and an independent investigation is required. This will be given in a subsequent paper.

LXX. *On the Stability of the Steady State of Forced Oscillation. By* Andrew Stephenson *.

1. IN the case of a simple system, that is a system in which the restoring force is exactly proportional to the 'displacement,' the motion under a periodic force is made up of a definite oscillation isochronous with the generator and an independent free motion of amplitude and phase determined by the initial conditions. If the system is subject to kinetic friction, the free element is gradually damped out and the motion approaches asymptotically to the state of steady forced oscillation. Although in practice the spring of a system varies to a certain extent with the amplitude, the results obtained for the ideal simple system are in general agreement with actual phenomena, and are tacitly assumed as being practically of universal application. From observation, however, of the behaviour of a system such as the simple pendulum when resonant to a force of nearly its own period, it is natural to question the validity of this assumption in certain cases, even for moderately small amplitudes; and it is our object here to seek out the circumstances in which the simple rules of the approximate theory do not apply.

When account is taken of the variation in the restoring force, the equation of motion is to a first approximation

$$x''+2\kappa x'+\mu^2(1-cx^2)x=a\cos(qt+\epsilon). \quad \text{(i.)}$$

There does not appear to be any practicable method of

* Communicated by the Author.