ON SOME INEQUALITIES.

By **Tsuruichi Hayashi** (Sendai, Japan).

Adunanza dell'11 gennaio 1920.

The SCHWARZ inequaltity

$$\int_a^b f^2 dx . \int_a^b \varphi^2 dx \geqq \left(\int_a^b f\varphi\, dx\right)^2,$$

and the TCHEBYCHEFF inequaltity

$$(b-a)\int_a^b f\varphi\, dx \geqq \int_a^b f\, dx . \int_b^b \varphi\, dx$$

are useful. Professor M. FUJIWARA has given a more general integral inequality of the form

$$\int_a^b f_1 \varphi_1 dx \int_a^b f_2 \varphi_2 dx \geqq \int_a^b f_1 \varphi_2 dx . \int_a^b f_2 \varphi_1 dx,$$

existing under certain conditions, and comprising the former two as its particular cases, f, f_1, f_2 and φ, φ_1, φ_2 being some functions of x.

Since the SCHWARZ inequality can be easily proved by applying the CAUCHY inequalty

$$\sum_{\nu=1}^{n} a_\nu^2 . \sum_{\nu=1}^{n} b_\nu^2 \geqq \left(\sum_{\nu=1}^{n} a_\nu b_\nu\right)^2,$$

it seems be desirable to establish an algebraical inequality from which FUJIWARA's can be derived [1]). The algebraical inequality has an intimate relation with Professor J. L. W. V. JENSEN's.

[1]) M. FUJIWARA. *Ein von* BRUNN *vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der* SCHWARZ*schen und der* TCHEBYCHEFF*schen Ungleichungen für bestimmte Integrale* [Tohoku Mathematical Journal, Sendai, vol. 13 (1918), pp. 228-235] and *Über eine Ungleichung für bestimmte Integrale* [Ibidem, vol. 15 (1919), pp. 285-288].

1. Let us compare the magnitudes of

$$\sum_{\nu=1}^{n} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{n} S_\nu$$

and

$$\sum_{\nu=1}^{n} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 2} .$$

If $n = 1$, these two are evidently equal.
If $n = 2$,

$$\sum_{\nu=1}^{2} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{2} S_\nu - \sum_{\nu=1}^{2} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{2} S_\nu r_{\nu 2} = S_1 S_2 (r_{21} - r_{11})(r_{22} - r_{12})$$

$$> 0 \quad \text{or} \quad < 0,$$

according as $r_{\nu 1}$ and $r_{\nu 2}$ both increase or both decrease with ν, or one of them increases and the other decreases with ν, provided that S_1 and S_2 have the same sign.

Now assume that

$$\sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{n-1} S_\nu \geqq \sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n-1} S_\nu r_{\nu 2} ,$$

when $r_{\nu 1}$ and $r_{\nu 2}$ both increase or decrease with ν, provided that all S's have the same sign. Then

$$\sum_{\nu=1}^{n} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{n} S_\nu - \sum_{\nu=1}^{n} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 2} = \left(\sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} r_{\nu 2} + S_n r_{n1} r_{n2} \right) \left(\sum_{\nu=1}^{n-1} S_\nu + S_n \right)$$

$$- \left(\sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} + S_n r_{n1} \right) \left(\sum_{\nu=1}^{n-1} S_\nu r_{\nu 2} + S_n r_{n2} \right) = \sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{n-1} S_\nu - \sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n-1} S_\nu r_{\nu 2}$$

$$+ S_n \left\{ \sum_{\nu=1}^{n-1} S_\nu \cdot r_{n1} r_{n2} + \sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} r_{\nu 2} - \sum_{\nu=1}^{n-1} S_\nu r_{\nu 2} \cdot r_{n1} - \sum_{\nu=1}^{n-1} S_\nu r_{\nu 1} \cdot r_{n2} \right\},$$

But if we rearrange the terms within the braces, we can show that it is equal to

$$\begin{aligned} & S_1 (r_{n1} - r_{11})(r_{n2} - r_{12}) \\ + & S_2 (r_{n1} - r_{21})(r_{n2} - r_{22}) \\ + & S_3 (r_{n1} - r_{31})(r_{n2} - r_{32}) \\ & \cdots\cdots\cdots\cdots\cdots \\ + & S_{n-1} (r_{n1} - r_{n-1,1})(r_{n2} - r_{n-1,2}), \end{aligned}$$

which is positive or negative according as all S's are positive or negative. Hence under the assumption, we have

$$\sum_{\nu=1}^{n} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{n} S_\nu \geqq \sum_{\nu=1}^{n} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 2},$$

when $r_{\nu 1}$ and $r_{\nu 2}$ both increase or both decrease with ν, provided that all S's have the same sign.

When one of $r_{\nu 1}$ and $r_{\nu 2}$ increases and the other decreases with ν, the sign of inequality becomes $<$ instead of $>$.

2. Let us again consider the case where $r_{\nu 1}$, $r_{\nu 2}$ and $r_{\nu 3}$ all increase or all decrease with ν, all S's having the same sign. Then by applying twice the theorem we can extend it to the following one:

$$\sum_{\nu=1}^{n} S_\nu r_{\nu 1} r_{\nu 2} r_{\nu 3} \cdot \left(\sum_{\nu=1}^{n} S_\nu\right)^2 \geqq \sum_{\nu=1}^{n} S_\nu r_{\nu 1} r_{\nu 2} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 3} \cdot \sum_{\nu=1}^{n} S_\nu \geqq \sum_{\nu=1}^{n} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 2} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 3}.$$

More generally we get the inequality

$$\sum_{\nu=1}^{n} S_\nu r_{\nu 1} r_{\nu 2} \ldots r_{\nu p} \cdot \left(\sum_{\nu=1}^{n} S_\nu\right)^{p-1} \geqq \sum_{\nu=1}^{n} S_\nu r_{\nu 1} \cdot \sum_{\nu=1}^{n} S_\nu r_{\nu 2} \ldots \sum_{\nu=1}^{n} S_\nu r_{\nu p},$$

when the p sequences $r_{\nu 1}, r_{\nu 2}, \ldots, r_{\nu p}$ all increase or all decrease with ν, provided that all S's have the same sign.

This inequality may be regarded in a certain sense as a generalized one of that which has been proved by Professor J. L. W. V. JENSEN in his paper, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes* [Acta Mathematica, vol. 30 (1906), p. 181], i. e. of the inequality

$$\sum_{\nu=1}^{n} a_\nu x_\nu^p \cdot \left(\sum_{\nu=1}^{n} a_\nu\right)^{p-1} \geqq \left(\sum_{\nu=1}^{n} a_\nu x_\nu\right)^p,$$

though all x's are positive and p is > 1 or < 0, and not necessarily a positive integer, in Professor JENSEN's case.

3. As Professor JENSEN has shown loc. cit. p. 187, the SCHWARZ inequality and some more general ones can be derived from the algebraical inequality above quoted. Just in the same way, we can derive the FUJIWARA inequality from our algebraical inequality proved in section 1. Thus if we put

$$S_\nu = b_{\nu 1} b_{\nu 2},$$

$$a_{\nu 1} = b_{\nu 1} r_{\nu 1},$$

$$a_{\nu 2} = b_{\nu 2} r_{\nu 2},$$

then

$$\sum_{\nu=1}^{n} a_{\nu 1} a_{\nu 2} \cdot \sum b_{\nu 1} b_{\nu 2} \geqq \sum a_{\nu 1} b_{\nu 2} \cdot \sum b_{\nu 1} a_{\nu 2},$$

when $\frac{a_{\nu 1}}{b_{\nu 1}}$ and $\frac{a_{\nu 2}}{b_{\nu 2}}$ both increase or both decrease with ν, provided that $b_{\nu 1} b_{\nu 2} > 0$.

Again if we put $a_{\nu 1}\delta_\nu$ and $b_{\nu 1}\delta_\nu$ for $a_{\nu 1}$ and $b_{\nu 1}$ respectively and then put $f_1(x)$, $\varphi_1(x)$, $f_2(x)$, $\varphi_2(x)$ and dx for $a_{\nu 1}$, $a_{\nu 2}$, $b_{\nu 1}$, $b_{\nu 2}$ and δ_ν and increase ν indefinitely, all the summations become the integrals by definition, and the FUJIWARA inequality comes forth.

Of course the case where the sign of inequality is reversed takes place when one of $\frac{a_{\nu 1}}{b_{\nu 1}}$ and $\frac{a_{\nu 2}}{b_{\nu 2}}$ increases and the other decreases with ν.

Similarly using the inequality in section 2, we get the inequalities

$$\sum a_{\nu 1} a_{\nu 2} \ldots a_{\nu p} \cdot \left(\sum b_{\nu 1} b_{\nu 2} \ldots b_{\nu p}\right)^{p-1}$$
$$\geqq \sum a_{\nu 1} b_{\nu 2} \ldots b_{\nu p} \sum b_{\nu 1} a_{\nu 2} \ldots b_{\nu p} \ldots \sum b_{\nu 1} b_{\nu 2} \ldots a_{\nu p},$$

$$\int f_1 f_2 \ldots f_p \, dx \cdot \left(\int \varphi_1 \varphi_2 \ldots \varphi_2 \, dx\right)^{p-1}$$
$$\geqq \int f_1 \varphi_2 \ldots \varphi_p \, dx \cdot \int \varphi_1 f_2 \ldots \varphi_p \, dx \ldots \int \varphi_1 \varphi_2 \ldots f_p \, dx,$$

under certain conditions.

If we particularize the last inequality by putting

$$f_1 = f_2 = \cdots = f_p = f,$$

and

$$\varphi_1 = \varphi_2 = \cdots = \varphi_p = \varphi^{\frac{1}{p-1}},$$

we get

$$\int f^p \, dx \cdot \left(\int \varphi^{\frac{p}{p-1}} \, dx\right)^{p-1} \geqq \left(\int f \varphi \, dx\right)^p,$$

which is the form obtained by Prof. O. HÖLDER in Göttinger Nachrichten, 1889, pp. 38-47, while p is a positive integer in our case.

If we put

$$f_1 = f_2 = \cdots = f_p = \{\varphi(x)\}^{\frac{1}{p}} f(x),$$

and

$$\varphi_1 = \varphi_2 = \cdots = \varphi_p = \{\varphi(x)\}^{\frac{1}{p}},$$

we get

$$\int \varphi f^p \, dx . \left(\int \varphi \, dx \right)^{p-1} \geqq \left(\int \varphi f \, dx \right)^p ,$$

which is the form mentioned by Mr. JENSEN, loc. cit., p. 187.

Sendai, Japan, June 1919.

T. HAYASHI.