ON NON-INTEGRAL ORDERS OF SUMMABILITY OF SERIES AND INTEGRALS

By S. CHAPMAN.

Communicated by G. H. HARDY.

[Received October 15th, 1910.—Read November 10th, 1910.]

Introduction.

1. In this paper I propose to develop the theory of Cesàro's mean value process for summing a series, when non-integral "orders of summability" are considered.

Disregarding the way in which Cesàro originally built up his method, I take as the basis of the extended theory the general formula which is arrived at* for the r-th mean, viz., $S_n^{(r)}/A_n^{(r)}$, where

$$S_n^{(r)} = s_n + \binom{r}{1} s_{n-1} + \binom{r+1}{2} s_{n-2} + \ldots + \binom{r+n-1}{n} s_0 \tag{1}$$

$$= u_n + \binom{r+1}{1} u_{n-1} + \binom{r+2}{2} u_{n-2} + \ldots + \binom{r+n}{n} u_0, \tag{2}$$

$$A_n^{(r)} = \binom{r+n}{n} = \frac{(r+1)(r+2)\ldots(r+n)}{n!}. \tag{3}$$

In these formulæ r may be supposed to have any real or complex values whatever, save negative integral values, which for obvious reasons are throughout excluded. Then, if

$$S_n^{(r)}/A_n^{(r)}$$

tends to a definite limit L_r as $n \to \infty$, the series will be said to be

* Bromwich, *Infinite Series*, §§ 122–128.

summable by Cesàro's method of order r, or, briefly, *summable* (Cr).* If no limit exists, but the same function of n oscillates finitely as $n \to \infty$, Σu_n is said to be *finite* (Cr).*

When r is restricted to be zero or a positive integer, it is well known that if Σu_n is summable (Cr), it is also summable $(C\,\overline{r+1})$, with the same "sum"; also that $u_n/n^r \to 0$ as $n \to \infty$, and $\Sigma u_n x^n \to L_r$ as $x \to 1$.

Moreover, if Σu_n is summable (Cr) and Σv_n is summable (Cs), r and s being zero, or positive integers, then the product series Σw_n, formed according to Cauchy's rule (*i.e.*, $w_n = u_0 v_n + u_1 v_{n-1} + \ldots + u_{n-1} v_1 + u_n v_0$) is summable $(C\,\overline{r+s+1})$, with a "sum" equal to the product of the sums of Σu_n and Σv_n.

In the present paper I have confined myself to real values of r,† positive or negative, integral or not. It will appear that the properties of the extended method of summation are quite different on the two sides of the point $r = -1$. All the ordinary properties which hold for positive integral values of r also hold for non-integral values of r which exceed -1. The marked changes in the formal theory for lower values of r are illustrated by the fact that *divergent* series exist (see § 7) which are summable (Cr), r being < -1, with the sum *zero*.

2. The extension of Cesàro's method to positive non-integral orders of r is of little advantage in assisting us to find the actual "sum" of a non-convergent series; it is almost invariably simpler to do this by using the Cesàro mean value process of some *integral* order (in §§ 4, 5 it is shewn that the sum is the same in either case). The value of non-integral orders of summability lies in the closer information which we may gain by their use concerning the *degree* or *amount* (speaking in a general way) of the non-convergence of series; and by which we can more narrowly predict the behaviour of a series when multiplied by other series, or by a sequence of convergence factors (see §§ 10, 13–17, *et seq.*).

When the order of summability is negative, on the other hand, Cesàro's method (for $r > -1$) is wholly inapplicable to non-convergent series. It is useful only for convergent series, and here again, not because it enables

---

* This convenient notation was introduced by Mr. Hardy in a paper which appeared in the *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 257. The notion of a series being *finite* (Cr) is due to M. Bohr, *Comptes Rendus*, 1909.

† Save in dealing with the class of series $\Sigma n^s e^{ain}$, in § 23 and § 26.

us to find their sum* (which may be supposed already known), but because of the insight into the nature of the convergence of series, which the method enables us to obtain.

Note.—The idea of non-integral orders of summability (with Cesàro's method) is not entirely novel, as I discovered after this paper was written. Dr. Marcel Riesz had stated several interesting theorems in the *Comptes Rendus*, June, 1909, in which such "orders" are considered. As he has not yet published more detailed statements and proofs of his theorems, it is difficult to say how far our work has coincided; but, so far as I am aware, this does not extend beyond the theorems of §§ 17, 18. My attention was drawn by Mr. Hardy to Dr. Riesz's writings after my own work was completed.

In two recent letters to me (December, 1910, and March, 1911) Dr. Riesz mentions that Knopp has also considered such orders of summability as are here dealt with (*Inaugural Dissertation*, Berlin, 1907, and *Sitzungsberichte der Berliner Math. Gesellschaft*, 1907; *Archiv der Math. u. Phys.*, Bd. 12). I have, unfortunately, not had an opportunity of reading Dr. Knopp's memoirs.

The extension of Cesàro's method to convergent series, by using negative values of r, is a particular case of the application of methods of summation,† in general, to convergent series, a treatment of the latter which appears never to have been considered before. This idea, amongst other recent extensions, in the theory of summability (in particular, Dr. Riesz's elegant and powerful method of summation‡) is discussed more generally in a paper by Mr. Hardy and myself (written since the present paper was completed) on "A General View of the Theory of Summable Series."§

Part I of this paper deals with the general theory of the extended mean value process, for all real values of r. Parts II and III contain those parts of the theory (and some of its applications) which relate more particularly to non-convergent and convergent series respectively.

Part IV contains a few theorems in the closely parallel theory of summable integrals; they are naturally suggested by the corresponding theorems for series, but their proofs take a more elegant form, and they were consequently thought worthy of inclusion.

* Of course the "sum" found by the extended Cesàro method must evidently agree with the sum as ordinarily defined, if serious complications are to be avoided in the use of the method.

† We refer to methods which are applicable *only* to convergent series; of course most methods which will sum non-convergent series will also sum convergent series, but no new information concerning the latter is gained from that fact.

‡ See § 3 of this paper for a brief description of this method, and reference (9), § 2, for the memoirs in which the method was first described.

§ *Quarterly Journal of Mathematics*, 191

For convenience I here append a list* of the memoirs to which I shall have occasion to refer. This list does not purport to be at all a complete bibliography of the subject.† Further, I would make a general reference to Dr. Bromwich's *Theory of Infinite Series*, §§ 122–128, where an account of Cesàro's method for positive integral orders of summability is given; hereafter I shall refer to this treatise by the initials (*I. S.*, p. ...).

PART I.

3. The following formulæ (4)–(6) will constantly be needed. They hold good whether r is integral or not,

$$\Sigma S_n^{(r)} x^n = (1-x)^{-r}\, \Sigma s_n x^n \tag{4}$$

$$= (1-x)^{-(r+1)}\, \Sigma u_n x^n, \tag{5}$$

$$\Sigma A_n^{(r)} x^n = (1-x)^{-(r+1)}. \tag{6}$$

All these series converge absolutely for $|x| < 1$, if Σu_n is summable (Cr) or finite (Cr).

---

* 1. E. Cesàro.—*Bulletin des Sciences Mathématiques*, 2e série, t. 14.
2. L. Fejér.—*Math. Annalen*, Bd. 58, 1903.
3. G. H. Hardy.—*Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 247.
4. C. N. Moore.—*Trans. Amer. Math. Soc.*, Vol. 8, 1907.
5. G. H. Hardy.—*Math. Annalen*, Bd. 64, 1907.
6. T. J. I'A. Bromwich.—*Math. Annalen*, Bd. 65, 1907.
7. G. H. Hardy.—*Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 255, 1908.
8. H. Bohr.—*Comptes Rendus*, January 11th, 1909.
9. M. Riesz.—*Comptes Rendus*, June, July, November, 1909.
10. G. H. Hardy.—*Proc. London Math. Soc.*, November, 1909.
11. L. Fejér.—*Comptes Rendus*, December, 1900; April, 1902.
12. L. Fejér.—*Mathematikai és Physikai Lapok*, 1902.
13. L. Fejér.—*Math. Annalen*, Bd. 58, 1904.
14. G. H. Hardy and S. Chapman.—*Quarterly Journal*, 1911.

† A more extended list of cognate memoirs will be found in the paper (14) of the preceding footnote.

From (5) it follows that

$$\Sigma u_n x^n = (1-x)^{r+1} \Sigma S_n^{(r)} x^n,$$

and hence
$$u_n = S_n^{(r)} - \binom{n+1}{1} S_{n-1}^{(r)} + \binom{r+1}{2} S_{n-2}^{(r)} - \ldots, \tag{7}$$

where (for $n \geqslant r+2$) there are $r+2$ or $n+1$ terms according as r is or is not a positive integer.

It is important to notice that, if $r > -1$, $A_n^{(r)}$ is positive ; if

$$-\overline{p+1} < r < -p,$$

p being a positive integer, then $A_n^{(r)}$ alternates in sign as n takes the values $0, 1, \ldots, p+1$, after which its sign is constant, that of $(-1)^p$. Also, if $r > 0$, $A_n^{(r)}$ steadily increases as $n \to \infty$; if $-1 < r < 0$, $A_n^{(r)}$ steadily decreases to 0 as $n \to \infty$.

Evidently
$$A_n^{(r)} = p_{n,r}\, n^r / \Gamma(1+r),$$

where, as $n \to \infty$, r remaining constant,

$$p_{n,r} \to 1.$$

Hence, since
$$\binom{r+m}{m} = A_m^{(r)},$$

$$\frac{S_n^{(r)}}{A_n^{(r)}} = \sum_{m=0}^{n-1} u_m \frac{A_{n-m}^{(r)}}{A_n^{(r)}} + \frac{u_n}{A_n^{(r)}}.$$

Therefore
$$\lim_{n\to\infty} \frac{S_n^{(r)}}{A_n^{(r)}} = \lim_{n\to\infty} \frac{1}{p_{n,r}} \sum_{m=0}^{n-1} p_{n-m,r}\, u_m \left(1-\frac{m}{n}\right)^r + \lim_{n\to\infty} \frac{u_n}{A_n^{(r)}}$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n-1} p_{n-m,r}\, u_m \left(1-\frac{m}{n}\right)^r,$$

since
$$\lim p_{n,r} = 1, \quad \lim u_n/n^r = 0 \quad \text{(see § 11)}.$$

Now let us consider the method of summation invented by Dr. Riesz. Here the "sum" of a series is defined by the limit, as n tends *continuously* to infinity, of

$$\Sigma_m^{(r)} = \sum_{m=0}^{[n]} u_m \left(1-\frac{\lambda(m)}{\lambda(n)}\right)^r ;$$

where $[n]$ denotes the greatest integer less than n, $\lambda(n)$ is a monotonic positive function of n (such as n or $\log n$) tending to infinity with n ; and n is the order of summability (R, λ)—in Mr. Hardy's notation. If the limit exists, the series is said to be *summable by Riesz's method of order* r, or, briefly, *summable* (R, λ, r).

When $\lambda(n) = n$,* $\Sigma_m^{(r)}$ takes the form

$$\sum_{m=0}^{[n]} u_m \left(1-\frac{m}{n}\right)^r,$$

the limit of which (since $p_{n,r} \to 1$ as $n \to \infty$) may be expected to be almost identical with that of $S_n^{(r)}/A_n^{(r)}$. As a matter of fact, the existence of either limit follows from and involves that of the other, and the two limiting values are identical; though this would not be the case if n in $\Sigma_n^{(r)}$ were only required to tend to infinity through positive integral values (as in $S_n^{(r)}/A_n^{(r)}$), instead of continuously.†

Thus Cesàro's method, and that particular case of Riesz's method for which $\lambda(n) = n$, are completely identical in range. *Therefore all the theorems proved in this paper for series summable* (Cr) *hold good for series summable* (R, n, r). Many general theorems seem easier to prove directly for Cesàro's than for Riesz's method; but it often happens that (as in § 24) the latter method is the easier to apply in order to find the sum and summability of particular series.

4. From (4)–(6) it appears that

$$\left.\begin{aligned} S_n^{(r)} &= S_n^{(r+1)} - S_{n-1}^{(r+1)} \\ A_n^{(r)} &= A_n^{(r+1)} - A_{n-1}^{(r+1)} \end{aligned}\right\}, \tag{8}$$

for all values of r. By Stolz's extension‡ of Cauchy's theorem it at once follows that, *if* Σu_n *is summable* (Cr), *it is also summable* ($C\,\overline{r+1}$), *provided that* $r > -1$.

Stolz's theorem is that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n},$$

provided that the latter limit exists, while b_n steadily increases to infinity, as $n \to \infty$. Put

$$a_n = S_n^{(r+1)}, \quad b_n = A_n^{(r+1)};$$

then b_n steadily increases with n if $r > -1$.

---

* We shall here confine ourselves to this case. For a treatment of the general case, which has very interesting properties, the reader is referred to (9) and (14) of our list.

† For this information I am indebted to the courtesy of Dr. Riesz, to whom the stated theorem is due. In the original announcement of the theorem in the *Comptes Rendus*, July and November, 1909, there was some ambiguity, owing to the crowding-out of certain important explanatory remarks. Dr. Riesz has apparently considered only real positive values of r; but his proof can be extended down to $r > -1$. In a shortly forthcoming article in the *Comptes Rendus*, Dr. Riesz intends to prove the theorem in full.

‡ Bromwich, *Infinite Series*, App., § 152, II.

5. Further, *if* Σu_n *is summable* (Cr), *then it is also summable* $(C\,\overline{r+1})$, *with the same sum, for* $r < -1$, *provided that* $\Sigma u_n x^n$ *has a finite upper limit as* $x \to 1$.

In particular, the theorem is true if, as in the case of all convergent series, $\Sigma u_n x^n$ tends to a finite limiting value as $x \to 1$.

By a limit theorem similar in form to that of Stolz (*I. S.*, § 152, I),

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n},$$

provided that the latter limit exists, that b_n steadily decreases to zero, and that $a_n \to 0$. The existence of the second limit above involves the existence of $\lim\limits_{n\to\infty} a_n$, for

$$| a_{n+1}-a_n | < K(b_n-b_{n+1});$$

and therefore $$| a_{n+m}-a_n | < K(b_n-b_{n+m}) < Kb_n,$$

which (since $b_n \to 0$ as $n \to \infty$) is a sufficient condition for the existence of $\lim a_n$.

In this limit theorem put $a_n = S_n^{(r+1)}$, $b_n = A_n^{(r+1)}$; b_n satisfies the stated condition provided $r < -1$, and the right-hand limit exists. Hence $\lim S_n^{(r+1)}$ exists, *i.e.*, the series $\sum\limits_{n=0}^{\infty} S_n^{(r)}$ is convergent. Thus, by Abel's theorem,

$$\lim_{n\to\infty} S_n^{(r+1)} = \lim_{x\to 1} \sum_{n=0}^{\infty} S_n^{(r)} x^n$$
$$= \lim_{x\to 1} (1-x)^{-(r+1)} \Sigma u_n x^n,$$

by (5). If $\overline{\lim\limits_{x\to 1}}\, \Sigma u_n x^n$ is finite, and $r < -1$, it follows that

$$\lim_{n\to\infty} S_n^{(r+1)} = 0,$$

and all the conditions of the limit theorem are fulfilled. Consequently $\lim\limits_{n\to\infty} S_n^{(r+1)}/A_n^{(r+1)}$ is equal to $\lim\limits_{n\to\infty} S_n^{(r)}/A_n^{(r)}$, which proves the theorem.

6. The theorem that, *if* Σu_n *is summable* (Cr), *then*

$$\lim_{x\to 1} \Sigma u_n x^n = L_r,$$

was proved by Dr. Bromwich.* His proof, of course, dealt only with positive integral values of r, but also applies to general values, under

---

* *Math. Annalen*, Bd. 65, 1907.

certain limitations. It rested on a well known theorem due to Cesàro, viz.,

$$\lim_{x\to 1} \frac{\sum_0^\infty a_n x^n}{\sum_0^\infty b_n x^n} = \lim_{n\to\infty} \frac{\sum_0^n a_n}{\sum_0^n b_n},$$

provided that the latter limit exists, Σb_n being a series of positive terms.

By (5) and (6), it follows that

$$\sum_0^\infty u_n x^n = \frac{\sum_0^\infty S_n^{(r)} x^n}{\sum_0^\infty A_n^{(r)} x^n}.$$

Hence $$\lim_{x\to 1} \Sigma u_n x^n = \lim_{n\to\infty} \frac{\sum_0^n S_n^{(r)}}{\sum_0^n A_n^{(r)}} = \lim_{n\to\infty} \frac{S_n^{(r+1)}}{A_n^{(r+1)}} = L_{r+1},$$

provided that the latter limit exists, and that $A_n^{(r+1)}$ is positive for all values of n.

Now, if L_r exists, and $r > -1$, both these conditions are satisfied, and $L_r = L_{r+1}$. Therefore, *if* Σu_n *is summable* (Cr) *and* $r > -1$,

$$\lim_{x\to 1} \Sigma u_n x^n = L_r.$$

The condition $r > -1$ is sufficient but not necessary, because the theorem holds true for *convergent* series which are summable (Cr), whether r is $>$ or < -1; by § 5, L_{r+1} for such a series is equal to L_r.

7. Since, if L_r exists for any value of $r > -1$, $\Sigma u_n x_n \to L_r$ as $x \to 1$, it follows that only convergent and oscillatory (not divergent) series are summable (Cr) for $r > -1$. But it is easy to give an example of a divergent series for which $L_r = 0$ for suitable values of $r < -1$, and for which, of course, $\Sigma u_n x^n \to \infty$ as $x \to 1$.

The simplest instance is the series for which $u_n = A_n^{(k)}$, where k is positive but not integral. Then

$$\Sigma S_n^{(-k+2)} x^n = (1-x)^{(-k-2+1)} \Sigma u_n x^n = (1-x)^{k+1} (1-x)^{-(k+1)} = 1.$$

Thus $S_n^{(-\overline{k+2})} = 0$ for $n \geqslant 1$. Now $A_n^{(-\overline{k+2})} \neq 0$ for any value of n, though $\to 0$ as $u \to \infty$. Hence $L_{-k-2} = 0$, and the series is summable $(C-\overline{k}-2)$, But it is not summable $(C-\overline{k}-\overline{1})$: L_{-k-1} is "properly infinite." It is easily seen that L_{-k-m} is 0 if m is a positive integer greater than unity, and is infinity if m is ositive but not integral.

8. We shall now prove the following very general theorem :—

If Σu_n is summable (Cr), where $r > -1$, then it is also summable (Cr'), where $r' > r$ (r and r' need not be integral); and the two sums are the same.

Cesàro, in the memoir already cited, stated, without proof, the following theorem :—

If $a_n/n^r \to a$, and $b_n/n^s \to b$, as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_{n-1} b_2 + a_n b_1}{n^{r+s+1}} = \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, ab.$$

He stated no limitations on r and s, but it appears at once from the proof (which is easy, and over which we shall not delay here) that the theorem is necessarily true only if $r > -1$, $s > -1$.

By (5), we have
$$\begin{aligned}\Sigma S_n^{(r+s)} x^n &= (1-x)^{-(r+s+1)} \Sigma u_n x^n \\ &= (1-x)^{-s} \Sigma S_n^{(r)} x^n \\ &= \Sigma A_n^{(s-1)} x^n . \Sigma S_n^{(r)} x^n.\end{aligned}$$

Hence
$$\frac{S_n^{(r+s)}}{A_n^{(r+s)}} = \frac{S_n^{(r)} A_0^{(s-1)} + S_{n-1}^{(r)} A_1^{(s-1)} + \ldots + S_1^{(r)} A_{n-1}^{(s-1)} + S_0^{(r)} A_n^{(s-1)}}{A_n^{(r+s)}}.$$

Remembering that

$$\lim_{n\to\infty} \frac{A_n^{(s-1)}}{n^{s-1}} = \frac{1}{\Gamma(s)},$$

$$\lim_{n\to\infty} \frac{S_n^{(n)}}{n^{(r)}} = \lim_{n\to\infty} \frac{S_n^{(r)}}{A_n^{(r)}} \frac{A_n^{(r)}}{n^r} = \frac{L_r}{\Gamma(1+r)},$$

$$\lim_{n\to\infty} \frac{A_n^{(r+s)}}{n^{r+s}} = \frac{1}{\Gamma(r+s+1)},$$

by an obvious application of the above theorem of Cesàro we see that

$$\begin{aligned}\lim_{n\to\infty} \frac{S_n^{(r+s)}}{A_n^{(r+s)}} &= \frac{\Gamma(s)\,\Gamma(r+1)}{\Gamma(r+s+1)} \frac{\Gamma(r+s+1)}{\Gamma(s)} \frac{L_r}{\Gamma(r+1)} \\ &= L_r,\end{aligned}$$

provided that $r > -1$, $s > 0$. Putting $r+s = r'$, we get the result as stated.

COROLLARY.—*Every series summable* (Cr) *where* $r < 0$, *and for which* $\Sigma u_n x^n$ *has a finite limit as* $x \to 1$, *is convergent.*

For, by repeated application of the theorem of § 5, we can shew that the series is summable (Ck) where $-1 < k < 0$; by the above theorem the series is then summable $(C0)$, *i.e.*, it is convergent.

Hence no oscillatory series summable (Cr), where $r > 0$, can also be summable (Cr) for $r < 0$, because $\Sigma u_n x^n$ has a finite limit as $x \to 1$, for such series. Thus abnormal series like that in § 7 differ fundamentally from those summable by positive means, in that for the former series the members of the sequence $L_{r+1}, L_{r+2}, \ldots, L_{r+m}, \ldots$ must cease to be finite before $r+m > -1$.

9. When positive integral values of r are alone considered, it is natural to speak* of the "degree of indeterminacy" k of a series, when k is the *least* positive integral value of r for which Σu_n is summable (Cr). But when r may range over the whole continuum of real number, there is not necessarily any *least* value.

The series $1-1+1-\ldots$ is an example; it is summable (Cr) provided $r > 0$, while it evidently is not summable $(C0)$.

If the series Σu_n is summable (Cr) for $r \geqslant k$, and if l is the lower limit of all such possible values of k, l will be called the *index of summability* of Σu_n; it will further be said to be *attained* or *unattained* according as Σu_n is or is not summable (Cl).

This definition is so framed to exclude such series as that of § 7, being said to have $-\infty$ as index; arbitrarily large negative values of r exist for which that series is summable (Cr); but a series will only be said to have $-\infty$ as its index when (like the geometrical series $1+t+t^2+\ldots$, where $|t| < 1$) it is summable (Cr) for *all* positive and negative values of r.

10. The extension to general real values of r, of Cesàro's multiplication theorem, will now be given.

THEOREM.—*If* Σu_n *is summable* (Cr), *and has sum* L_r, *and* $\Sigma u'_n$ *is summable* (Cr') *and has sum* $L'_{r'}$, *then* Σw_n *is summable* $(C\,\overline{r+r'+1})$, *and has sum* $L_r L'_{r'}$, *provided that* $r > -1$, $r' > -1$; *where*

$$w_n = u_0 u'_n + u_1 u'_{n-1} + \ldots + u_n u'_0.$$

By (5), we have
$$(1-x)^{-(r+1)} \Sigma u_n x^n = \Sigma S_n^{(n)} x^n,$$
$$(1-x)^{-(r'+1)} \Sigma u'_n x^n = \Sigma S_n'^{(n')} x^n.$$

* As does Dr. Bromwich in his treatise already cited.

Hence $$(1-x)^{-(r+r'+2)}\Sigma w_n x^n = \Sigma S_n^{(r)} x^n . \Sigma S_n'^{(r')} x^n$$
$$\equiv \Sigma S_n''^{(r+r'+1)} x^n,$$

where $$S_n''^{(r+r'+1)} = S_0^{(r)} S_n'^{(r')} + S_1^{(r)} S_{n-1}'^{(r')} + \ldots + S_n^{(r)} S_0'^{(r')}.$$

Hence, by Cesàro's limit theorem, quoted in § 8, we have

$$\lim_{n\to\infty} \frac{S_n''^{(r+r'+1)}}{A_n^{(n+n'+1)}} = \lim_{n\to\infty} \frac{S_n^{(r)}}{A_n^{(n)}} \lim \frac{S_n'^{(r')}}{A_n^{(r')}}$$
$$= L_r L'_{r'},$$

provided that $r > -1$, $r' > -1$.

Thus the index of Σw_n does not exceed by more than unity the sum of the indices of Σu_n and $\Sigma u'_n$.

This theorem only gives an upper limit to the index of Σw_n; the real value, as will be seen in Part III of this paper, often falls below this upper limit.

The index of $(1-1+1-\ldots)^2 = 1-2+3-\ldots$ does not exceed 1, by the above theorem. It is easily seen to be 1 unattained, so that in this case the value equals the upper limit. The usual form of the multiplication theorem, where only positive integral values of r are considered, would give the upper limit of the degree of indeterminacy as 3, while the *value* of the "degree" is 2. The use of the index of a series, combined with the above generalized theorem, will usually furnish much closer information about a product series than the older method did. (*I. S.*, § 126, Ex. 2.)

Again, if $a_0 - a_1 + a_2 - \ldots$ is a convergent series with sum s (*i.e.*, index $\not> 0$), then

$$a_0 - (a_0 + a_1) + (a_0 + a_1 + a_2) - \ldots$$

has index $\not> 0$, and sum $\frac{1}{2}S$; for it is the product of $(1-1+1-\ldots)$ and $(a_0 - a_1 + a_2 - \ldots)$.

11. We shall now prove the generalization of an important theorem which, for the case when r is integral, is given in Dr. Bromwich's *Infinite Series*, § 127.

THEOREM.—*If* Σu_n *is summable* (Cr), *then* $\lim_{n\to\infty} u_n/n^r = 0$, *provided* $r > -1$.

Since, by (6), $$1 = (1-x)^{r+1}\Sigma A_n^{(r)} x^n,$$
we have, for $n > 1$,

$$0 = A_n^{(r)} - \binom{r+1}{1} A_{n-1}^{(r)} + \binom{r+1}{2} A_{n-2}^{(r)} - \ldots.$$

Making similar use of (5), we have

$$u_n = [S_n^{(r)} - L_n A_n^{(r)}] - \binom{r+1}{1}[S_{n-1}^{(r)} - L_r A_{n-1}^{(r)}] + \ldots$$
$$= A_n^{(r)}\epsilon_n + A_{n-1}^{(r)}\binom{r+1}{1}\epsilon_{n-1} + A_{n-2}^{(r)}\binom{r+1}{2}\epsilon_{n-2} + \ldots,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Remembering the properties of $A_n^{(r)}$ noticed in § 3, it is evident that

$$\frac{u_n}{n^r} = \delta_n + \binom{r+1}{1}\left(1-\frac{1}{n}\right)^r \delta_{n-1} + \binom{r+1}{2}\left(1-\frac{2}{n}\right)^r \delta_{n-2} + \dots,$$

where $\delta_n \to 0$ as $n \to \infty$.

[*Note.*—The $(n+1)$-th term does not vanish, as the above formula would seem to indicate, but it is easily seen to tend to 0 as $n \to \infty$, provided $r > -1$. For brevity, therefore, it is here neglected, only the first n terms being considered.]

An arbitrarily small positive number ϵ being assigned, an integer m can be determined such that for all values of $n > 2m$, h_m the upper limit of

$$|\delta_n|, \ |\delta_{n-1}|, \ \dots, \ |\delta_{n-m}|,$$

is $< \epsilon/3\kappa$, where

$$\kappa = 1 + \left|\binom{r+1}{1}\right| + \left|\binom{r+1}{2}\right| + \dots + \left|\binom{r+1}{m}\right|.$$

Further, $$\left(1-\frac{p}{n}\right)^r < \frac{1}{2^r} \quad \text{for} \quad m \leqslant p < \frac{n}{2},$$

while $\Sigma\left|\binom{r+1}{p}\right|$ is convergent for $r > -1$, because

$$\left|\binom{r+1}{p}\right| = |A_p^{(-r-2)}|,$$

and $$\sum^{\infty} |A_p^{(-r-2)}| < K \sum^{\infty} \frac{1}{p^{r+2}}.$$

Thus, if $r > -1$, $\Sigma\left|\binom{r+1}{p}\right|$ converges with the series $\Sigma \frac{1}{p^{r+2}}$. Let R_m be its remainder after m terms. Then*

$$\sum_{p=m}^{[\frac{1}{2}n]} \left|\binom{r+1}{p}\right| \left(1-\frac{p}{n}\right)^r < \frac{R_m}{2^r}.$$

If K' be the upper limit of $|\delta_1|$, $|\delta_2|$, ..., m can be chosen large enough to satisfy the former condition and also

$$\frac{K' R_m}{2^r} < \frac{\epsilon}{3}.$$

* $[\frac{1}{2}n]$ means $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

Lastly,* $$\sum_{p=[\frac{1}{2}n]}^{n} \left| \binom{r+1}{p} \right| \left(1-\frac{p}{n}\right)^r |\delta_{n-p}| < K'' \sum_{[\frac{1}{2}n]}^{n} \frac{\left(1-\frac{p}{n}\right)^r}{p^{r+2}}.$$

Here r may be negative, so that $\left(1-\frac{p}{n}\right)^r$ need not have any upper limit if $n-p$ remains finite. But the last expression evidently is less than

$$\frac{K'''}{n^{r+1}} \int_{\frac{1}{2}}^{1} \frac{(1-x)^r\,dx}{x^{r+2}},$$

where K''' is a constant. Since $r > -1$, n_0 can be found so that for $n > n_0$ the value of the last expression is less than $\frac{1}{3}\epsilon$.

Collecting these results, it is evident that by taking the expression for $\frac{u_n}{n^r}$ in three parts

$$\left\{ \sum_{p=1}^{m} + \sum_{p=m}^{[\frac{1}{2}n]} + \sum_{p=[\frac{1}{2}n]}^{n} \right\} |\delta_{n-p}| \left(1-\frac{p}{n}\right)^r \left| \binom{r+1}{p} \right|,$$

it can (by taking $n > n_0 > 2m$) be made less than ϵ. Thus under the sole condition $r > -1$ the theorem holds as stated.

We can now give a simple example of a series which has its index attained. The series $\sum_{n=1}^{\infty} (-1)^{n+1} (\log \overline{n+1})^{-a}$ is convergent if $a > 0$. But since for no negative value of r does $\frac{u_n}{n^r} \to 0$, the series is only summable for values of $r \geqslant 0$, *i.e.*, its index is 0 attained.

PART II.

Oscillatory Series ($r > 0$).

§§ 12. Introductory.
13–15. Convergence factors.
16. An extension of Abel's theorem.
17. The convergence factors n^{-s}.
18. The index of $\Sigma u_n/(n+1)$ does not exceed that of Σu_n, less 1.
20. The index of a Fourier's series and its derivative.
23. The index of the series $1^s - 2^s + 3^s - \dots$.

12. As already stated, since all series summable (Cr), where $r > 0$, are such that $\Sigma u_n x^n$ has a finite limit as $x \to 1$, no such series can be strictly divergent. Hence only finitely or infinitely oscillatory series will be considered in this section of the paper.

Neither finitely oscillatory series, nor infinitely oscillatory series, are necessarily summable

* K'' is a finite constant depending on K'.

(Cr) for any positive value of r. As examples we may quote $1-t+t^3-\ldots$ ($t>1$) among series with infinite oscillation, and among finitely oscillatory series, the one for which

$$\Sigma u_n x^n \equiv \Sigma (-1)^n x^{a^n}.$$

Mr. Hardy has shewn* that for this series $\Sigma u_n x^n$ has no definite limit, and therefore Σu_n as so defined cannot be summable (Cr) for $r>0$.

Another interesting possibility has been shewn to exist, by Mr. Hardy, viz., that series Σu_n may be found with as large an index as we please, while $u_n \to 0$ and $\lim_{x\to 1} \Sigma u_n x^n$ exists; thus the series $\Sigma (e^{in^b}/n^a)$, where $0<b<1$, $0<a<1$, is summable (Ck) if $a+(k+1)b>1$.

Mr. Littlewood† has constructed an example of a series Σu_n for which $u_n \to 0$ and $\lim_{x\to 1} \Sigma u_n x^n$ exists, which is not summable for *any* value of r; his series is $\Sigma (e^{i(\log n)^2}/n^a)$.

13. A very considerable amount of work has recently been published on extensions, in various directions, of Abel's theorem on the limiting value of $\Sigma u_n x^n$ as $x\to 1$. Those extensions which make use of the property of Cesàro-summability generally take the form of the statement of conditions under which an oscillatory series is reduced to convergence when its terms are multiplied by corresponding members of a sequence of variable quantities (hence called by Dr. C. N. Moore *convergence factors*); and further, the additional restrictions to be placed on these convergence factors in order that, as their independent variable tends to some specified limit, the sum of the convergent series obtained by their aid may tend to the Cesàro-sum of the original series. The latest results are due to Mr. Hardy and Dr. Bromwich. See references, 2–10, § 2.

By the consideration of non-integral means, it is possible to place less stringent conditions on the convergence factors, just as we were enabled by the same means to gain more precise information concerning the index of the product of two summable series. Hence as a contribution to the abstract theory of summable oscillatory series such an extension of the known theorems seems to possess some interest.

The complete theorem falls naturally into two parts, A and B.

14. THEOREM A.—*If Σu_n is summable (Cr), or finite (Cr), where $r>0$, then $\Sigma u_n v_n$ converges, provided that*

$$(a) \quad \lim_{n\to\infty} n^r v_n = 0,$$

and

$$(b) \quad \Sigma n^r \,|\Delta^{r+1} v_n| < K;‡$$

* *Quarterly Journal of Mathematics*, Vol. 38, p. 269, 1907.

† J. E. Littlewood, "The Converse of Abel's Theorem on Power Series," *Proc. London Math. Soc.* (present volume).

‡ This, of course, implies the absolute convergence of $\Sigma n \,\Delta^{r+1} v_n$.

also its sum equals that of the series $\Sigma S_n^{(r)} \Delta^{r+1} v_n$, *which is absolutely convergent.*

Here $$\Delta^{r+1} v_n \equiv \lim_{p \to \infty} \Delta_p^{r+1} v_n,$$

provided this limit exists ; where

$$\Delta_p^{r+1} v_n \equiv v_n - \binom{r+1}{1} v_{n+1} + \binom{r+1}{2} v_{n+2} - \ldots,$$

to $p-n+1$ terms.

If condition (*a*) is satisfied, then $\Delta^{r+1} v_n$ exists. For the series which defines it is absolutely convergent when all the v's are replaced by unity, when $r > -1$; and for $r > 0$ we have $|v_n| < K$ for all values of n.

The following lemma is first required :—

LEMMA.—*Conditions* (*a*) *and* (*b*) *being satisfied, and* r *being* > 0, *we have*

$$\lim_{v \to \infty} \sum_{n-1}^{p} n^r \Delta_p^{r+1} v_n = \sum_{n=1}^{\infty} n^r \Delta^{r+1} v_n.$$

By virtue of (*a*), we can choose p_0, an arbitrarily small positive number ϵ having been assigned, such that for $m > p_0$,

$$| v_m m^r | < \epsilon ;$$

also we have $$\binom{r+1}{m} < \frac{K}{m^{r+2}},$$

where K is a constant for all values of m.

Hence* if $p > p_0$,

$$n^r | \Delta^{r+1} v_n - \Delta_p^{r+1} v_n | = n^r \left| v_{p+1} \binom{r+1}{p-n+1} - v_{p+2} \binom{r+1}{p-n+2} + \ldots \right|$$

$$< K\epsilon \left(\frac{n}{p}\right)^r \left\{ \frac{1}{(p-n+1)^{r+2}} + \frac{1}{(p-n+2)^r} + \ldots \right\}$$

$$< K\epsilon \frac{1}{(p-n+1)^{r+1}},$$

by an easy integral summation, provided $n < p$.

* K is used here, and throughout the subsequent work, in the sense introduced by Borel.

Hence $$\left| \sum_{n=1}^{p} n^r (\Delta_p^{r+1} v_n - \Delta^{r+1} v_n) \right| < K\epsilon \sum_{n=1}^{p} \frac{1}{(p-n+1)^{r+1}}$$

$$< K\epsilon \sum_{1}^{\infty} \frac{1}{n^{r+1}} < K\epsilon,$$

where K is independent of p. The truth of the lemma is an immediate consequence.

15. To prove the main proposition we proceed thus :—

By a transformation which is easy, but tedious to write out, it follows that

$$\sum_{n=0}^{p} u_n v_n = \sum_{n=0}^{p} S_n^{(r)} \Delta_p^{r+1} v_n.$$

by making use of the identity (7).*

Now, provided that Σu_n is summable or finite (Cr), we have

$$\left| \sum_{n=0}^{p} S_n^{(r)} (\Delta_p^{r+1} v_n - \Delta^{r+1} v_n) \right| < K \left| \sum_{n=0}^{p} n^r (\Delta_p^{r+1} v_n - \Delta^{r+1} v_n) \right|;$$

K is a constant not depending on p. Hence, by the lemma just established,

$$\lim_{p\to\infty} \sum_{n=0}^{p} S_n^{(r)} \Delta_p^{r+1} v_n = \sum_{n=0}^{\infty} S_n^{(r)} \Delta^{r+1} v_n,$$

which is absolutely convergent by comparison with the series

$$\Sigma n^r \,|\, \Delta^{r+1} v_n \,|.$$

Therefore $$\sum_{0}^{\infty} u_n v_n = \sum_{0}^{\infty} S_n^{(r)} \Delta^{r+1} v_n,$$

as the theorem states.

16. THEOREM B.—*If the convergence factors v_n are functions of a variable x, and are such that $\lim_{x\to c} v_n = 1$, then, provided that conditions (a) and (b) hold for all values of $x > c$, the K of condition (b) being a constant, we have*

$$\lim_{x\to c} \Sigma u_n v_n = L_r.$$

We first notice that $\lim_{x\to c} \Delta^{r+1} v_n = 0$ for all values of n; for $\Delta^{r+1} v_n$ is

* A similar transformation is set out at length in Dr. Bromwich's paper in *Math. Annalen*, Bd. 65.

an absolutely convergent series, and so also is

$$S' = 1 - \binom{r+1}{1} + \binom{r+1}{2} - \ldots = (1-1)^{r+1} = 0.$$

Since $|v_n| < K$ for all values of $x > c$, and all values of n, $\Delta^{r+1} v_n$ is uniformly convergent for $x > c$, by Weierstrass's M-test, S' being the comparison series. Hence

$$\lim_{x \to c} \Delta^{r+1} v_n = S' = 0.$$

Using a device which is frequently of service in this kind of work, we consider the convergent series for which

$$a_0 = 1, \quad a_2 = a_3 = \ldots = 0, \quad s_0' = s_1' = \ldots = 1, \quad S_n'^{(r)} = A_n^{(r)}.$$

Thus, by Theorem A,

$$\sum_0^\infty a_n v_n = v_0 = \sum_0^\infty S_n'^{(r)} \Delta^{r+1} v_n = \sum_0^\infty A_n^{(r)} \Delta^{r+1} v_n.$$

Hence
$$(\Sigma u_n v_n - L_r v_0) = \sum_0^\infty [S_n^{(r)} - L_r A_n^{(r)}] \Delta^{r+1} v_n,$$

and since
$$\frac{S_n^{(r)}}{A_n^{(r)}} \to L_r \quad \text{as} \quad n \to \infty,$$

we have
$$|S_n^{(r)} - L_r A_n^{(r)}| < \epsilon_{n_0} n^r \quad \text{for} \quad n > n_0;$$

ϵ_{n_0} can, of course, be taken as small as we please by taking n_0 sufficiently large.

Consequently,

$$\left| \sum_0^\infty (S_n^{(r)} - L_r A_n^{(r)}) \Delta^{r+1} v_n \right| < \left| \sum_0^{n_0} (S_n^{(r)} - A_n^{(r)} L_r) \Delta^{r+1} v_n \right| + \epsilon_{n_0} \left| \sum_{n_0}^\infty n^r \Delta^{r+1} v_n \right|$$

$$< \delta + \epsilon_{n_0} \sum_0^\infty n^r \Delta^{r+1} v_n$$

$$< \delta + K\epsilon_{n_0},$$

where δ can be taken as small as we please by choosing a sufficiently small interval for $x - c$, since

$$\lim_{x \to c} \Delta^{r+1} v_n = 0,$$

and n_0 is fixed.*

* $\Delta^{r+1} v_n$ does not necessarily $\to 0$, as $x \to c$, uniformly for all values of n.

Thus
$$\lim_{x \to c} \sum_{0}^{\infty} (S_n^{(r)} - A_n^{(r)} L_r) \Delta^{r+1} v_n = 0,$$
whence it follows that
$$\lim_{x \to c} \Sigma u_n v_n = L_r \lim_{x \to c} v_0 = L_n.$$

17. After $v_n = x^n$, $c = 1$, which evidently satisfy the conditions of Theorems A and B, and lead to part of the general theorem of § 6, the most important convergence factors are $v_n = (n+1)^s$. Consequently we shall prove that they satisfy the general conditions, and so deduce the following theorem* :—

If Σu_n is summable or finite (Cr) $r > 0$, then $\Sigma u_n (n+1)^{-s}$ is convergent, provided $s > r$.

Condition (*a*) is obviously satisfied. It only remains to shew that
$$\Sigma n^r \,|\, \Delta^{r+1} (n+1)^{-s} \,|$$
is convergent.

We have
$$\Gamma(s) m^{-s} = \int_0^\infty e^{-mx} x^{s-1} \, dx.$$

Hence
$$\Gamma(s) \Delta^{r+1} v_n = \sum_{m=0}^{\infty} \left[\int_0^\infty e^{-(m+n+1)x} x^{s-1} \, dx \, (-1)^m \binom{r+1}{m} \right].$$

By a well known test,† the order of summation and integration can be inverted, and we thus obtain
$$\Gamma(s) \Delta^{r+1} v_n = \int_0^\infty e^{-(n+1)x} x^{s-1} \left[\sum_{m=0}^{\infty} (-1)^m \binom{r+1}{m} e^{-mx} \right] dx$$
$$= \int_0^\infty e^{-(n+1)x} x^{s-1} (1-e^{-x})^{r+1} \, dx.$$

Thus $\Delta^{r+1} n^{-s}$ is positive, and decreases as n increases, just as when r is integral.

None of the ordinary tests seem to suffice to prove that
$$\Sigma n^r \Delta^{r+1} (n+1)^{-s}$$
is absolutely convergent; we therefore proceed as follows. Consider the integral
$$I_\epsilon \equiv \int_\epsilon^\infty x^{s-1} (1-e^{-x})^{r+1} \sum_{n=m}^{p+m} n^r e^{-(n+1)x} \, dx.$$

* See the note to § 2 of the Introduction, p. 371.

† See, for example, Bromwich's *Infinite Series*, § 176 A.

A finite constant K_m can be found such that the n-th term of the series $\sum_{n=0}^{p} (n+m)^r e^{-nx}$ is less than the corresponding term of the series

$$K_m \sum_{n=0}^{\infty} A_n^{(r)} e^{-nx} = \frac{K_m}{(1-e^{-x})^{r+1}} \quad (x \geqslant \epsilon > 0)$$

for all values of n, m being fixed. Here K_m is independent of ϵ, but is a function of m. Evidently

$$K_m < K'(m+1)^r,$$

where K' is independent of both m and ϵ. Therefore

$$\sum_{n=m}^{m+p} n^r e^{-(n+1)x} \leqslant \frac{K_m e^{-(m+1)x}}{(1-e^{-x})^{r+1}}$$

for all values of p. Consequently

$$I_\epsilon < K_m \int_\epsilon^\infty x^{s-1} e^{-(m+1)x} dx$$

or
$$\sum_{n=m}^{m+p} n^r \int_\epsilon^\infty x^{s-1}(1-e^{-x})^{r+1} e^{-(n+1)x} dx < \frac{K_m \Gamma(s)}{(m+1)^s} < \frac{K' \Gamma(s)}{(m+1)^{s-r}}$$

for all values of p and all positive values of ϵ (< 1, say). Now let $\epsilon \to 0$. Since the number of terms on the left is finite, we have

$$\sum_{n=m}^{m+p} n^r \lim_{\epsilon \to 0} \int_\epsilon^\infty x^{s-1}(1-e^{-x})^{r+1} e^{-(n+1)x} dx = \sum_{n=m}^{m+p} n^r \int_0^\infty x^{s-1}(1-e^{-x})^{r+1} e^{-(n+1)x} dx$$

$$= \Gamma(s) \sum_{n=m}^{m+p} n^r \Delta^{r+1} v_n \leqslant \frac{K' \Gamma(s)}{(m+1)^{s-r}}.$$

Hence, since K' is independent of p, and $s > r$, the series $\Sigma n^r \Delta^{r+1} v_n$ is absolutely convergent.

The theorem above has an immediate corollary: *If Σu_n is summable by Cesàro's method and has an index of summability k (> 0), then $\sum_0^\infty u_n (n+1)^{-l}$ is convergent, provided $l > k$.*

Obviously this corollary enables us to obtain less rapidly diminishing convergence factors of the type n^{-s} than the theorems dealing only with positive integral orders of summation; and the analysis by which the result has been obtained is only very slightly more complex than that needed for those theorems.

The corollary gives us some information about the index of a series (supposed summable). If k is the lower limit of the positive numbers l, for which $\Sigma u_n n^{-l}$ is convergent, the index of Σu_n is $\geqslant k$. If Σu_n is summable of order r, then $u_n/n^r \to 0$ as $u \to \infty$; hence $\Sigma u_n/n^{r+2}$ is absolutely convergent. Therefore such values of l always exist and have a lower limit. But the series may be reducible to convergence by such factors and yet not be summable, of course.

If the index does exist, however, we can find a lower limit to its possible values. Thus $1^s-2^s+3^s\ldots$ cannot have an index $< s$ $(s > 0)$. Also, since $u_n/n^s = \pm 1$, if the index is s, it is unattained.

Many further types of convergence factors might be investigated, but I shall not here multiply examples of such, as the general consequences of the extension of the theorems to non-integral values of r have been sufficiently indicated.

18. Dr. H. Bohr* and Dr. Marcel Riesz† have given the theorem that, if Σu_n is summable (Cs), then $\Sigma u_n/(n+1)$ is summable $(C\,\overline{s-1})$, s being a positive integer. The following extension‡ of this theorem is interesting in itself, and will be of use later.

THEOREM.—*If* $\sum\limits_0^\infty u_n$ *is summable or finite* (Ck) *whether* k *is integral or not,* POSITIVE OR NEGATIVE, *then* $\Sigma u_n/(n+1)$ *is summable* $(C\,\overline{k+1})$.

Let T_n be the function related to the series $\Sigma u_n/(n+1)$ in the same way as $S_n^{(k-1)}$ is related to Σu_n. Then, by (2) and (3), we have

$$T_n = \sum_{r=0}^{n} A_{n-r}^{(k-1)} \frac{u_r}{n+1} = \sum_{r=0}^{n} S_r^{(k)} B_{n,\,r},$$

substituting for u_r from (7) and making a transformation similar to that in § 15. Here $B_{n,r}$ denotes

$$\frac{A_{n-r}^{(k-1)}}{r+1} - \binom{k+1}{1}\frac{A_{n-r-1}^{(k-1)}}{r+2} + \binom{k+1}{2}\frac{A_{n-r-2}^{(k-1)}}{r+3} - \ldots$$

to $n-r+1$ terms (as the theorem has previously been proved for the case when k is a positive integer, and, as k cannot be 0 or a negative integer, it will be supposed that k is non-integral). Evidently

$$\sum_{n=r}^{\infty} B_{n,r}x^{n+1} = (1-x)^{-k}\left\{\frac{x^{r+1}}{r+1} - \binom{k+1}{1}\frac{x^{r+2}}{r+2} + \ldots\right\}$$

$$= (1-x)^{-k}\int_0^x x^r(1-x)^{k+1}\,dx \quad (x < 1). \qquad (a)$$

* *Comptes Rendus*, January 11th, 1909.

† *Ibid.*, June 21st, 1909. These theorems were not stated as above, and also included further results which are not considered here; but the form above is immediately deducible from the theorems.

‡ See the note to § 2 of the Introduction, p. 371.

Integrate $(r-1)$ times by parts. We get

$$\sum_{n=r}^{\infty} B_{n,r} x^{n+1}$$

$$= \frac{1}{(1-x)^k}\left\{-\frac{x^r(1-x)^{k+2}}{k+2} - \frac{r}{k+2}\frac{x^{r-1}}{k+3}(1-x)^{k+3} - \dots\right.$$

$$\left. - \frac{r!\,x(1-x)^{k+r+1}}{(k+2)(k+3)\dots(k+r+1)} + \frac{r!}{(k+2)\dots(k+r+1)}\int_0^x (1-x)^{k+r}dx\right\}$$

$$= \frac{1}{(1-x)^k}\left\{-\frac{x^r}{k+2}(1-x)^{k+3} - \dots - \frac{r!\,(k+1)!}{(k+r+2)!}\left[(1-x)^{k+r+2}-1\right]\right\}. \quad (\beta)$$

The coefficients of x^n on the right vanish, as we know from (α), up to $n = r+1$; also it is easy to see that the coefficient of x^{r+1} is $(r+1)^{-1}$, and of x^{r+2} is $(k-r-1)(r+1)^{-1}(r+2)^{-1}$. The coefficients of the higher terms come, as is seen from (β), from the expansion of the term

$$\frac{r!\,(k+1)!}{(k+r+2)!}\frac{1}{(1-x)^k} = \frac{r!\,(k+1)!}{(k+r+2)!}\sum_0^\infty A_n^{(k-1)}x^n.$$

Hence

$$B_{n,n} = \frac{1}{n+1}, \qquad B_{n,n-1} = \frac{k-n}{n(n+1)}, \qquad B_{n,r} = \frac{r!\,(k+1)!}{(k+r+2)!}A_{n+1}^{(k-1)},$$

for $r \leqslant n-2$. Therefore

$$\frac{T_n}{A_n^{(k-1)}} = \frac{A_{n+1}^{(k-1)}}{A_n^{(k-1)}}\sum_{r=0}^{n-2}\frac{r!\,(k+1)!}{(k+r+2)!}S_r^{(k)} + \frac{k-n}{n(n+1)}\frac{S_{n-1}^{(k)}}{A_n^{(k-1)}} + \frac{1}{n+1}\frac{S_n^{(k)}}{A_n^{(k-1)}}.$$

The *sum* of the last two terms is readily seen to tend to 0, though they separately do not $\to 0$ as $n \to \infty$. Also

$$\frac{r!\,(k+1)!}{(k+r+2)!} = \frac{k+1}{(k+r+1)(k+r+2)}\frac{1}{A_r^{(k)}}.$$

Hence $$\frac{T_n}{A_n^{(k-1)}} = (k+1)\frac{A_{n+1}^{(k-1)}}{A_n^{(k-1)}}\sum_{r=0}^{n-2}\frac{1}{(k+r+1)(k+r+2)}\frac{S_r^{(k)}}{A_r^{(k)}} + \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Since $|S_r^{(k)}/A_r^{(k)}| < K$ for all value of r, by comparison with the series $\Sigma 1/r^2$, we see that

$$\lim_{n\to\infty}\frac{T_n}{A_n^{(k-1)}} = (k+1)\sum_{r=0}^{\infty}\frac{1}{(k+r+1)(k+r+2)}\frac{S_r^{(k)}}{A_r^{(k)}},$$

the latter series being absolutely convergent. This proves the theorem.

We thus have as a corollary:—*The index of a series* $\sum_0^\infty u_n/(n+1)$ *does not exceed* $k-1$, *where* k *is the index of* Σu_n.

By repeated application of the theorem, we have: *The index of a series* Σu_n *being* k, *the index of* $\Sigma u_n/(n+1)^m$ *is* $\not> k-m$, m *being a positive integer* ($k-m$ *must not be a negative integer*).

As applied to convergent (or finitely oscillatory) series, the theorem must thus be stated: Σu_n *being a convergent or finitely oscillatory series* [such are easily seen to be always finite (Cr) for any positive value of r], *the series* $\Sigma\{u_n/(n+1)\}$ *is summable* (Cr) *for any value of* $r > -1$, *i.e., its index* (whose existence is thus proved) *is* $\not> -1$.

For the series is finite ($C\epsilon$) where ϵ is any positive number, however small; hence $\Sigma u_n/(n+1)$ is summable ($C\,\overline{\epsilon-1}$).

19. One of the most interesting of the applications of Cesàro's method was that made by Fejér in a number of papers* on Fourier's series. He shewed that, if $f(x)$ is integrable and periodic in 2π, then the Fourier's series

$$\sum_0^\infty (a_n \cos x + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a)\,da,$$

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a) \begin{matrix} \cos \\ \sin \end{matrix} na\,da$$

is summable ($C1$), and has the sum† $\frac{1}{2}[f(x+0)+f(x-0)]$ at any point x at which the latter limit is definite.

Fejér considered only the case when $f(x)$ has a Riemann integral; wide generalizations of his theorems have been published by Dr. Hobson and Lebesgue, in which $f(x)$ is subject to the less stringent condition of Lebesgue-integrability.‡

These theorems may still further be extended§ by the consideration of non-integral orders of summability, and we so obtain the following theorem, which is of some theoretical interest:—

If $f(x)$ *be any function, limited or unlimited, in* $(-\pi, \pi)$, *which has a Lebesgue integral, and therefore a corresponding Fourier series, then*

---

* See memoirs 11–13 of the list appended to § 2.

† This expression, of course, denotes $\lim_{h\to 0} \frac{1}{2}[f(x+h)-f(x-h)]$.

‡ For references to original memoirs and a full account of the whole subject, reference may be made to Hobson's *Theory of Functions of a Real Variable*, ch. vii.

§ Only the main outline of the proof will be given; it is similar in many ways to the proof in §§ 469 *et seq.* of Dr. Hobson's treatise, where the series is shewn to be summable, and to this place reference may be made to complete the detailed proof.

the latter series is summable (Ck) for any value of $k > 0$, and the sum is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at any point x at which this limit exists. Further, the convergence as $n \to \infty$ of the sum-function $S_n^{(k)}/A_n^{(k)}$ is uniform in any interval contained within an interval in which $f(x)$ is limited and at every point of which $f(x+0)+f(x-0)$ exists.

Thus at such points, if a Fourier's series is not convergent, it only just oversteps the bounds of convergence. Its index of summability is then zero.

20. We shall suppose, for convenience, that $0<k<1$; this, of course, implies no ultimate restriction. The point x is understood to be one at which the limit $[f(x+0)+f(x-0)]$ exists as a definite number. We have

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a') \frac{\sin \frac{2n+1}{2}(a'-x)}{\sin \frac{1}{2}(a'-x)}\, da'.$$

Write $a' = x+2a$. Remembering that $f(x)$ is defined as a periodic function, for values of x not necessarily in the interval $(-\pi, \pi)$, we have

$$s_n(x) = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} [f(x+2a)+f(x-2a)] \frac{\sin(2n+1)a}{\sin a}\, da.$$

Hence, denoting $S_n^{(k)}/A_n^{(k)}$, for the point x, by $L_n(x)$, we have

$$L_n(x) = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} [f(x+2a)+f(x-2a)]\, \phi(a, n)\, da,$$

where $$\phi(a, n) = \frac{1}{A_n^{(k)}} \sum_{r=0}^{n} A_r^{(k-1)} \frac{\sin(2\overline{n-r}+1)a}{\sin a}.$$

Since $$\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)a}{\sin a}\, da = \frac{\pi}{2},$$

we have $$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \phi(a, n)\, da = \tfrac{1}{2}.$$

Therefore $$L_n(x) - \tfrac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} F(a, x)\, \phi(a, n)\, da,$$

where $$F(a, x) = f(x+2a)+f(x-2a)-f(x+0)-f(x-0),$$

so that $$\lim_{a\to 0} F(a, x) = 0.$$

Now, by Abel's lemma, since $0 < k < 1$,

$$\left| \sum_{r=0}^{n} A_r^{(k-1)} \sin(2n+1)a \right| < K,$$

where K is a finite constant independent of n and a, provided that

$$0 < \epsilon \leqslant a \leqslant \frac{\pi}{2}.$$

Hence

$$|\phi(a, n)| < \frac{K}{A_n^{(k)}}$$

for

$$0 < \epsilon \leqslant a \leqslant \frac{\pi}{2}.$$

Thus we have

$$\left| \frac{1}{\pi} \int_\epsilon^{\frac{1}{2}\pi} F(a, x)\, \phi(a, n) da \right| < \frac{K}{A_n^{(k)}} \int_\epsilon^{\frac{1}{2}\pi} |F(a, x)|\, da,$$

whence it follows that n_0 can be so chosen that, for $n > n_0$,

$$\left| \frac{1}{\pi} \int_\epsilon^{\frac{1}{2}\pi} F(a, x)\, \phi(a, n) da \right| < \sigma;$$

and moreover, if x lies within an interval contained in an interval in which $f(x)$ is limited and at every point of which $f(x+0)+f(x-0)$ exists, n_0 can be so chosen, independently of x, that this inequality is satisfied for all points x in this interval.

We shall now show that

$$\frac{1}{\pi} \int_0^\epsilon |\phi(a, n)|\, da < K,$$

where K is a constant which is independent of n.*

Since for $0 < a < \pi/n$,

$$\left| \frac{\sin (2\,\overline{n-r}+1)\, a}{\sin a} \right| < 2n,$$

it is easily seen that

$$\int_0^{\pi/n} |\phi(a, n)|\, da < \frac{2n}{A_n^{(k)}} \frac{\pi}{n} \{A_0^{(k-1)} + \ldots + A_n^{(k-1)}\} = \frac{2\pi}{A_n^{(k)}} A_n^{(k)} = 2\pi.$$

Hence we have only to shew that

$$\int_{\pi/n}^\epsilon |\phi(a, n)|\, da < K;$$

* K will have this meaning throughout the remainder of § 20; but its value is not necessarily the same in each inequality, or even in different parts of the same inequality.

this involves some rather tedious but not very difficult analysis, and is best done (as was kindly pointed out to me by Mr. Littlewood) by replacing the *sum* $\phi(a, n)$ by the analogous *integral* $\psi(a, n)$, where

$$\psi(a, n) = \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} \sin a} \int_1^{n+1} x^{k-1} \sin(2n+1-2x)\, a\, dx.$$

Evidently

$$\psi(a, n) = \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} a^k \sin a} \int_a^{(n+1)a} y^{k-1} \sin(\overline{2n+1}\, a - 2y)\, dy.$$

Since we are considering values of a ranging from π/n to ϵ, the lower limit of the last integral will vary between these values, while the upper limit will range from π to $n^{1-k}\pi$. Hence, writing $2n+1 = n'$,

$$\psi(a, n) = \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} a^k \sin a} \left\{ \int_a^{\pi} y^{k-1} \sin(n'a - 2y)\, dy + \int_{\pi}^{(n+1)a} y^{k-1} \sin(n'a - 2y)\, dy \right\}.$$

Since $0 < k < 1$, the first of the integrals within the last bracket has a finite upper limit independent of n, and the same is true of the second, because the infinite integral

$$\int_{\pi}^{\infty} y^{k-1} \sin(n'a - 2y)\, dy$$

is convergent (though not absolutely). Consequently

$$|\psi(a, n)| < \frac{K}{A_n^{(k)} a^k \sin a} < \frac{K'}{n^k a^{k+1}};$$

and therefore $\displaystyle\int_{\pi/n}^{\epsilon} |\psi(a, n)|\, da < K' n^{-k} \int_{\pi/n}^{\epsilon} a^{-k-1}\, da < \frac{Kn^k}{n^k} = K.$

Hence it only remains to shew that

$$\int_{\pi/n}^{\epsilon} |\phi(a, n) - \psi(a, n)|\, da < K.$$

Now, by means of Stirling's expression for the gamma function (*I.S.*, p. 462), it is not difficult to prove that for $n \geqslant 1$, $k > 0$,

$$|\Gamma(k) A_n^{(k-1)} - n^{k-1}| < An^{k-\frac{3}{2}},$$

where A is a constant independent of n. Therefore

$$\left| \phi(a, n) - \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} \sin a} \sum_{m=1}^{n} m^{k-1} \sin(2\overline{n-m}+1)a \right|$$

$$< \frac{K}{A_n^{(k)} \sin a} \sum_{m=1}^{n} m^{k-\frac{3}{2}} < \frac{K'}{n^k \sin a} n^{k-\frac{1}{2}} < \frac{K}{an^{\frac{1}{2}}};$$

and since
$$\int_{\pi/n}^{\epsilon} \frac{K\,da}{an^{\frac{1}{2}}} < \frac{A}{n^{\frac{1}{2}}} \log n,$$

which tends to zero as $n \to \infty$, our problem is finally reduced to proving that

$$\frac{1}{A_n^{(k)}} \int_{\pi/n}^{\epsilon} \frac{da}{\sin a} \left\{ \sum_{m=1}^{n} m^{k-1} \sin(2\overline{n-m}+1)a - \int_1^{n+1} x^{k-1} \sin(n'-2x)a\,dx \right\}$$

has a finite upper limit as $n \to \infty$.

We notice that

$$m^{k-1} \sin(n'-2m)a - \int_m^{m+1} x^{k-1} \sin(n'-2x)a\,dx$$

$$= m^{k-1} \sin(n'-2m)a - \int_0^1 (m+y)^{k-1} \sin(n'-2m-2y)a\,dy$$

$$= m^{k-1} \sin(n'-2m)a \left[1 - \int_0^1 \left(1+\frac{y}{m}\right)^{k-1} \cos 2ya\,dy\right]$$

$$+ \int_0^1 (m+y)^{k-1} \sin 2ya\,dy \cos(n'-2m)a.$$

But
$$\left(1+\frac{y}{m}\right)^{k-1} = 1 + \frac{\delta}{m},$$

where δ has a finite upper limit for all values of y and m considered. Hence the above difference is equal to

$$m^{k-1} \sin(n'-2m)a \left[1 - \frac{\sin 2a}{2a} + \frac{K}{m}\right]$$

$$- \int_0^1 (m+y)^{k-1} \sin(n'-2m)a \sin 2ya\,dy,$$

where K has a finite upper limit for all values of m and a under consideration.

Summing from $m = 1$ to $m = n$, we have

$$\frac{1}{A_n^{(k)}} \left\{ \sum_{m=1}^{n} m^{k-1} \sin(n'-2m)\,a - \int_1^{n+1} x^{k-1} \sin(n'-2x)\,a\,dx \right\}$$

$$= \frac{K}{n^k} \left[\sum_{m=1}^{n} m^{k-1} \cos(n'-2m)\,a \left(K'a^2 + \frac{K''}{m}\right) \right.$$

$$\left. - \int_0^1 \left\{ \sum_{m=1}^{n} (m+y)^{k-1} \sin(n'-2m)\,a \right\} \sin 2ya\,dy \right],$$

K' being a constant depending on a but not on m. Now

$$\left| \sum_{m=1}^{n} m^{k-1} \cos(n'-2m)\,a \right| < An^{k-1}/\sin a,$$

$$\left| \sum_{m=1}^{n} (m+y)^{k-1} \sin(n'-2m)\,a \right| < A'n^{k-1}/\sin a,$$

$$\left| \sum_{m=1}^{n} m^{k-2} \cos(n'-2m)\,a \right| < A''n^{k-1}.$$

Hence

$$\frac{1}{A_n^{(k)}} \left\{ \sum_{m=1}^{n} m^{k-1} \sin(n'-2m)\,a - \int_1^{n+1} x^{k-1} \sin(n'-2x)\,a\,dx \right\}$$

$$< A_1 a + A_2 + \int_0^1 A' \frac{\sin 2ya}{\sin a}\,dy < K;$$

and therefore the integral of the left-hand side, from $a = \pi/n$ to $a = \epsilon$, is also less than K.

This completes the proof that

$$\int_0^\epsilon |\phi(a, n)|\,da < K,$$

where K is independent of n.

We can now see that

$$\int_0^\epsilon F(a, x)\,\phi(a, n)\,da < K\,\overline{F(\epsilon, x)},$$

where $\overline{F(\epsilon, x)}$ is the upper limit of $F(a, x)$ in the interval $0 \leqslant a \leqslant \epsilon$. Now as $\epsilon \to 0$, $\overline{F(\epsilon, x)} \to 0$, and this convergence to the limit is, moreover, uniform in any interval at every point of which $f(x+0)+f(x-0)$ exists, $f(x)$ being limited in the interval. Therefore the same is true also of

$$\int_0^\epsilon F(a, x)\,\phi(a, n)\,da.$$

Collecting our results, we see that

$$\lim_{n\to\infty} L_n(x) = \tfrac{1}{2}[f(x+0)+f(x-0)],$$

and that this convergence to the limit—*i.e.*, the summability (Ck)—is uniform in such an interval as has been specified. This proves the theorem.*

21. Fejér has also shown that, if $f'(x)$ satisfy the condition of integrability, then the series obtained by term-by-term differentiation of the Fourier's series for $f(x)$ is summable $(C1)$ with a sum equal to $f'(x)$, save at $\pm\pi$, if $f(x)$ has a finite discontinuity there. It is easily proved that the differentiated series has an index $\not> 0$.

For the Fourier's series for $f'(x)$ is

$$a'_0+\Sigma(a'_n \cos nx + b'_n \sin nx),$$

where
$$a'_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(a)\,da = \frac{q-p}{2\pi},$$

$$a'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(a)\cos na\,da = \frac{q-p}{\pi}+nb_n,$$

$$b'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(a)\sin na\,da = -na_n,$$

where p, q are the limiting values of $f(x)$ at $-\pi$, π.

Thus $$f'(x) = \frac{q-p}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{q-p}{\pi}\cos nx + n(b_n\cos nx - a_n \sin nx)\right].$$

Now the series $\frac{q-p}{\pi}[\tfrac{1}{2}+\Sigma\cos nx]$ may very easily be shewn† to have index 0 and sum 0 save at the points $x = 0, 2\pi, \ldots$. Hence, by subtraction it follows that

$$f'(x) = \Sigma n(b_n \cos nx - a_n \sin nx),$$

---

* This theorem has been proved in an entirely different way by Dr. Riesz (*Comptes Rendus*, November 22, 1909), as was pointed out to me by Mr. Hardy.

In a paper shortly to be published in the *Quarterly Journal of Mathematics*, and entitled "Notes on the General Theory of Summability, with applications to Fourier's and other Series," I have proved the same theorem (with various others) as a particular example of a systematic theory.

† This is a very special case of the theorems of §§ 25, 26, but of course a much simpler proof can be given in this particular case.

except at 0 and 2π, if $q \neq p$, and that this series has index $\not> 0$; and this series is the one got by term-by-term differentiation of the Fourier's series for $f(x)$.

22. In Dr. Bromwich's *Infinite Series*, it is proved that the series $1^s-2^s+3^s-\ldots$, where s is integral, is summable $(C\,\overline{s+1})$. The method there employed in the proof may easily be adapted to shew that the series is *finite* (Cs). It therefore seems probable that the index of the series is s unattained, and that this is also true even when s is not integral. This theorem seems by no means easy to establish directly; but the following chain of argument, based on theorems already published* (without proofs) by M. Bohr and Dr. Marcel Riesz, appears to meet the case.

M. Bohr has shewn that if Σu_n is finite (Cs), s being integral, then $\Sigma u_n/n^a$ is summable (Cs), if $a > 0$. Next, Dr. Riesz has proved that if Σa_n is summable (Cs), s being integral, then $\Sigma a_n/n^k$ (where $k > 0$, integral or not) is summable $(C\,\overline{s-k})$. Hence $1^p-2^p+3^p-\ldots$, where

$$p = s-a-k,$$

is summable $(C\,\overline{s-k})$. Therefore, r being any positive number, integral or not, an integer s, and numbers k and a, can be so chosen that $s-k-a=r$, whence it follows that $1^r-2^r+3^r-\ldots$ is summable $(C\,\overline{r+a})$, where a is only subject to the condition of being positive, however small. Hence the index of the series is n, and it is evidently (by § 11) unattained.

By the theorem of § 18 the same result holds good also when s is negative, the series then being convergent.

Though we are compelled to adopt the above indirect proof to establish the theorem that the index of $1^s-2^s+3^s-\ldots$ (s positive or negative) is s, when Cesàro's method of summation is used, it is not very difficult to prove the same result directly when, instead of Cesàro's, we use Dr. Riesz's method of summation in the particular case $\lambda(n) = n$; and, as we have already mentioned, the two methods are coextensive (§ 3).

The direct proof is given in § 26 of Part III of this paper. It applies not only to convergent series (when s is negative), but also to all values of s, integral or not, positive or negative, real or complex.

Moreover, the proof is easily modified, as is there explained, so as to apply to the case of the more general series $\Sigma n^s e^{ai n}$, where $0 < a < 2\pi$.

* See (8) and (9) in the list of references.

We thus find that, for all values of s, the series

$$\Sigma n^s \cos na,$$

$$\Sigma n^s \sin na,$$

are summable (R, n, s') if $s' > s$, and $0 < a < 2\pi$.

Further, it is readily proved that for values of a such that

$$|\sin a| > \epsilon > 0,$$

ϵ being any fixed positive quantity, these series are uniformly summable with respect to a.

If we care to consider complex values of s', corresponding to complex orders of summability, the theorems still hold good provided that the real part of s' is greater than the real part of s.

PART III.

Convergent Series.

23. It seems advisable here to say a few words on the use and meaning of the index of a convergent series. Absolutely convergent series are usually regarded as being essentially more convergent than conditionally or semi-convergent series (as the very names imply). But, as we shall shew, a series may be absolutely convergent and yet have index 0 (attained, of course), so that a semi-convergent series may have a lower index than an absolutely convergent series. The index only measures the evenness and the rate of the approach of the partial sum s_n to its limit.

My original object in considering the summability of convergent series was to endeavour to prove the multiplication theorems for an absolutely convergent series with a like series or with a semi-convergent series, as particular cases of the general multiplication theorem of § 10. But it is evident that the latter theorem would then require the index of an absolutely convergent series to be -1, and § 11 shews that this is not usually the case. Thus in the case of an absolutely convergent series a knowledge of its index seems to be of little interest; in the case of semi-convergent series, however, such knowledge does add to our insight into the non-convergence of the product of two such series. In his original memoir, Cesàro shewed that the product is at most simply indeterminate; if we know the indices r and s, we know also that the index of their product is at most $r+s+1$, so that if $r+s < 1$, the product series is convergent. It must, however, be noted that if r or s be < -1, on account of the limitations of the multiplication theorem, it must be replaced by -1.

24. The results of § 18 and § 23 give, in combination, the result that the index of $1^{-s}-2^{-s}+3^{-s}-\dots$ is $-s$ unattained, since if $-s+1 = r > 0$, $1^r-2^r+3^r-\dots$ has index 0, and therefore

$$1^{r-1}-2^{r-1}+3^{r-1}-\dots \equiv 1^{-s}-2^{-s}+3^{-s}-\dots$$

has index $r-1$; if $-s+1 < 0$, $-s+m$ can similarly be taken (in integral), the theorem of § 18 being repeatedly applied.

By § 10, taken in conjunction with this, certain multiplication theorems due to Cajori and Pringsheim follow immediately; as, for instance, that the product of $1^{-r}-2^{-r}+3^{-r}-\dots$ and $1^{-s}-2^{-s}+3^{-s}-\dots$, where $r < 1$, $s < 1$, is a convergent series provided $r+s > 1$; for the index of the product series is not greater than $-r-s+1 < 0$. More generally, the continued product of n series of the same type, with indices $-r_1, -r_2, \dots, -r_n$, each r being $\leqslant 1$, is convergent provided that

$$r_1+r_2+\dots+r_n > n-1.$$

When $r_1 = r_2 = \dots = r_n$, we have another of Cajori's theorems, viz., that the n-th power of the series $1^{-s}-2^{-s}+3^{-s}-\dots$ is convergent, provided $s > 1-1/n$.

25. We now proceed to consider the summability (R, n, r) of the series $1^s-2^s+3^s-\dots$.

(*a*) First we shall consider convergent series, *i.e.*, series for which s is negative; or rather, for ease in writing, we shall deal with the series $1^{-s}-2^{-s}+3^{-s}-\dots$, where s is positive. Denoting

$$\sum_{\nu=0}^{[n]} u_\nu \left(1-\frac{\nu}{n}\right)^{-r}$$

by $\Sigma_n^{(-r)}$, we have

$$\Sigma_n^{(-r)} = \frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \int_0^\infty x^{s-1} y^{r-1} \left[\sum_{m=1}^{[n]} (-1)^{m-1} e^{-mx} e^{-(n+1-m)y}\right] dx\,dy,$$

where a product of two repeated absolutely convergent integrals has been transformed into a double integral; this change is legitimate. The above expression is arrived at by applying the well known formula

$$\Gamma(s)\, m^{-s} \int_0^\infty e^{-mx} x^{s-1} dx,$$

to the expression of u_ν and $(n-\nu)^{-r}$.

Continuing, we have

$$\Sigma_n^{(-r)} = \frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \int_0^\infty x^{s-1} y^{r-1} \frac{e^{-x} e^{-y}}{e^{-x}+e^{-y}} [e^{-ny} + (-1)^{[n]-1} e^{-nx}]\, dx\, dy$$

$$= \frac{n^r}{\Gamma(r)\,\Gamma(s)} \left\{ \int_0^\infty \left[\int_0^\infty \frac{x^{s-1} e^{-x} dx}{1+e^y e^{-x}} \right] y^{r-1} e^{-ny} dy \right.$$

$$\left. + (-1)^{[n]-1} \int_0^\infty \left[\int_0^\infty \frac{y^{r-1} e^{-y} dy}{1+e^x e^{-y}} \right] x^{s-1} e^{-nx} dx \right\},$$

changing to repeated integrals again. Now

$$\int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^y e^{-x}}\, dx \equiv f_s(y) < \Gamma(s),$$

whatever the value of y (> 0), and

$$\int_0^\infty \frac{y^{r-1} e^{-y}}{1+e^x e^{-y}}\, dy \equiv f_r(x) < \Gamma(r),$$

whatever the value of x (> 0). Thus

$$\Sigma_n^{(r)} = \frac{n^r}{\Gamma(r)\,\Gamma(s)} \left[\int_0^\infty f_s(y)\, y^{r-1} e^{-ny}\, dy + (-1)^{[n]-1} \int_0^\infty f_r(x)\, x^{s-1} e^{-nx}\, dx \right].$$

Now
$$\frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty f_s(y)\, y^{r-1} e^{-ny} dy < 1,$$

whatever be the value of n. If we write it in the form

$$\frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^{y/n} e^{-x}}\, dx \right] y^{r-1} e^{-y}\, dy,$$

we see that it steadily increases with n, and hence tends to a definite limit S, say. Again,

$$\frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{y^{r-1} e^{-y}}{1+e^x e^{-y}}\, dy \right] x^{s-1} e^{-nx}\, dx$$

$$= \frac{n^{r-s}}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{y^{r-1} e^{-y}}{1+e^{x/n} e^{-y}}\, dy \right] x^{s-1} e^{-x}\, dx.$$

As before, this repeated integral tends to a finite limit as $n \to \infty$, and consequently the whole of the last expression $\to \infty$, S, or 0 according as r is $>$, $=$, or $< s$. Thus, if $r > s$, $\Sigma_n^{(-r)}$ is infinitely oscillatory, while the series is finite $(R, n, -s)$, and summable $(R, n, -r)$ when $r < s$. The index of summability (Rn) is therefore $-s$, and the sum is

$$\lim_{n\to\infty} \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{x^{s-1} e^{-x} dx}{1+e^{y/n} e^{-x}}\right] y^{r-1} e^{-y} dy$$

$$= \lim_{n\to\infty} \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty y^{r-1} e^{-y} f_s\left(\frac{y}{n}\right) dy.$$

Choose Y so that $$\int_Y^\infty y^{r-1} e^{-y} dy < \epsilon,$$

ϵ being an arbitrarily small positive number. Then

$$\frac{1}{\Gamma(r)\,\Gamma(s)} \int_Y^\infty y^{r-1} e^{-y} f_s\left(\frac{y}{n}\right) dy < \frac{\epsilon}{\Gamma(r)}.$$

Again, N_0 can be so chosen that for $n > N_0$, $0 \leqslant y \leqslant Y$,

$$\left| f_s\left(\frac{y}{n}\right) - \int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^{-x}} dx \right| < \epsilon.$$

Thus, for $n > N_0$,

$$\left| \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty y^{r-1} e^{-y} f_s\left(\frac{y}{n}\right) dy - \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^{-x}} dx \right| < \epsilon \left(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(s)}\right).$$

Hence the " sum " of the series is

$$\frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^{-x}} dx,$$

which is readily seen to equal the ordinary value of the series.

26. (*b*) We now pass on to the more difficult case when s is not negative. By using contour integrals the same method may be extended to quite general values of s (real or complex, integral or not). Instead of using the formula (in which s is supposed negative)

$$\Gamma(-s)\, m^{+} = \int_0^\infty e^{-mx} x^{-s-1} dx,$$

we employ the expression

$$\Gamma(s) \frac{i}{2\pi} \int_C (-x)^{-s-1} e^{-mx} dx$$

for m^s. The integral is supposed to be taken along a curve commencing at $+\infty$, circulating round the origin in the counter-clockwise direction, and returning again to $+\infty$; and $(-x)^{-s}$ is to mean $e^{-s\log(-x)}$, where the real value of $\log(-x)$ is to be taken when x is negative, and the logarithm is to be rendered one-valued by the stipulation that the variable is not to cross the real axis at any point on the positive side of the origin.

All the integrals being absolutely convergent, on account of the presence of e^{-x} or e^{-y} in the integrand, the transformations used in the first part (*a*) of the investigation are valid when s has any value. It follows, by making trifling modifications in the preceding work, that quite generally the series

$$1^s - 2^s + 3^s - \dots$$

is summable (R, n, r), provided that

$$R(r) > R(s);$$

R here means "real part of." And the "sum" is

$$\Gamma(s)\frac{i}{2\pi}\int_C \frac{(-x)^{-s-1}e^{-x}}{1+e^{-x}}\,dx.$$

The existence and position of the lines of summability which were introduced by Dr. H. Bohr* are rendered very evident for the case of the series considered, in the last portion of the proof.

(*c*) A slight further modification of the proof suffices to deal with the more general series

$$\Sigma n^s e^{ain}.$$

If s is negative and equal to $-s'$ (say), the expression for $\Sigma_n^{(-r)}$ which was found in part (*a*) of this section becomes

$$\frac{n^r}{\Gamma(r)\Gamma(s')}\left\{\int_0^\infty\left[\int_0^\infty \frac{x^{s'-1}e^{-(x-ai)}}{1-e^{y}e^{-(x-ai)}}\,dx\right]y^{r-1}e^{-ny}\,dy\right.$$

$$\left.-\int_0^\infty\left[\int_0^\infty \frac{y^{r-1}e^{-y}}{1-e^{x-ai}e^{-y}}\,dy\right]x^{s'-1}e^{-n(x-ai)}\,dx\right\}.$$

Provided that $|\sin a| \geqslant \epsilon > 0$, this may be treated as in (*a*), and the

* See reference 8 in the list of memoirs appended to § 2.

series thus shewn to be summable $(R, n, -r)$, provided $r < s$, with the sum

$$\frac{e^{ai}}{\Gamma(s')}\int_0^{\infty}\frac{x^{s'-1}e^{-x}\,dx}{1-e^{-(x-ai)}}.$$

Moreover, for all such values of a, the convergence of $\Sigma_n^{(-r)}$ to its limit is uniform, *i.e.*, the series is uniformly summable $(R, n, -r)$ with respect to a.

In the general case, when s is not negative, for the same range of a corresponding modifications of the proof indicated in part (*b*) of this section may be made, and are sufficient to prove the general results relating to the series

$$\Sigma n^s \cos na,$$

$$\Sigma n^s \sin na,$$

which were stated at the end of § 23.

The expression to be considered in this last and most general case, is easily seen to be

$$-\frac{\Gamma(r)\,\Gamma(s)}{4\pi^2 n^r}\left\{\int_{C'}\left[\int_C \frac{(-x)^{-s-1}e^{-(x-ai)}}{1-e^{y}e^{-(x-ai)}}\,dx\right](-y)^{-r-1}e^{-ny}\,dy\right.$$

$$\left.-e^{nai}\int_{C'}\left[\int_C \frac{(-y)^{-r-1}e^{-y}}{1-e^{x-ai}e^{-y}}\,dy\right](-x)^{-s-1}e^{-nx}\,dx\right\},$$

where C is a contour of the kind described in the x-plane, and C' is a similar contour in the y-plane. The second term is oscillatory, as n increases, and the condition that it shall vanish is that

$$R(r) > R(s), \qquad 0 < a < 2\pi.$$

As before, this repeated integral tends to a finite limit as $n \to \infty$, and consequently the whole of the last expression $\to \infty$, S, or 0 according as r is $>$, $=$, or $< s$. Thus, if $r > s$, $\Sigma_n^{(-n)}$ is infinitely oscillatory, while the series is finite $(R\,\overline{-s})$, and summable $(R\,\overline{-r})$ where $r < s$. Thus the index of summability by Riesz's method is $-s$, and the sum is

$$\lim_{n\to\infty}\frac{1}{\Gamma(r)\,\Gamma(s)}\int_0^{\infty}\left[\int_0^{\infty}\frac{x^{s-1}e^{-x}}{1+e^{y/n}e^{-x}}\,dx\right]y^{r-1}e^{-y}\,dy,$$

for $0 < r < s$.

27. Some little attention has been paid to the effect on the summability and the "sum" (by Cesàro's and other methods, such as Borel's) of the

insertion of zero terms between the original terms of a series. For brevity it is convenient to speak of such insertion as "dilution" (by an obvious analogy). Such dilution may be termed *uniform* if between every pair of original terms of a series is placed a constant number of zero terms, and non-uniform when this is not the case.

Non-uniform dilution may destroy the property of Cesàro-summability of a series, as is shewn by the example (due to Mr. Hardy) quoted in § 13. Here the series $1-1+1-\ldots$ is rendered non-summable by suitable dilution. Non-uniform dilution, however, need not affect the summability or the sum. An example of this is given in Bromwich's *Infinite Series*, p. 388, Ex. 1. An example of how what we may term semi-uniform dilution may affect the sum of a series is given on p. 263 of the same treatise.

Uniform dilution, on the other hand, can affect neither the sum nor the summability of a series. For it is not difficult to prove that if Σu_n is summable (Cr), where r is integral, then the uniformly diluted series Σv_n, obtained by inserting $(m-1)$ zero terms between each original pair, is also summable (Cr). A proof can easily be based on equation (2) of § 124 of Dr. Bromwich's treatise. Assuming this theorem on the summability, the fact that

$$\Sigma v_n x^n = \Sigma u_n x^{mn},$$

and

$$\Sigma u_n x^n,$$

have the same limiting value as $x \to 1$, shews that the two sums are the same.

Though I have not troubled to write out a formal proof, the above theorem on summability of a diluted series can probably be extended to the case when r is not integral.

For further remarks on the summability of diluted series we may refer to § 29 of the paper (15) cited in § 2.

28. THEOREM.—*An absolutely convergent series Σu_n, such that $nu_n \to 0$ as $n \to \infty$, has an index of summability $\not> -1$.*

Evidently all absolutely convergent series of terms arranged in descending order of magnitude come under this theorem. The proof is not difficult, but as the theorem holds not only for Cesàro's method of summation, but also for Dr. Riesz's much more general method, a general proof will be given.

To make the one proof suffice for Cesàro's and Riesz's methods, we shall consider the limit of

$$\Sigma_n^{(r,\lambda)} = \sum_{m=1}^{n} u_m p_{n-m} \left(1 - \frac{\lambda_m}{\lambda_n}\right)^r,$$

where p_{n-m} is unity if Riesz's mode of summation be considered, while it tends to unity as $n \to \infty$ if Cesàro's mode be used (in which case $\lambda_m = m$).

Consider the first θn terms, where $0 < \theta < 1$; by Tannery's theorem their sum tends to the sum of the series, as $n \to \infty$. Now the p's are always finite, and moreover $nu_n \to 0$ as $n \to \infty$, if μ is the function inverse to λ, *i.e.*, if

$$\lambda_m = M, \quad m = \mu(M).$$

Hence the sum of the remaining terms of $\Sigma_n^{(r,\lambda)}$ is less than

$$K\epsilon_n \int_\theta^1 f(x)(1-x)^r \frac{dx}{x};$$

here K denotes a constant independent of n, $\epsilon_n \to 0$ as $n \to \infty$, and

$$f(x) = \frac{M}{\mu(M)} = \frac{\lambda_m}{m}, \quad x = \frac{M}{N}.$$

Thus, if $\overline{\lim\limits_{n\to\infty}} (\lambda_m/m)$ be finite, the sum of the remaining terms of $\Sigma_n^{(r,\lambda)}$ tends to zero as $n \to \infty$, provided that $r > -1$; the result follows.

It may be noticed here that the same theorem holds for summable integrals.

If the condition $nu_n \to 0$ as $n \to \infty$ is not satisfied, by § 11 the index must be $\geqslant -1$. It is evident that we can so dilute an absolutely convergent series as to make the transformed series* $\Sigma u'_n$ such that $n^k u'_n$ does not tend to 0 as $n \to \infty$, for any positive value of k whatever. The transformed series, being convergent, has by the theorem of § 11 the index 0 attained.

The general multiplication theorem of § 10 would only give

$$0+0+1 = 1$$

as the upper limit of the possible values of the index of the product series formed from two such transformed absolutely convergent series. By Cauchy's theorem it is thus evident that in this case the upper limit so given is greater than the actual value of the index by at least unity.

In conclusion, we may remark that series may be found with as large

* Choose any sequence of positive numbers $k_0, k_1, k_2, \ldots$ tending to 0, and any sequence of positive integers $b_0, b_1, b_2, \ldots$, such that $b_n^{k_n} u_n \to \infty$ as $n \to \infty$. Then dilute the series so that $u'_{b_n} = u_n$, all other values of u'_n being 0.

a negative index as we please, or even with $-\infty$ for index. It is not difficult to prove that if $|n^t u_n| < K$ for all values of n, where $t > 1$, the index of Σu_n is $\not> -t$; while the geometrical series $1-x+x^2-\ldots$, where $|x| < 1$, has $-\infty$ for its index, *i.e.*, is summable $(C\overline{-k})$ however large k may be.

PART IV.

Summable Integrals.

29. The extension of Cesàro's methods to the summation of infinite integrals was first considered by Mr. Hardy;* only a first "mean" of an integral was there taken, however. Very shortly afterwards Dr. C. N. Moore published a paper† on the same subject—also considering only summation $(C1)$—in which he proved many properties of integrals so summable; in particular he dealt with the introduction of convergence factors into integrals summable $(C1)$.

The general extension is obvious. Thus an integral is said to be summable (Cr) if the limit as $x \to \infty$ of

$$\frac{r!}{x^r}\int_a^x\int_a^{a_1}\int_a^{a_2}\ldots\int_a^{a} f(\xi)\,d\xi\,da_r\,da_{r-1}\ldots da,$$

exists and is finite; r is necessarily a positive integer here. The above repeated integral can immediately be transformed into

$$\int_a^x f(\beta)\left(1-\frac{\beta}{x}\right)^r d\beta,$$

which is evidently analogous to Dr. Riesz's method of summation of series, for the case $\lambda_n = n$. (Mr. Hardy has considered the integral analogue of Dr. Riesz's most general method of summation.‡) In the latter form r need not be integral, and consequently this form is adopted as the basis of the general theory of summable integrals.

30. I shall first state the theorem for integrals analogous to Cesàro's

* *Quarterly Journal of Mathematics*, Vol. 35, p. 54.

† *Trans. Amer. Math. Soc.*, Vol. 8, 1907.

‡ I think it extremely probable that Dr. Riesz has himself considered the same problem, though I have not seen any reference to it in his published works.

theorem for sequences, which was used in § 8.

If $\dfrac{f(x)}{x^r} \to F$, $\dfrac{g(x)}{x^s} \to G$, *as* $x \to \infty$, *then provided that* $r > -1$, $s > -1$,

$$\lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_a^x f(t)\, g(x-t)\, dt,$$

is equal to
$$\frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, FG.$$

Here $f(x)$ and $g(x)$ need not be continuous. As far as I am aware, the above theorem is new. Its proof presents no difficulty, and will not be set out here.

The first theorem required in this subject is the one analogous to that of § 8, and which shews, of course, that in the case of convergent integrals the condition of consistency is fulfilled.

If the integral $\int_a^\infty \phi(x)\, dx$ *is summable* (Cr), *with sum* S, r *being any real number* > -1, *integral or not, then it is summable* (Cr') *with the same sum, provided that* $r' > r$.

This theorem has been established independently by Mr. Hardy for $r > 0$, by a somewhat similar method of proof.

Write $r' = r+s+1$, so that $s > -1$. In the analogue of Cesàro's theorem, substitute

$$f(x) = \int_a^x \phi(\beta)\,(x-\beta)^r\, d\beta,$$

$$g(x) = x^s.$$

Then $f(x)/x^r \to S$, $g(x)/x^s \to 1$, as $x \to \infty$. Hence

$$\frac{1}{x^{r+s+1}} \int_a^x (x-a)^s \int_a^a \phi(\beta)\,(a-\beta)^r\, d\beta\, da,$$

tends as $x \to \infty$ to
$$\frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, S.$$

If $\phi(\beta)$ is continuous we may evidently invert the order of integration, and obtain

$$\frac{1}{x^{r+s+1}} \int_a^x \phi(\beta) \int_\beta^x (a-\beta)^r\,(x-a)^s\, da\, d\beta$$
$$= \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, \frac{1}{x^{r+s+1}} \int_a^x \phi(\beta)\,(x-\beta)^{r+s+1}\, d\beta.$$

Hence $$S = \lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_a^x \phi(\beta)(x-\beta)^{r+s+1} d\beta.$$

Since $r+s+1 = r'$, this proves the theorem.

31. The well known theorems of Cauchy, Abel, Mertens, and Cesàro, on the multiplication of series, have interesting analogues for integrals. Mr. Hardy has shewn* that if $a(x)$, $b(x)$ be continuous functions of x, such that
$$A(x) = \int_0^x a(x)\,dx, \qquad B(x) = \int_0^x b(x)\,dx,$$

have definite limits as $x \to \infty$, then if
$$c(x) = \int_0^x a(u)\,b(x-u)\,du,$$

the integral $$C(x) = \int_0^x c(x)\,dx,$$

is at any rate simply summable, if not convergent.

This theorem is a particular case of the following:—*if* $\int_0^\infty a(x)\,dx$ *is summable* (Cr), *with sum* A, *and* $\int_0^\infty b(x)\,dx$ *is summable* (Cs), *with sum* B, *then if* $r > -1$, $s > -1$, *the integral* $\int_0^\infty c(x)\,dx$ *is summable* $(C\,\overline{r+s+1})$, *with sum* AB.

In the analogue of Cesàro's theorem, write
$$f(x) = \int_0^x a(u)(x-u)^r\,du,$$
$$g(x) = \int_0^x b(u)(x-u)^s\,du.$$

Then $$f(x)/x^r \to A, \qquad g(x)/x^s \to B.$$

Therefore $$\frac{1}{x^{r+s+1}} \int_0^x f(z)\,g(x-z)\,dz$$

has the limit $$\frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+1)}\,AB,$$

---

* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301.

as $x \to \infty$. Now

$$\int_0^x f(z)\, g(x-z)\, dz = \int_0^x \int_0^z a(y)(z-y)^r\, dy \int_0^{x-z} b(u)(x-z-u)^s\, du\, dz$$

$$= \int_0^x b(u)\, du \int_0^{x-u} a(y)\, dy \int_y^{x-u} (z-y)^r (x-z-u)^s\, dz,$$

changing the order of integration, which is evidently legitimate under the stated conditions.

Put $$\theta = \frac{z-y}{x-u-y}.$$

We get

$$\int_0^x f(z)\, g(x-z)\, dz = \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)} \int_0^x b(u)\, du \int_0^{x-u} a(y)(x-u-y)^{r+s+1}\, dy,$$

if $r > -1$, $s > -1$. Again invert the order of integration. Using our former result, we have

$$\lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_0^x (x-y)^{r+s+1}\, dy \int_0^y b(u)\, a(y-u)\, du$$

$$= \lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_0^x (x-y)^{r+s+1}\, c(y)\, dy$$

$$= AB,$$

which proves the theorem.