On the Analogues of the Nine-Points Circle in Space of Three Dimensions, and connected Theorems. By SAMUEL ROBERTS.

[*Read January* 12*th*, 1888.]

1. When one analogue to a plane theorem exists in space of three dimensions, we are apt to find others which have rival claims. This is so with respect to the nine-points circle in many of its relations to the associated triangle. For two spheres present themselves immediately as entitled *primâ facie* to the rank of analogues. The middle points of the sides of a triangle may be regarded either as the centres of circles described on the sides as diameters, or as the centroids of the sides. We may therefore consider, as corresponding to the nine-points circle, the sphere passing through the centres of the circles circumscribed about the faces of a tetrahedron (or, say, through the centres of the spheres circumscribed about the faces and having their planes respectively for diametral planes), or else the sphere passing through the centroids of the faces.

The difficulty, however, at once occurs, that the altitudes of a tetrahedron form four generators of the same system of a hyperboloid, and do not co-intersect, except when it degenerates to a cone. If, therefore, this common intersection be insisted on, we must forego obtaining any general analogue, or else subject the tetrahedron to conditions.

M. Prouhet, adopting the latter alternative, obtained an analogue in the case of the "orthogonal" tetrahedron, in which the altitudes pass through the same point. This sphere passes through the centroids and orthocentres of the faces. I shall hereafter refer particularly to a paper by Signor Carmelo Intrigila, in which he extends the analogy to the general tetrahedron. His sphere of twelve points passes through the centroids of the faces.

2. There appear to be some reasons for giving the first rank as an analogue to the sphere which passes through the centres of the circles circumscribed about the faces.

The orthocentre of a triangle possesses, in addition to the property from which it derives its name, a further characteristic which equally defines it, viz., it is the isogonal conjugate of the centre of the circumscribed circle with respect to the triangle. Moreover, in regard to the triangle the following theorems exist:—

(*a*) If a pair of points are isogonal conjugates, the orthogonal projections of the points on the sides lie on a circle.

This circle has for a diameter the major axis of the ellipse inscribed in the triangle, and having the points in question for its foci.

(*b*) If two circles are given, the centres of all circles which are bisected by either of the circles and orthogonally cut by the other, lie on a circle coaxal with the given circles, and whose centre is the middle point between their centres.

(*c*) If any circle be described about a given point as centre, and circles be described on the intercepts made by it on the sides of a triangle as diameters, the radical centre of the circles is the isogonal conjugate of the centre of the first-named circle with respect to the triangle.

(*d*) Combining these results, we see that, if a pair of points are isogonal conjugates with respect to a triangle, and two circles be described about them as centres, such that each cuts orthogonally the circles described on the intercepts made on the sides by the other as diameters, the six orthogonal projections of the points on the sides lie on a circle whose centre is the middle point between the centres of the first-named circles, and which is coaxal with them.

If one of the pairs of circles is circumscribed about the triangle, the other is the "polar" circle, and the derived circle is the nine-points circle.

The latter circle may therefore be described as the locus of the centres of circles which are bisected by one or other of the circumscribed and polar circles, and cut orthogonally by the other.

3. Now, the theorems (*a*), (*b*), (*c*), and (*d*) have close analogues in solid space. In fact, we only have to substitute "sphere" for "circle," "tetrahedron" for "triangle," "face" for "side," and adapt in minor ways the phraseology.

Thus the following theorem may be enunciated (*Educational Times*, August, 1887) :—(*D*) If a pair of points are isogonal conjugates with regard to a tetrahedron, and two spheres be constructed about them as centres such that either of the spheres cuts orthogonally the spheres constructed on the intercepts made on the faces by the other as diametral sections, the eight orthogonal projections of the pair of points on the faces lie on a sphere whose centre is the middle point between the centres of the first-named spheres, and which has a common section with them.

The derived sphere has for its diameter the major axis of the ellipsoid of revolution inscribed in the tetrahedron and having the pair of points for its foci (Neuberg, "Sur le Tétraèdre," t. XXXVII., *Mémoires publiés par l'Académie Royale de Belgique*, 1884).

Prof. Neuberg has further pointed out that the isogonal conjugate of the centre of the circumscribed sphere is the centre of the sphere inscribed in the tetrahedron formed by connecting its four orthogonal projections on the faces of the original tetrahedron (*Educational Times*, December, 1887).

Let A, B, C, D be the vertices of a tetrahedron, and let O_2 be the isogonal conjugate of O, the centre of the circumscribed sphere, and O_1 the middle point between O and O_2. Then, producing AO_1, BO_1, CO_1, DO_1, respectively, to A', B', C', D', so that $AO_1 = O_1A'$, $BO_1 = O_1B'$, &c., we get the vertices of another tetrahedron symmetric with, equal, and inversely homothetic to the original one, and which has O_2 for the centre of the circumscribed sphere, O being its isogonal conjugate with respect to the new tetrahedron.

Hence, by the symmetry of the figure, the sphere about O_1 as centre, according to the before mentioned conditions, is common to the two tetrahedrons, and the sixteen orthogonal projections of O, O_2 on the faces lie on the sphere about O_1 as centre. The sixteen points are the extremities of eight diameters of this sphere, and the line joining O or O_2 to a vertex of the tetrahedron is perpendicular to the plane through the orthogonal projections of O_2 or O, as the case may be, on the faces meeting at that vertex.

The sixteen points correspond to the twelve points determined on the nine-points circle of a triangle, by a precisely analogous construction, and comprising amongst them the nine-points from which the circle derives its name.

4. To bring out more clearly the analogies of these constructions, I have recourse to certain equations which retain marked geometrical characters. It will be convenient to collect in the first instance some results relating to a triangle.

Let the sides opposite the vertices A, B, C of a triangle be denoted by $\overline{bc}$, $\overline{ca}$, $\overline{ab}$ respectively; then in triangular coordinates the equation of a circle whose radius is ρ and the coordinates of whose centre are α_1, β_1, γ_1 is

$$\Sigma\, \overline{ab}^2\, \alpha\beta - \Sigma\left\{\alpha_1\left(\overline{ab}^2\beta + \overline{ac}^2\gamma\right)\right\}\Sigma\alpha + \left(\Sigma\,\overline{ab}^2\,\alpha_1\beta_1 + \rho^2\right)(\Sigma\alpha)^2 = 0,$$

where the symbol Σ denotes the summation of similar combinations of the quantities involved.

When the coordinates of any point are substituted for α, β, γ in the above left-hand expressions, the result is minus the square of the tangent from that point to the circle, or minus the power of the point with respect to the circle. If, then, T, T_1, T_2 are the tangents from the vertices A, B, C, we may write the equation in the form

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma T^2\alpha \,.\, \Sigma\alpha = 0.$$

Let t^2, t_1^2, t_2^2 be the powers of the vertices with respect to circles drawn on the opposite sides respectively as diameters. The equations of these circles will be

$$\Sigma \overline{ab}^2 \alpha\beta - t^2\alpha\,\Sigma\alpha = 0,\ \Sigma \overline{ab}^2 \alpha\beta - t_1^2\beta\,\Sigma\alpha = 0,\ \Sigma \overline{ab}^2 \alpha\beta - t_2^2\gamma\,\Sigma\alpha = 0 \ldots(1).$$

The equation of the circle cutting them orthogonally is

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma\left(\frac{1}{t^2\Sigma\frac{1}{t^2}}(\overline{ab}^2\beta + \overline{ac}^2\gamma)\right)\Sigma\alpha + \frac{1}{\Sigma\frac{1}{t^2}}(\Sigma\alpha)^2 = 0.$$

This is the polar circle. The equation may be obtained by taking the Jacobian of the left-hand expressions of the equations (1), or perhaps more readily by first determining the coordinates of the centre of the circumscribed circle in the form $\frac{t^2}{2p^2}$, $\frac{t_1^2}{2p_1^2}$, $\frac{t_2^2}{2p_2^2}$, where p, p_1, p_2 are the altitudes of the triangle. The coordinates of the isogonal conjugate are therefore $1/t^2\Sigma\frac{1}{t^2}$, $1/t_1^2\Sigma\frac{1}{t^2}$, $1/t_2^2\Sigma\frac{1}{t^2}$.

The equation of the circumscribed circle, $\Sigma\overline{ab}^2\alpha\beta = 0$, may be written in the form

$$\Sigma \overline{ab}^2 \alpha\beta - \Sigma\left(\frac{t^2}{2p^2}(\overline{ab}^2\beta + \overline{ac}^2\gamma)\right)\Sigma\alpha + 2R^2(\Sigma\alpha)^2 = 0 \ \ldots\ldots(2).$$

The equation of the circle passing through the centres of the circles described on the sides as diameters (*i.e.*, the nine-points circle) is therefore

$$2\Sigma \overline{ab}^2 \alpha\beta - \Sigma\left(\frac{1}{t^2\Sigma\frac{1}{t^2}}(\overline{ab}^2\beta + \overline{ac}^2\gamma)\right)\Sigma\alpha + \frac{1}{\Sigma\frac{1}{t^2}}(\Sigma\alpha)^2 = 0 \ \ldots(3).$$

For the nine-points circle is coaxal with the circumscribed and polar circles, and its centre has for triangular coordinates

$$\tfrac{1}{2}\left(1/t^2\Sigma\frac{1}{t^2} + t^2/2p^2\right),\ \tfrac{1}{2}\left(1/t_1^2\Sigma\frac{1}{t^2} + t_1^2/2p_1^2\right),\ \tfrac{1}{2}\left(1/t_2^2\Sigma\frac{1}{t^2} + t_2^2/2p_2^2\right)$$

To obtain its equation, therefore, we have only to add the left-hand member of (2) to the left-hand member of the equation of the polar circle, and equate the result to zero. But in (2) the terms following $\Sigma\overline{ab}^2\alpha\beta$ are identically zero, and we may omit them.

We have also the following equalities

$$\Sigma \frac{t^2}{2p^2} = 1,$$

$$\frac{t^2}{2p^2}\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\overline{bc}^2 = \frac{t^2}{2p^2}\overline{ac}^2 + \frac{t_1^2}{2p_1^2}\overline{bc}^2 = \frac{t_1^2}{2p_1^2}\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\overline{bc}^2 = 2R^2,$$

which may be verified by means of

$$2t^2 = \overline{ab}^2 + \overline{ac}^2 - \overline{bc}^2, \quad 2t_1^2 = \overline{ab}^2 + \overline{bc}^2 - \overline{ac}^2, \quad 2t_2^2 = \overline{bc}^2 + \overline{ca}^2 - \overline{ab}^2.$$

By substituting $t^2/2p^2$, $t_1^2/2p_1^2$, $t_2^2/2p_2^2$ for α, β, γ in (2) and (3), we get

$$R^2 + R_1^2 - d^2 = 1/\Sigma\frac{1}{t^2} = 2\left(R_2^2 - \frac{d^2}{4}\right) \;\dots\dots\dots\dots\dots(4),$$

where R_1, R_2 are the radii of the circles, and d is the distance between the centres of the circumscribed and polar circles.

The equations of the polar and nine-points circles may also be written

$$\Sigma\overline{ab}^2\alpha\beta - \Sigma t^2\alpha\,\Sigma\alpha = 0, \quad 2\Sigma\overline{ab}^2\alpha\beta - \Sigma t^2\alpha\,\Sigma\alpha = 0;$$

from which we get

$$R^2 - d^2 - 3/\Sigma\frac{1}{t^2} = R^2 \;\dots\dots\dots\dots\dots\dots\dots\dots\dots(5).$$

Eliminating $\Sigma\frac{1}{t^2}$ and d^2, we get

$$R^2 = 4R_2^2,$$

and eliminating R and $\Sigma\frac{1}{t^2}$,

$$R^2 + 2R_1^2 = d^2,$$

lastly,

$$\Sigma\frac{1}{t^2} + \frac{1}{R_1^2} = 0.$$

5. Now, referring to a tetrahedron and tetrahedral coordinates, we find precisely similar forms. The equation of a sphere whose radius is ρ, and the coordinates of whose centre are α_1, β_1, γ_1, δ_1, is

$$\Sigma\,\overline{ab}^2\alpha\beta - \Sigma\left\{\alpha_1\left(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta\right)\right\}\Sigma\alpha + \left(\Sigma\overline{ab}^2\alpha_1\beta_1 + \rho^2\right)(\Sigma\alpha)^2 = 0,$$

where $\overline{ab}$ denotes the edge connecting the vertices A, B of the tetrahedron $ABCD$, and so on.

If T, T_1, T_2, T_3 are the tangents from the vertices respectively to the sphere, its equation may be written

$$\Sigma\,\overline{ab}^2\alpha\beta - \Sigma\left(T^2\alpha\right)\Sigma\alpha = 0.$$

Denoting the tangents from the vertices to the spheres having for diametral sections the circles circumscribed about the opposite faces by t, t_1, t_2, t_3, the equations of those spheres are

$$\Sigma\,\overline{ab}^2\,\alpha\beta - t^2\alpha\,\Sigma\alpha = 0, \quad \Sigma\,\overline{ab}^2\,\alpha\beta - t_1^2\beta\,\Sigma\alpha = 0, \quad \Sigma\,\overline{ab}^2\,\alpha\beta - t_2^2\gamma\,\Sigma\alpha = 0,$$
$$\Sigma\,\overline{ab}^2\,\alpha\beta - t_3^2\delta\,\Sigma\alpha = 0.$$

And the equation of the sphere cutting these orthogonally (the Jacobian of the left-hand expressions equated to zero) is

$$\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\left(\frac{1}{t^2\,\Sigma\,\dfrac{1}{t^2}}\,(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta)\right)\Sigma\alpha + \frac{1}{\Sigma\,\dfrac{1}{t^2}}\,(\Sigma\alpha)^2 = 0 \ldots (6).$$

The equation of the circumscribed sphere $\Sigma\,\overline{ab}^2\,\alpha\beta = 0$ is identical with

$$\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\left(\frac{t^2}{2p^2}\,(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta)\right)\Sigma\alpha + 2R^2\,(\Sigma\alpha)^2 = 0,$$

R being the radius, p, p_1, p_2, p_3 being the altitudes of the tetrahedron, and the tetrahedral coordinates of the centre being $t^2/2p^2$, $t_1^2/2p_1^2$, &c., as I shall presently show.

Consequently, the equation of the sphere passing through the centres of the spheres having the circles circumscribed about the faces for diametral sectors is, as in the case of the triangle,

$$2\Sigma\,\overline{ab}^2\,\alpha\beta - \Sigma\left(\frac{1}{t^2\Sigma\,\dfrac{1}{t^2}}\,(\overline{ab}^2\beta + \overline{ac}^2\gamma + \overline{ad}^2\delta)\right)\Sigma\alpha + \frac{1}{\Sigma\,\dfrac{1}{t^2}}\,(\Sigma\alpha)^2 = 0 \ldots (7).$$

Also the following equalities hold—

$$\Sigma\,\frac{t^2}{2p^2} = 1,$$

$$\frac{t_1^2}{2p_1^2}\,\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\,\overline{ac}^2 + \frac{t_3^2}{2p_3^2}\,\overline{ad}^2 = \frac{t^2}{2p^2}\,\overline{ab}^2 + \frac{t_2^2}{2p_2^2}\,\overline{bc}^2 + \frac{t_3^2}{2p_3^2}\,\overline{bd}^2$$
$$= \frac{t^2}{2p^2}\,\overline{ac}^2 + \frac{t_1^2}{2p_1^2}\,\overline{bc}^2 + \frac{t_3^2}{2p_3^2}\,\overline{cd}^2 = \frac{t^2}{2p^2}\,\overline{ad}^2 + \frac{t_1^2}{2p_1^2}\,\overline{bd}^2 + \frac{t_2^2}{2p_2^2}\,\overline{cd}^2 = 2R^2.$$

By means of the general expression in tetrahedral coordinates for the distance between two points, we have

$$t_3^2 = \frac{\Sigma\,\overline{ad}^2\,\overline{bc}^2\,(\overline{ab}^2 + \overline{ac}^2 - \overline{bc}^2) - 2\,\overline{ab}^2\,\overline{bc}^2\,\overline{ac}^2}{\Sigma\,\overline{bc}^2\,(\overline{ab}^2 + \overline{ac}^2 - bc^2)}$$
$$= \frac{\overline{ad}^2\sin 2A + \overline{bd}^2\sin 2B + \overline{cd}^2\sin 2C - 4\Delta_3}{\sin 2A + \sin 2B + \sin 2C},$$

if A, B, C are the angles of the face ABC and Δ_3 is the area.

And there are corresponding expressions for t^2, t_1^2, t_2^2.

Let the faces opposite the vertices A, B, C, D of the tetrahedron of reference be denoted by Δ, Δ_1, Δ_2, Δ_3 respectively, and let V be its volume. Then

$$\Sigma \frac{t^2}{2p^2} = \frac{1}{18V^2}\left(t^2\Delta^2 + t_1^2\Delta_1^2 + t_2^2\Delta_2^2 + t_3^2\Delta_3^2\right).$$

But $$t^2 = \frac{1}{16\Delta^2}\left\{\Sigma\, \overline{ba}^2\, \overline{cd}^2\left(\overline{bc}^2 + \overline{bd}^2 - \overline{cd}^2\right) - 2\,\overline{bc}^2\,\overline{bd}^2\,\overline{cd}^2\right\},$$

&c. &c.,

and $$16\,\Sigma t^2\Delta^2 = 2\,.\,144V^2.$$

Hence $$\Sigma \frac{t^2}{2p^2} = 1.$$

Since the *four-plane* coordinates of the isogonal conjugate to the centre of the circumscribed sphere are as $\frac{p}{t^2}$, $\frac{p_1}{t_1^2}$, $\frac{p_2}{t_2^2}$, $\frac{p_3}{t_3^2}$, those of the centre in question are as $\frac{t^2}{p}$, $\frac{t_1^2}{p_1}$, $\frac{t_2^2}{p_2}$, $\frac{t_3^2}{p_3}$; and the *tetrahedral* coordinates are therefore $\frac{t^2}{2p^2}$, $\frac{t_1^2}{2p_1^2}$, $\frac{t_2^2}{2p_2^2}$, $\frac{t_3^2}{2p_3^2}$.

If, then, the radii of the spheres (6) and (7) are R_1, R_2, and d denotes the distance between the centres of the circumscribed sphere and (6), we have

$$R^2 + R_1^2 - d^2 = \frac{1}{\Sigma \frac{1}{t^2}} = 2\left(R_2^2 - \frac{d^2}{4}\right)$$

The equations of the spheres (6) and (7) may be written also

$$\Sigma\, \overline{ab}^2\, \alpha\beta - \Sigma\,(t^2\alpha)\,\Sigma\alpha = 0, \quad 2\,\Sigma\, \overline{ab}^2\, \alpha\beta - \Sigma\,(t^2\alpha)\,\Sigma\alpha = 0\,;$$

from which we derive the additional equation

$$R^2 - d^2 - \frac{4}{\Sigma \frac{1}{t^2}} = R_1^2.$$

These expressions differ in form from the expressions (4), (5), only by the substitution of the numerator 4 for 3 over $\Sigma \frac{1}{t^2}$ in the last equation. This slight difference, however, creates ultimately a fault in the general analogy of more importance. Eliminating $1/\Sigma \frac{1}{t^2}$ and

d^2, we get

$$12R_2^2 = 3R^2 + R_1^2.$$

Eliminating $1/\Sigma\frac{1}{t^2}$ and R_2^2, we get

$$3R^2 + 5R_1^2 = 3d^2;$$

and similarly $$\frac{1}{\Sigma\frac{1}{t^2}} + \tfrac{2}{3}R_1^2 = 0, \quad R^2 = 5R_2^2 + \frac{d^2}{4}.$$

6. It thus becomes necessary, in order to maintain the analogies, to admit the claim of the sphere passing through the centroids of the faces. Analogues, however, have different ranks, and I think the sphere we have been discussing is entitled to precedence. I propose to state, as fairly as I can, the case of the other analogue in the light of Signor Intrigila's paper "Sul Tetraedro" (*Rendiconti della Societá Reale di Napoli*, Anno 22, 1883).

The four altitudes p, p_1, p_2, p_3 of a tetrahedron $ABCD$ are four generators of the same system of a hyperboloid. Also the normals to the faces at the several orthocentres are generators of the other system belonging to the same hyperboloid. This theorem is attributed to Joachimstal (*Grunert*, t. xxxii., p. 109). Signor Intrigila determines by an easy geometrical construction the centre I of the hyperboloid.

If O, O_1 are the centres of the circumscribed sphere and the sphere through the centroids of the faces, he shows that I, O_1 and the centre of gravity G of the tetrahedron lie in one straight line, G being the middle point.* Moreover, I and G are the centres of direct and inverse similitude of the spheres. The modulus is 3. These relations are analogous to those of the circumscribed and nine-points circle, and do not hold in the previous case.

In tetrahedral coordinates, since $\frac{t^2}{2p^2}$, $\frac{t_1^2}{2p_1^2}$, &c. are the coordinates of the centre O, and those of G are $\frac{1}{4}$, $\frac{1}{4}$, &c., we find for the coordinates of O_1

$$\tfrac{1}{3}\left(1-\frac{t^2}{2p^2}\right), \quad \tfrac{1}{3}\left(1-\frac{t_1^2}{2p_1^2}\right), \text{ \&c.},$$

and for those of I,

$$\tfrac{1}{2}\left(1-\frac{t^2}{p^2}\right), \quad \tfrac{1}{2}\left(1-\frac{t_1^2}{p_1}\right), \text{ \&c.}$$

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* The author quotes Joachimstal as giving this result also (*Nouvelles Annales*, t. xviii., 1859, p. 266).

The corresponding coordinates in the case of the triangle are the coordinates of the centre of the nine-points circle,

$$\tfrac{1}{2}\left(1-\frac{t^2}{2p^2}\right),\quad \tfrac{1}{2}\left(1-\frac{t_1^2}{2p_1^2}\right),\ \&c.,$$

and those of the orthocentre

$$\left(1-\frac{t^2}{p^2}\right),\quad \left(1-\frac{t_1^2}{p_1^2}\right),\ \&c.$$

Signor Intrigila shows that the sphere which passes through the centroids of the faces also divides the distances of the vertices of the tetrahedron from the centre I of the hyperboloid into two parts whose ratio is 2, and passes through a point on each face, which is the harmonic conjugate of the orthocentre of the face with respect to the orthogonal projections thereon of the opposite vertex, and the centre I of the hyperboloid. Thus twelve noteworthy points are determined.

The paper of Signor Intrigila contains, beyond these results, several theorems of a more general character, but lying outside the field of analogy to the plane case. He also makes application of his results to equifacial and orthogonal tetrahedra, demonstrating in the latter case that his theorems relative to the general tetrahedron become identical with those of M. Prouhet, as indeed they evidently must since the sphere in question is determined by passing through the centroids of the faces.

7. In the case of the "orthogonal" tetrahedron, we find a third analogue of the nine-points circle distinct from the two already mentioned.

The equation of the polar sphere is

$$\Sigma\,\overline{ab}^2\,\alpha\beta-\Sigma\,T^2\alpha\,.\,\Sigma\,\alpha=0,$$

where $\Sigma\,T^2\alpha$ means, as before,

$$T^2\alpha+T_1^2\beta+T_2^2\gamma+T_3^2\delta,$$

and the coefficients are the powers of the vertices relative to the sphere, and where

$$\overline{ab}^2=T^2+T_1^2,\quad \overline{ac}^2=T^2+T_2^2,\quad \overline{ad}^2=T^2+T_3^2,\ \&c.,$$

which imply, of course, the usual relations

$$\overline{ab}^2+\overline{cd}^2=\overline{ac}^2+\overline{bd}^2=\overline{ad}^2+\overline{bc}^2=\Sigma\,T^2,$$

and further

$$2T^2=\overline{ab}^2+\overline{ac}^2-\overline{bc}^2=\overline{ab}^2+\overline{ad}^2-\overline{bd}^2=\overline{ac}^2+\overline{ad}^2-\overline{cd}^2,$$

&c. &c.

But these are the expressions determining the powers of the vertices of each face with respect to the circles drawn on the opposite sides as diameters. The sections of the polar sphere by the plane of the faces are, therefore, the polar circles of those faces. The equation

$$2\Sigma\overline{ab}^2\,\alpha\beta - \Sigma\, T^2\alpha\ \Sigma\alpha = 0$$

represents the sphere passing through the intersection of the polar and circumscribed spheres, and having its centre midway between their centres, *i.e.*, at the centre of gravity of the tetrahedron. The sections of this sphere by the planes of the face are the nine-points circles of the faces. Professor Wolstenholme has noted this and other properties of this sphere ["Exercices sur le Tétraèdre," *Nouvelles Annales* (2), x., 451, 452]. It is evidently a close analogue of the nine-points circle, although the centres of the polar sphere and the circumscribed sphere are not isogonal conjugates.

It is instructive to consider the following figure. If through any point P we draw lines AP, BP, CP, DP from the vertices of a tetrahedron, and produce them respectively to the points A', B', C', D', so that $AP = PA'$, &c., the last-named points are the vertices of an equal and inversely homothetic tetrahedron. If further, we make the condition that $B'C'$ shall meet AD in A'', $B'D'$ shall meet AC in B'', and so forth, the point P must be the common centroid of the two tetrahedrons, and the six points, say, A'', B'', C'', E'', F'', H'' can be connected in four ways with two corresponding vertices, as (A, A'), (B, B'), &c., so as to form parallelepipeds. If the tetrahedron is orthogonal, we may have $C''F''$, $H''B''$ perpendicular to $F''H''$, and $A''F''$, $E''B'$ perpendicular to $E''F''$, &c. Consequently, A'', B'', C'', E'', F'', H'' lie on a sphere, as before stated. But we may also regard the figure as *in plano*, or as the orthographic projection, and those six points lie on a conic whose centre is G. If the conic becomes a circle, it is in the projection, the nine-points circle of each of two projected faces, and the projections of the two vertices not on those faces will be the centres of the circumscribed circles and the orthocentres.

The imposition of conditions on the tetrahedron manifestly increases the number of analogues. We might establish another, for instance, by taking the isogonal conjugate of the intersection of the altitudes, constructing a sphere which should cut orthogonally the spheres having for diametral sections the circular intercepts on the faces made by the polar sphere. The interest, however, attaching to these analogies is much diminished by the limitations involved.

Thursday, February 9th, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Messrs. A. E. Hough Love and G. G. Morrice were admitted into the Society.

The following communications were made:—

Further remarks on the Theory of Distributions: Captain P. A. MacMahon, R.A.

The Free and Forced Vibrations of an Elastic Spherical Shell containing a given Mass of Liquid: A. E. H. Love, B.A.

On the Volume generated by a Congruency of Lines: R. A. Roberts, M.A.

On Isoscelians: R. Tucker, M.A.

The following presents were received:—

"Educational Times," for February, 1888.

"Bulletin de la Société Mathématique de France," Tome xv., No. 7.

"Beiblätter zu den Annalen der Physik und Chemie," Band xi., Stück 12; Band xii., Stück 1.

"Journal für die reine und angewandte Mathematik," Band cii., Heft 3.

"Annali di Matematica," Tome xv., Fasc. 4.

"Bollettino delle Pubblicazioni Italiane ricevute per Diritto di Stampa," Nos. 49 and 50; Index, Nos. 1 and 2; Firenze, 1888.

"Archives Néerlandaises des Sciences exactes et naturelles," Tome xxii., Liv. 2 and 3; Haarlem, 1887.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. iii., Fasc. 6, 7, and 8.

"Mémoires de la Société des Sciences physiques et naturelles de Bordeaux," Tome ii., Cahier 2; Tome iii., Cahier 1; Paris, 1886.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. viii., No. 2; Coimbra, 1887.

"Atti del Reale Istituto Veneto di Scienze, Lettere, ed Arti," Tomo v., Disp. 2 to 9.

"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde de Juin 1885 à Mai 1886," 8vo; Bordeaux, 1886.

"Elektrische Untersuchungen," von W. G. Hankel, Achtzehnte Abhandlung—"Fortsetzung der Versuche über das Elektrische verhalten der Quärz- und der Boracitkrystalle," 4to; Leipzig, 1887.

"Untersuchungen über die Papillae Foliatae et Circumvallatae des Kaninchen und Feldhasen," von Dr. Otto Drasch, 4to; Leipzig, 1887.

"Appendix to Mathematical Questions and Solutions from the 'Educational Times,'" Vol. xlv., by A. Mukhopâdhyây, M.A.

"On the Differential Equation of a Trajectory" (Reprinted from the "Journal of the Asiatic Society of Bengal," Vol. lvi., Part ii., No. 1, 1887), by A. Mukhopâdhyây, M.A.; Calcutta, 1887.

"On Monge's Differential Equation to all Conics" (Reprinted from the "Journal of the Asiatic Society of Bengal," Vol. lvi., Part ii., No. 2, 1887), by A. Mukhopâdhyây, M.A.; Calcutta, 1887.

"American Journal of Mathematics," Vol. x., No. 2; Baltimore, January, 1888.