AN EXTENSION OF BOREL'S EXPONENTIAL METHOD OF SUMMATION OF DIVERGENT SERIES APPLIED TO LINEAR DIFFERENTIAL EQUATIONS

By E. CUNNINGHAM.

[Received December 6th, 1904.—Read December 8th, 1904.]

THE object of the following discussion is to make somewhat more precise the connexion between Laplace's definite integral solution of linear differential equations with rational coefficients and of "rank"* unity in the neighbourhood of $z = \infty$ and the development of this solution in the form of a power series with an exponential factor. Poincaré has shewn† that these developments are "asymptotic expansions" of the integrals from which they are derived, and as such serve to calculate numerically the value of the integral for large values of the variable. But from the function theory point of view asymptotic expansions are of little value, inasmuch as they do not represent unique analytic functions. In fact, if we consider the integrals of a given differential equation, not only does an asymptotic expansion fail to represent a definite solution, but a definite solution has different expansions for different phases of the complex variable.

Now it seems clear that there should be some more definite relation between the divergent series and the integrals of the equation than the foregoing seems to suggest, and it appears that some approach to this connexion may be made on the lines of Borel's theory of *summable* (*sommable*) divergent series.‡

The divergent expansions we have to consider are of the form

$$e^{ax}x^{-\rho}\left\{u_0+\frac{u_1}{x}+\frac{u_2}{x^2}+\dots\right\}.$$

Leaving out of account for the present the exponential factor, we shall extend Borel's theory to include series of the form

$$x^{-\rho}\left\{u_0+\frac{u_1}{x}+\dots\right\},$$

* Forsyth, *Linear Differential Equations*, p. 271.

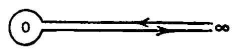
† *Acta Math.*, t. VIII., p. 296.

‡ E. Borel, *Leçons sur les Séries Divergentes*. pp. 97 *ff*.

ρ not being an integer. We require first a generalized form of the Γ function. It is well known that for all values of z

$$\Gamma(z) = \frac{i}{2\sin\pi z}\int e^{-t}(-t)^{z-1}\,dt,*$$

the integral being taken along a contour which goes from infinity along the axis of t to the origin, makes a small positive circuit round the origin, and returns to infinity along the axis. Integration along such a path as this will be denoted by the suffix A. The path may be shown diagrammatically by the figure



Integrating with respect to a along a like contour, we have

$$\int_A \frac{u_r a^{z+r}}{\Gamma(z+r+1)} e^{-a} da = (-1)^{-(z+r)}\{-2i\sin\pi(z+r+1)\}\,u_r$$
$$= e^{-\pi i(z+r+1)}\{e^{\pi i(z+r+1)} - e^{-\pi i(z+r+1)}\}\,u_r$$
$$= u_r(1-e^{-2\pi i z}),$$

or
$$u_r = (1-e^{-2\pi i z})^{-1}\int_A \frac{u_r a^{z+r}}{\Gamma(z+r+1)} e^{-a} da.$$

Consider now a series

$$u = u_0 + u_1 + u_2 + \ldots$$

which may be divergent; and let

$$u_z(a) = \frac{a^z u_0}{\Gamma(z+1)} + \frac{a^{z+1} u_1}{\Gamma(z+2)} + \ldots + \frac{a^{z+r} u_r}{\Gamma(z+r+1)} + \ldots.$$

Then we have the formal equation

$$\frac{1}{1-e^{-2\pi i z}}\int_A e^{-a} u_z(a)\,da = \left[\Sigma\int \frac{u_r a^{z+r}}{\Gamma(z+r+1)} e^{-a} da\right]\frac{1}{1-e^{-2\pi i z}} = \Sigma u_n.$$

Further
$$u_z\left(\frac{a}{x}\right) = \left(\frac{a}{x}\right)^z \frac{u_0}{\Gamma(z+1)} + \ldots,$$

and
$$\frac{1}{1-e^{-2\pi i z}}\int_A e^{-a} u_z\left(\frac{a}{x}\right) da = \Sigma\,\frac{u_n}{x^{n+z}} = x^{-z}\sum_0^\infty u_n x^{-n},$$

the equation being once more merely formal so far.

We proceed to show that, if we agree to call the definite integral on the left-hand side the sum of the divergent series on the right, Borel's propositions regarding the sums of divergent series are equally true of the more general series and their sums so defined.

* Whittaker's *Analysis*, p. 181.

We shall assume that z is not an integer; for otherwise the expression on the left is illusory. If z is a positive integer, Borel's method of summation is applicable without modification. If z is a negative integer, the series begins with a finite number of positive integral powers of x and the remainder of the series can be summed by Borel's method.

Assuming then that z is not an integer, let

$$x^{-z}\sum_0^\infty u_n x^{-n}$$

be a series diverging for all values of x, however large.

Let $f(\xi)$ denote the associated series

$$\xi^z \sum_0^\infty u_n \xi^n/\Gamma(z+n+1),$$

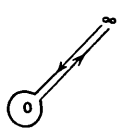
and let this series be convergent within a finite circle about $\xi = 0$; and, by continuation, define an analytic function, existing over the whole plane, isolated singular points excepted; and suppose that none of these singular points occur within a certain angle Θ bounded by two lines through $\xi = 0$ between which lies the real axis, nor within a circle of finite radius including $\xi = 0$. Further, suppose that a quantity k, real, finite, and positive, can be found such that

$$e^{-k\xi}\,|f^{(\lambda)}(\xi)|\ \ldots\ \lambda = 0, 1, \ldots$$

tends uniformly to zero as ξ becomes large within this sector.

Now
$$\int e^{-a} f\left(\frac{a}{x}\right) da = \int_B e^{-bx} f(b)\cdot\frac{db}{x},$$

where on the left the integration is taken as before, and on the right a straight line inclined to the real axis takes the place of that axis in the path of integration. Such integrations will in future be expressed by the suffix B. The path may be shown diagrammatically by the figure



The integral on the right exists if the straight line in question lies within the angle Θ; and, if this is so, that path may be taken along the real axis without changing the value of the integral. We must also have the real part of $x \geqslant k$. The existence of the former integral is contingent upon (a/x) lying within the angle Θ; a being real and positive, x must lie within an angle such that x^{-1} lies within Θ. Let the region which fulfils this condition, and also $R(x) \geqslant k$, be called $\bar{\Theta}$; in the future the variation of x will be supposed restricted to the region $\bar{\Theta}$.

The series $x^{-z}\sum_{n=0}^{\infty} u_n x^{-n}$ will be called "absolutely summable" within the region $\bar{\Theta}$, and either of the above written integrals will be denoted by

$$(1-e^{-2\pi i z})\,u,$$

u being called the "sum" of the above series.

It will now be shewn that the series

$$x^{-z}\sum_{n=1}^{\infty} u_n x^{-n},$$

obtained by omitting the first term of the previous series, is likewise absolutely summable within $\bar{\Theta}$, and has for sum $u-u_0x^{-z}$.

The associated series is here

$$\phi(a,\,x) = x^{-z}\left[\frac{u_1}{x}\,\frac{a^z}{\Gamma(z+1)}+\frac{u_2}{x^2}\,\frac{a^{z+1}}{\Gamma(z+2)}+\dots\right]$$

$$= \frac{\partial}{\partial a}\left(f\left\{\frac{a}{x}\right\}\right)-x^{-z}u_0\,\frac{a^{z-1}}{\Gamma(z)},$$

the series $\frac{\partial}{\partial \xi}\,\{f(\xi)\}$ being convergent within a finite circle, and giving by continuation throughout the plane the analytic function $f'(\xi)$. Hence, since $e^{-k\xi}\,|f'(\xi)|$, $e^{-k\xi}\,|f''(\xi)|$, ... tend uniformly to zero within Θ, the integral

$$\int_A e^{-a}\phi(a,\,x)\,da$$

exists, and the said series is absolutely summable within $\bar{\Theta}$.

Its sum

$$= \frac{1}{1-e^{-2\pi i z}}\int_A e^{-a}\left[\frac{\partial}{\partial a}\left(f\left\{\frac{a}{x}\right\}\right)-x^{-z}u_0\frac{a^{z-1}}{\Gamma(z)}\right]da$$

$$= -\,u_0x^{-z}+\frac{1}{1-e^{-2\pi i z}}\left[e^{-a}f\left(\frac{a}{x}\right)\right]_A+\int_A e^{-a}f\left(\frac{a}{x}\right)da$$

$$= u-u_0x^{-z}.$$

Conversely, if a series be absolutely summable within $\bar{\Theta}$, the series obtained by prefixing a term is so also; and its sum is the result of adding the term prefixed to the original sum. The proof of this is a natural extension of Borel's proof for the more limited definition, and need not be given in full.*

From these two propositions it follows immediately that the addition or subtraction of any finite number of terms at the beginning of a series

---

* *Leçons sur les Séries Divergentes*, pp. 101–2.

does not affect its summability; and, therefore, that the interchange of any finite number of terms in a series affects neither its summability nor its sum. Next it is clear that two series of the same index z or of indices differing only by integers may be added or subtracted term by term, the result being a series absolutely summable in the region common to the two series, its sum being the sum or difference of the sums of the two separately.

We may similarly extend Borel's proposition as to the multiplication of two such series. Let

$$x^{-\rho}\left\{u_0+\frac{u_1}{x}+\ldots\right\},\qquad x^{-\sigma}\left\{v_0+\frac{v_1}{x}+\ldots\right\}$$

be two series having a common region within which they are absolutely summable. Then, if

$$w_n = u_0 v_n+\ldots+u_n v_0,$$

the series

$$x^{-(\rho+\sigma)}\left\{w_0+\frac{w_1}{x}+\ldots\right\}$$

will be absolutely summable within that common region, and, if u, v, w denote the sums of the three series respectively, $w = uv$.

Letting $u(a)$, $v(b)$ denote the associated series

$$\left(\frac{a}{x}\right)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}+\frac{u_1}{\Gamma(\rho+2)}\left(\frac{a}{x}\right)+\ldots\right\},$$

$$\left(\frac{b}{x}\right)^{\sigma}\left\{\frac{v_0}{\Gamma(\sigma+1)}+\frac{v_1}{\Gamma(\sigma+2)}\left(\frac{b}{x}\right)+\ldots\right\},$$

the product of the sums of the series is given by

$$(1-e^{-2\pi i\rho})(1-e^{-2\pi i\sigma})uv = \int_A e^{-a}u(a)\,da\int_A e^{-b}v(b)\,db.$$

Inasmuch as $e^{-a}u(a)$ and $e^{-b}v(b)$ tend uniformly to zero, as a and b become infinite along the given contours, this product may be written as a double integral, viz.,

$$\int_A\int_A e^{-(a+b)}u(a)\,v(b)\,da\,db.$$

Call this integral W, and let the variables be changed to

$$c = a+b,\qquad \gamma = a-b.$$

Then

$$W=\int e^{-c}\left[\int u\left(\frac{c+\gamma}{2}\right)v\left(\frac{c-\gamma}{2}\right)\frac{d\gamma}{2}\right]dc.$$

The range of integration with respect to c and γ will be specified a little later.

Let $\int u\left(\frac{c+\gamma}{2}\right) v\left(\frac{c-\gamma}{2}\right) \frac{d\gamma}{2}$ be denoted by $\bar{w}(c)$; so that

$$W = \int e^{-c} \bar{w}(c)\, dc.$$

This integral from its formation is known to be valid.

Within the common circle of convergence of $u(a)$, $v(b)$ these two series may be multiplied together, giving an absolutely converging double series of powers of a and b. The typical term in the product is

$$\frac{u_r v_s a^{\rho+r} b^{\sigma+s}}{(x)^{\rho+r+\sigma+s}\Gamma(\rho+r+1)\Gamma(\sigma+s+1)}$$

or

$$\frac{u_r v_s (c+\gamma)^{\rho+r} (c-\gamma)^{\sigma+s}}{(2x)^{\rho+r+\sigma+s}\Gamma(\rho+r+1)\Gamma(\sigma+s+1)}.$$

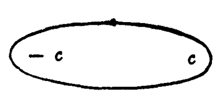
Call this term w_{rs}.

Now as to the paths to be described by the variables c, γ during the integration :—

For a given value of c, $= a+b$, γ describes a path from $-c$ to $+c$, a small positive circle round the latter point, returns along itself to $-c$, and closes the path by a small positive circle round $-c$.

The variation of c is then represented by the diagram used above :—

The γ contour may be represented diagrammatically by



and an integration along such a contour will be denoted by $\int_C$.

Supposing $|c|$ so small that the corresponding γ contour lies wholly within the region for which $\frac{1}{2}(c+\gamma)$ and $\frac{1}{2}(c-\gamma)$ lie within the circles of convergence of $u(a)$, $v(b)$, the integral $w(c)$ can be integrated term by term. The typical term is

$$\tfrac{1}{2}\int_C w_{rs}\, d\gamma = \frac{\frac{1}{2}u_r v_s}{(2x)^{\rho+r+\sigma+s}} \frac{1}{\Gamma(\rho+r+1)\Gamma(\sigma+s+1)} \int_C (c+\gamma)^{\rho+r}(c-\gamma)^{\sigma+s}\, d\gamma.$$

Put now $$c+\gamma = 2ct,$$

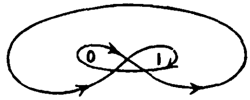
so that $$c-\gamma = 2c(1-t).$$

Then $\quad \int_C (c+\gamma)^{\rho+r}(c-\gamma)^{\sigma+s}\,d\gamma = \int_{C'} t^{\rho+r}(1-t)^{\sigma+s}\,dt\,(2c)^{\rho+r+\sigma+s+1},$

where in $\int_{C'}$ the contour is of the same kind as in $\int_{C}$, but the points $-c$ and c are replaced by 0 and 1. The integral last written is equal to

$$(2c)^{\rho+r+\sigma+s+1}\int\frac{t^{\rho+r}(1-t)^{\sigma+s}\,dt}{e^{2\pi i(\rho+r+\sigma+s)}-1}$$

taken along the contour represented diagrammatically by



and the integral in this expression is Pochhammer's generalization of the Eulerian function of the first kind.* It is equal to

$$-\frac{4\sin\{(\rho+r+1)\pi\}\,\sin\{(\sigma+s+1)\pi\}\,e^{\pi i(\rho+r+s+2)}\,\Gamma(\rho+r+1)\,\Gamma(\sigma+s+1)}{(e^{2\pi i(\rho+r+\sigma+s)}-1)\,\Gamma(\rho+r+\sigma+s+2)}$$

$$=\frac{(1-e^{-2\pi i\rho})(1-e^{-2\pi i\sigma})}{1-e^{-2\pi i(\rho+\sigma)}}\,\frac{\Gamma(\rho+r+1)\,\Gamma(\sigma+s+1)}{\Gamma(\rho+r+\sigma+s+2)}.$$

Thus $\quad \frac{1}{2}\int w_{rs}\,d\gamma = \frac{u_r v_s c^{\rho+r+\sigma+s+1}}{x^{\rho+r+\sigma+s}1}\,\frac{(1-e^{-2\pi i\rho})(1-e^{-2\pi i\sigma})}{(\rho+r+\sigma+s+2)(1-e^{-2\pi i(\rho+\sigma)})}.$

If now all the terms of $\bar{w}(c)$ which have the same value for $(r+s)$ be grouped together, we obtain

$$\bar{w}(c) = \frac{(1-e^{-2\pi i\rho})(1-e^{-2\pi i\sigma})}{1-e^{-2\pi i(\rho+\sigma)}}\sum_0^\infty \frac{w_n}{x^{\rho+\sigma+n}}\,\frac{c^{\rho+\sigma+n+1}}{\Gamma(\rho+\sigma+n+2)},$$

provided c lies within a certain circle.

Since $u\{\frac{1}{2}(c+\gamma)\}$, $v\{\frac{1}{2}(c-\gamma)\}$ are analytic functions which exist for all values of c and γ, so too is $\bar{w}(c)$; thus, without the circle of convergence of the series for $\bar{w}(c)$, the continuation of this series will give the value of

$$\frac{1}{2}\int_C u\{\tfrac{1}{2}(c+\gamma)\}\,v\{\tfrac{1}{2}(c-\gamma)\,d\gamma.$$

It is now necessary to see that $\bar{w}(c)$ satisfies the same conditions that $u(a)\,v(b)$ satisfy, viz., that $e^{-c}|\bar{w}^{(\lambda)}(c)|$ tends uniformly to zero as c tends to infinity, when x lies within the region we have called $\bar{\Theta}$.

* *Math. Ann.*, t. xxxv.

Within this region we have

$$|u\{\tfrac{1}{2}(c+\gamma)\}| < |\alpha e^{\frac{1}{2}(c+\gamma)}|$$

and $$|v\{\tfrac{1}{2}(c-\gamma)\}| < |\beta e^{\frac{1}{2}(c-\gamma)}|,$$

α, β being assignable finite constants.

Hence $$|u\{\tfrac{1}{2}(c+\gamma)\}\, v\{\tfrac{1}{2}(c-\gamma)\}| < |e^c \alpha\beta|;$$

therefore $$|\bar{w}(c)| < 2\left|\tfrac{1}{2}\int_{-c}^{c} |\alpha\beta e^c|\, d\gamma\right| < 2\alpha\beta\,|ce^c|,$$

or, putting $c/x = \xi$, $$\left|\frac{\bar{w}(\xi x)}{x}\right| < 2\alpha\beta\,|\xi e^{\xi x}|.$$

The left-hand side is now independent of x, and this inequality holds for all points within the region $\bar{\Theta}$. Let κ be the least value of $R(x)$ within this region. Then $$e^{-c}|\bar{w}(c)|/x < 2\alpha\beta\,|\xi e^{-\xi(x-\kappa)}|.$$

Hence, provided $R(x) > \kappa$, $e^{-c}\bar{w}(c)$ tends to zero when ξ, and therefore c, tends to infinity. Similarly, we may show that, since $u'(a)$, $v'(b)$ also satisfy this condition at infinity, so also does $w'(c)$, and so on.

Consider now the integral

$$\int_A e^{-c}\bar{w}(c)\,dc = [e^{-c}\bar{w}(c)]_A + \int_A e^{-c}\frac{\partial}{\partial c}\bar{w}(c)\,dc = \int_A e^{-c}\frac{\partial}{\partial c}\{\bar{w}(c)\}\,dc,$$

what we have just proved being sufficient to ensure the existence of this integral. But

$$\frac{\partial}{\partial c}\{\bar{w}(c)\} = \Sigma \frac{w_n}{x^{\rho+\sigma+n}} \cdot \frac{c^{\rho+\sigma+n}}{\Gamma(\rho+\sigma+n+1)}\, \frac{(1-e^{-2\pi\iota\rho})(1-e^{-2\pi\iota\sigma})}{1-e^{-2\pi\iota(\rho+\sigma)}};$$

so that $$\{1-e^{-2\pi\iota(\rho+\sigma)}\}\, uv = \int_A e^{-c}\,dc\, w(c)$$

where $w(c)$ is the function defined by the series

$$\Sigma \frac{w^n}{x^{\rho+\sigma+n}}\, \frac{c^{\rho+\sigma+n}}{\Gamma(\rho+\sigma+n+1)}.$$

This series is exactly the series associated with the divergent series $\Sigma\, w_n/x^{\rho+\sigma+n}$, and, inasmuch as we have proved that

$$\underset{c=\infty}{\mathrm{Lt}}\, \{e^{-c}\,|w^{(\lambda)}(c)|\} = 0$$

within the region $\bar{\Theta}$, this series is absolutely summable, and we have shewn its sum to be uv.

Next consider the series formed by differentiating the series $x^{-\rho}\Sigma u_n x^n$ term by term.

For convenience put $x = 1/z$. Then, with the extended definition of absolute summability which has been adopted, we may apply all Borel's propositions as to the summability of the derived series to the series

$$z^\rho\{u_0+u_1z+\ldots\}.$$

The proofs are so identical with those which he gives that they need not be repeated here.*

Thus, if the last written series be denoted by $u(z)$ and its sum by u, the divergent series

$$u'(z) = u_0\rho z^{\rho-1}+u_1(\rho+1)z^\rho+\ldots$$

is absolutely summable within the same region as $u(z)$ and has for sum du/dz. Reverting to the original series in powers of $1/x$,

$$\frac{d}{dx}\{u(x)\} = -x^{-2}u'(z).$$

But x^2 may be looked upon as the limiting form of a series absolutely summable within any assigned region, and $u'(z)$ has been shewn to be so within a certain region. Thus $u'(x)$ is absolutely summable within that region, and has for sum

$$-z^2\frac{du}{dz} = \frac{du}{dx}.$$

Thus the propositions as to differentiation are extended to series proceeding in descending powers of x.

We may therefore state the following proposition, which includes practically all that has been developed, and which is the generalization of Borel's theorem :—†

Let u, u, w, ... be series absolutely summable for $x = x_0$ and each of the typical form

$$x^{-\rho}\left\{u_0+\frac{u_1}{x}+\ldots\right\},$$

and let $$P(u, v, w, \ldots, u^{(\lambda)}, v^{(\lambda)}, w^\lambda, x)$$

be a polynomial in the series u, v, w and their derivatives, the coefficients

* Borel, *Leçons*, pp. 108–115; *Ann. de l'Ecole Norm.*, 1899, pp. 94–5.

† Borel, *Leçons*, p. 114.

being developable for $|x| > |x_0|$ in the form $x^{-m}\left\{a_0 + \frac{a_1}{x} + \dots\right\}$, m being any index and the series being convergent.

Then, if the polynomial P is calculated as if the series were absolutely convergent, and the terms whose indices differ by integers are collected together, the result is an aggregate of series absolutely summable for $x = \theta x_0$, $\theta > 1$: and, if the sums of these series be substituted in their place, the result is what would be obtained by substituting in P in the first instance the sums of the series $u, v, w, \dots$.

Further, it is clear that the sum of an absolutely summable series is identically zero when, and only when, each coefficient vanishes separately. Hence, if the equation $P(u, v, \dots, x) = 0$ is satisfied formally by the absolutely summable series $u, v, w, \dots$, the analytic functions defined by these series also render the polynomial P identically zero.

We now proceed to consider expressions of the form

$$e^{ax}x^{-\rho}\left\{u_0 + \frac{u_1}{x} + \dots\right\},$$

where $x^{-\rho}\{u_0 + \dots\}$ is absolutely summable within a certain region, and is denoted by u, its sum being $\bar{u}$.

Two such expressions with the same exponential factor may clearly be added term by term to give a like expression. Two series $e^{ax}.u$ and $e^{\beta x}.v$, if multiplied formally, give $e^{(a+\beta)x}uv$, and the product uv is absolutely summable, giving the product of the sums of u and v, that is, $\bar{u}\bar{v}$.

Consider now the process of differentiation applied to such expressions. Differentiating formally, we have

$$\frac{d}{dx}\{e^{ax}.u\} = e^{ax}\{u' + au\}.$$

Now u' is absolutely summable and may be added term by term to au; so that

$$\frac{d}{dx}\{e^{ax}u\} = e^{ax}u^{(1)}$$

where $u^{(1)}$ is absolutely summable. Further, the sum of

$$e^{ax}u^{(1)} = e^{ax}\{\bar{u}' + a\bar{u}\} = \frac{d}{dx}\{e^{ax}\bar{u}\}.$$

Thus the process of differentiation gives a series of like form whose sum is the derivative of the sum of the original expression.

It is now clear that the general proposition stated above may be extended to the case where u, v, w represent absolutely summable series, each multiplied by a factor of the form e^{ax}, and where P may also contain explicitly expressions of the same form.

We may now bring what has been said above into line with the normal series satisfying linear differential equations with rational coefficients, of rank 1 at infinity, and of order n. Subject to the condition that a certain algebraic equation has its roots all different, we know that there are n expressions of the form

$$e^{ax}x^{-\rho}\left\{u_0+\frac{u_1}{x}+\ldots\right\}$$

which formally satisfy the equation.

Assuming that this series is absolutely summable for certain values of x, it follows that the sum, viz.,

$$\frac{1}{1-e^{-2\pi i\rho}}\int_A e^{ax}e^{-a}\left(\frac{a}{x}\right)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}+\frac{u_1}{\Gamma(\rho+2)}\frac{a}{x}+\ldots\right\}da,$$

is an integral of the equation for those values of x.

Now this integral may easily be changed into Laplace's definite integral solution.

Put $a=-x(t-a)$. Then the above sum becomes at once

$$\frac{1}{1-e^{-2\pi i\rho}}(-1)^{\rho+1}\int e^{tx}x(t-a)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}-\frac{u_1}{\Gamma(\rho+2)}(t-a)+\ldots\right\}dt.$$

The contour now consists of a line from infinity to the point $t=a$, encircling that point and returning to infinity in the direction whence it came, namely, the direction such that $x(t-a)$ is real and negative. Calling this path B' and integrating by parts, the sum becomes

$$\frac{1}{1-e^{-2\pi i\rho}}(-1)^{\rho+1}\left[e^{tx}(t-a)^{\rho}\left\{\frac{u_0}{\Gamma(\rho+1)}-\ldots\right\}\right]_{B'}$$

$$+\frac{1}{1-e^{-2\pi i\rho}}(-1)^{\rho}\int_{B'}e^{tx}(t-a)^{\rho-1}\left\{\frac{u_0}{\Gamma(\rho)}-\frac{u_1}{\Gamma(\rho+1)}(t-a)+\ldots\right\}dt.$$

The assumption as to the absolute summability of the series causes the quantity within brackets in the integrated part to vanish at the infinite limits. Thus we are led to the conclusion that, provided

$$e^{tx}\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[(t-a)^{\rho-1}\left\{\frac{u_0}{\Gamma(\rho)}-\frac{u_1}{\Gamma(\rho+1)}(t-a)+\ldots\right\}\right]$$

converges uniformly to zero when t becomes infinite in a certain direction for all values of x within a certain region $\bar{\Theta}$, the integral

$$\int_{B'} e^{tx}(t-a)^{\rho}\left\{\frac{u_0}{\Gamma(\rho)}-u_1\frac{t-a}{\Gamma(\rho+1)}+\ldots\right\}dt$$

exists and is an integral of the given equation.

Now it has been shown directly from the recurrence formula for the coefficients u_r that the series within the last written integral satisfies the equation known as Laplace's transformation of the given equation,* and hence that a finite number λ exists, such that, as t becomes infinite by real positive values,

$$\mathrm{Lt}_{t=\infty}\, e^{-t\mu}\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[(t-a)^{\rho-1}\left\{\frac{u_0}{\Gamma(\rho)}-\ldots\right\}\right]=0$$

for all values of μ such that $R(\mu)>\lambda$.†

We have thus arrived directly from the normal series at an analytic function satisfying the equation within a given region of the x plane, and therefore throughout its region of existence; that is to say, we have shown that a normal series formally satisfying an equation defines a unique integral of that equation by the method here developed, and that these series may be added, multiplied, and differentiated within a certain region as if they were absolutely convergent.

Poincaré's proposition that they asymptotically represent the definite integrals is included in the fact of their absolute summability. The proof is essentially the same as that which Poincaré gives.‡

As was stated at the outset, the object of this discussion has been rather to make closer the connection between the divergent normal series and the ordinary integrals of the differential equation than to obtain fresh knowledge of the integrals from the divergent series apart from the known integrals. One tangible result, at least, emerges from this reversal of the procedure which begins with the definite integral, viz.:—If two differential equations with rational coefficients, and each of rank 1, are satisfied formally by one normal series. these equations have a common integral, even if that series be divergent; and consequently, if one equation be irreducible, the second admits of all the integrals of that one as integrals of itself.

* *V.* Schlesinger, t. I., § III.

† *V.* Forsyth, *Linear Differential Equations*, pp. 319-322.

‡ See also, Le Roy, "Mémoire sur les Séries div.," *Ann. de la Fac. des Sci. de Toul.*, 1900, p. 427.

Suppose two such equations to be denoted by R and S, S being irreducible, and R containing all the integrals of S, and being therefore of higher order than S.

If the Laplace transformations of these two equations be called U, V respectively, then at least one integral of V must satisfy U; and, if V is irreducible, all the integrals of V must satisfy U. In this case, therefore, U must be of higher order than V, unless the equations are identical.

But the order of the Laplace transformation is the degree of the first coefficient in the given equation. Hence, if a linear equation of rank 1 with rational coefficients is irreducible, and also its Laplace transformation, and a normal series exists satisfying this equation and another equation of rank 1 with rational coefficients, this other equation must not only be of higher order, but must have its coefficients of higher degree. In particular, it follows that no two equations of Laplace's type, *i.e.*, with linear coefficients, can be satisfied by a common normal series.