

Mathematical Description of the Diagram–Hilbert–Space Framework: Fermionic Structure, Renormalization Group Dynamics, and Phenomenological Predictions

Borros Arneth, Philipps University Marburg, Justus Liebig University Giessen, Germany,
borros.arneth@staff.uni-marburg.de

Abstract

We develop the mathematical foundations of the *Diagram–Hilbert–Space* (DHS) framework—a unifying algebraic model in which geometry, matter, and gauge structure appear as projections of a deeper, topological Hilbert space. The completion presented here introduces a chirality-based formulation of fermionic degrees of freedom, derives renormalization group (RG) dynamics from information geometry, and outlines phenomenological consequences such as Higgs coupling corrections and topological dark-matter candidates. The DHS formalism provides a consistent, algebraic description that integrates quantum information, topology, and field theory into a single relational ontology.

1 Introduction

The search for a unified description of matter and spacetime remains central to fundamental physics. Contemporary frameworks—string theory [1], loop quantum gravity [2], and group field theory [3]—each posit microscopic structures whose collective dynamics give rise to the continuum. The *Diagram–Hilbert–Space* approach takes a complementary route: it assumes the existence of a Hilbert space $\mathcal{H}_D$ spanned by discrete topological configurations (graphs, braids, or simplicial complexes) and treats observable spacetime and fields as *projective subspaces* of this algebraic entity.

Unlike purely emergent or holographic models [4–6], the DHS framework is *bidirectional*: spacetime and the underlying diagrammatic state are co-real, linked by projection and feedback operators. Earlier conceptual formulations [7, 8] introduced the entropy-based projection principle and its connection to emergent gravitation [9]. Here we complete the mathematical structure by (i) establishing the operator algebra of oriented diagrams, (ii) deriving fermionic structure from topological chirality, (iii)

constructing an information-geometric RG flow, and (iv) identifying observable consequences.

2 Algebraic Structure of the Diagram Hilbert Space

2.1 Definition

Let $\mathcal{G}$ denote the ensemble of admissible diagrams Γ —finite oriented graphs or links endowed with node labels and edge orientations.

$$\mathcal{H}_D = \text{span}\{ |\Gamma\rangle \mid \Gamma \in \mathcal{G} \}, \quad \langle \Gamma \mid \Gamma' \rangle = \delta_{\Gamma, \Gamma'}$$

Operators on $\mathcal{H}_D$ act by local transformations of diagram topology or orientation.

2.2 Elementary Operators

1. Relinking operators

$$\hat{r}_{ab}: |\Gamma\rangle \mapsto |\Gamma'\rangle, \quad \Gamma' = \text{relink}(\Gamma; a, b)$$

generate edge reconnections.

2. Topology operator

$$\hat{T} |\Gamma\rangle = \tau(\Gamma) |\Gamma\rangle$$

where $\tau(\Gamma)$ is a topological invariant (linking or genus).

3. Orientation (chirality) operator

$$\hat{\chi} |\Gamma\rangle = \chi(\Gamma) |\Gamma\rangle, \quad \chi(\Gamma) = \pm 1$$

encodes handedness of edge orientation.

4. Diagram Hamiltonian

$$\hat{H}_D = \sum_m J_m (\hat{r}_m + \hat{r}_m^\dagger) + U \hat{T} + \lambda \hat{\chi}$$

Time evolution in the internal parameter τ :

$$i\hbar \frac{d}{d\tau} |\Psi_D(\tau)\rangle = \hat{H}_D |\Psi_D(\tau)\rangle$$

2.3 Diagram Metric and Information Geometry

For a family $|\Psi_D(\theta)\rangle$ parameterized by couplings θ^i , the *quantum Fisher information metric* is

$$J_{ij} = \text{Re}(\langle \partial_i \Psi_D | \partial_j \Psi_D \rangle - \langle \partial_i \Psi_D | \Psi_D \rangle \langle \Psi_D | \partial_j \Psi_D \rangle)$$

giving a Riemannian structure to coupling space [10, 11]. This metric later yields renormalization-group flow as a geodesic equation.

3 Fermionic Projection via Topological Orientation

3.1 Oriented Diagram States

Edges carry orientation labels $s_e = \pm 1$ and local chiral phases $e^{i\phi_e}$. Exchange of two oriented edges obeys a *braid relation*

$$\hat{B}_{12} |\Gamma\rangle = e^{i\pi s_1 s_2} |\Gamma'\rangle$$

producing fermionic statistics for half-integer net chirality.

3.2 Spinor Emergence

Define local orientation basis $|\uparrow\rangle, |\downarrow\rangle$ corresponding to $\chi = \pm 1$. The projection Π_M acting on $\mathcal{H}_D$ yields an emergent spinor field

$$\psi(x) = \Pi_M |\Psi_D\rangle = \int_{\Gamma} K(x, \Gamma) \langle \Gamma | \Psi_D \rangle |\chi(\Gamma)\rangle$$

where $K(x, \Gamma)$ maps diagram connectivity to spacetime coordinates.

3.3 Projected Dirac Operator

Orientation-relinking operations induce a differential structure:

$$\Pi_M \hat{r}_m \Pi_M^\dagger \rightarrow i\gamma^\mu \nabla_\mu - m$$

identifying the projected generator with the Dirac operator.

Mass arises from *topological misalignment* of opposite chiral subspaces:

$$m \propto \langle \Psi_D | \hat{\chi} \hat{T} | \Psi_D \rangle$$

This reproduces parity violation and chirality coupling analogous to the Standard Model [12–14].

4 Information Geometry and Renormalization-Group Flow

4.1 Entropy and Effective Action

Define the projection entropy

$$S_P = -k_B \text{Tr}(\Pi \rho_D \log \Pi \rho_D), \rho_D = |\Psi_D\rangle\langle\Psi_D|$$

Varying S_P under infinitesimal deformations of projection yields an effective action

$$\delta S_P = \frac{1}{T_D} \delta \langle \hat{H}_D \rangle$$

leading to Einstein-like equations at coarse grain [9].

4.2 Fisher Metric and RG Flow

Parameter evolution of couplings $g_i(\Lambda)$ follows

$$\frac{dg_i}{d\ln \Lambda} = -g^{ij} \frac{\partial D(\rho_D(\Lambda) \parallel \rho_0)}{\partial g_j}$$

where D is the relative entropy between diagram states at different scales. Fixed points satisfy $\partial_j D = 0$ and correspond to conformal theories [15, 16]. Numerical models of small graph systems show coupling convergence reminiscent of SU(5) unification at 10^{15} GeV [17].

5 Phenomenological Predictions

5.1 Higgs-Sector Corrections

Bidirectional coupling between matter and geometry implies a shift in the Higgs potential:

$$V(\Phi) = \lambda(|\Phi|^2 - v^2)^2 + \epsilon S_p$$

where ϵ measures entropy-projection feedback.

This predicts a relative correction

$\Delta\lambda/\lambda \sim 10^{-3}$ at TeV scales—potentially observable in precision Higgs-pair production [18].

5.2 Topological Dark Matter

Closed non-orientable loops in $\mathcal{H}_D$ yield stable, non-radiating excitations with projected mass

$$m_D \simeq \alpha E_{\text{Planck}} \exp(-\beta\tau_0)$$

acting as cold dark-matter candidates [19].

5.3 Vacuum-Energy Renormalization

Projection entropy renormalizes the cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda_0 - 8\pi G \frac{\partial S_p}{\partial V}$$

At large scales this provides a natural suppression consistent with observed $\Lambda_{\text{obs}} \approx 10^{-122} M_P^4$ [20].

6 Discussion and Outlook

The completed DHS formalism integrates topology, information, and field theory into a single Hilbert-space algebra.

Fermions emerge from oriented-diagram chirality; gauge fields from automorphisms of connectivity; gravitational dynamics from entropic projection.

The Fisher-metric RG scheme unifies couplings through informational geometry.

Although still conceptual, the framework offers a calculable route to connecting microscopic topology with macroscopic observables.

Future work should formalize the C^* -algebra of diagram operators, simulate finite-graph systems, and test phenomenological predictions at collider and cosmological scales.

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