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2700. There can be no doubt that the copper linings, when in place, were full of currents at the time of action, and that when away no such currents would exist in the air or glass replacing them. There is also full reason to admit, that the divergence and convergence of the magnetic lines of force supposed above (2697.) would satisfactorily account for such currents in them, supposing the indirect action of the cores were assumed. If that supposition be rejected, then it seems to me that the whole of the bodies present, the magnet, the helix, the core, the copper lining, or the air or glass which replaces it, must all be in a state of tension, each part acting on every other part, being in what I have occasionally elsewhere imagined as the electro-tonic state (1729.).

2701. The advance of the copper makes the lines of magnetic force diverge, or, so to say, drives them before it (2697.). No doubt there is reaction upon the advancing copper, and the production of currents in it in such a direction as makes them competent, if continued, to continue the divergence. But it does not seem logical to say, that the currents which the lines of force cause in the copper, are the cause of the divergence of the lines of force. It seems to me, rather, that the lines of force are, so to say, diverged, or bent outward by the advancing copper (or by a connected wire moving across lines of force in any other form of the experiments), and that the reaction of the lines of force upon the forces in the particles of the copper causes them to be resolved into a current, by which the resistance is discharged and removed, and the line of force returns to its place. I attach no other meaning to the words *line of force* than that I have given on a former occasion (2149.).

Royal Institution, Dec. 14, 1849.

X. *On the Geometrical Interpretation of Quaternions.*

By William Spottiswoode, *M.A., of Balliol College, Oxford**.

§ 1. *Fundamental Laws.*

THE following investigations refer to the same subject as that treated by Professor Donkin in vol. xxxvi. of this Journal; and are offered, not as at all preferable to his, but simply as indicating another mode in which the question may be viewed; it being desirable to exhibit a subject, which is somewhat new, in more than one way, in order that as much light as possible may be thrown upon it. The present paper will be interesting (if it is so at all) principally because its results are substantially the same as those of the paper just referred to,

* Communicated by the Author.

although obtained by an entirely independent process. I have stated my views as briefly as possible, because Prof. Donkin's paper renders any more lengthened discussion superfluous; and if any expressions occur which appear to indicate a view of algebraic symbols, &c. different from his, they have been used merely because they are the ordinary terms; and I should wish them to be understood as far as possible in his way, if for no better reason, at least in order that the two methods may be compared.

The calculus of quaternions is a generalization of algebra, in which sets of four ordinary algebraical quantities are used instead of single quantities. Each such set of four quantities is called a quaternion; the nature and laws of combination of which are the object of the present investigations. The corresponding laws in ordinary algebra will be assumed as known.

Let a quaternion be defined to be a set of four algebraic quantities considered with reference to their order of position, and let it be expressed by the following equation,

$$Q=(w, x, y, z), \quad \ldots \ldots \quad (1.)$$

in which Q, or its equivalent, is called the quaternion, and w, x, y, z its constituents. As this definition involves no law of connexion between the constituents, it is clear that the equivalence of any number of quaternions must involve the equivalence of their several constituents; so that the equations

$$Q=Q_1=Q_2=\ldots \quad \ldots \ldots \quad (2.)$$

involve the following,

$$\left.\begin{array}{l} w=w_1=w_2=\ldots \\ x=x_1=x_2=\ldots \\ y=y_1=y_2=\ldots \\ z=z_1=z_2=\ldots \end{array}\right\} \quad \ldots \ldots \quad (3.)$$

and conversely (3.) will involve (2.). The same principle gives rise to the following law for the addition and subtraction of quaternions:

$$\Sigma Q_n=(\Sigma w_n, \Sigma x_n, \Sigma y_n, \Sigma z_n); \quad \ldots \quad (4.)$$

particular cases of which are

$$nQ=(nw, nx, ny, nz) \quad \ldots \ldots \quad (5.)$$

$$Q-Q=0=(0, 0, 0, 0). \quad \ldots \ldots \quad (6.)$$

The following consideration will assist further investigations. The quaternion

$$(w, 0, 0, 0) \quad \ldots \ldots \ldots \quad (7.)$$

is a system consisting of the quantity w, followed by no other quantities, *i. e.* associated with nothing but itself; in other words, it is simply equivalent to the ordinary algebraic quantity w; so that by means of the law of addition of quaternions, it will be allowable to write,

$$\left.\begin{aligned}(w,x,y,z)&=(w,0,0,0)+(0,x,0,0)+(0,0,y,0)+(0,0,0,z)\\&=w+(0,x,0,0)+(0,0,y,0)+(0,0,0,z)\end{aligned}\right\}\quad(8.)$$

With respect to the last three terms of this expression, it will be necessary to introduce some new symbols. Thus, for instance, if T, T′, T″ indicate the operations of transposition defined by the following equations,

$$\left.\begin{aligned}T\,(x,0,0,0)&=(0,x,0,0)\\T'(y,0,0,0)&=(0,0,y,0)\\T''(z,0,0,0)&=(0,0,0,z)\end{aligned}\right\},\quad . \; . \; . \quad (9.)$$

the equation (8.) might be written

$$Q=w+Tx+T'y+T''z. \quad . \; . \; . \; . \quad (10.)$$

And, if the laws of the combination of the symbols T, T′, T″ were known, the general laws of the combination of quaternions would be at once deducible.

It will however be more advantageous to use some symbols of transposition rather different from those above noticed; let then

$$\left.\begin{aligned}iQ&=(-x,\,w,-z,\,y)\\jQ&=(-y,\,z,\,w,-x)\\kQ&=(-z,-y,\,x,\,w)\end{aligned}\right\};\quad . \; . \; . \; . \quad (11.)$$

from these definitions of the symbols of transposition, i, j, k, it is easy to deduce the following relations:

$$\left.\begin{aligned}i.iQ=j.jQ=k.kQ=i.j.kQ&=(-w,-x,-y,-z)=-Q\\j.kQ=-k.jQ&=iQ\\k.iQ=-i.kQ&=jQ\\i.jQ=-j.iQ&=kQ\end{aligned}\right\}\quad(12.)$$

in which the expression for $-Q$ may be deduced from (5.) by writing -1 for n. These relations may be symbolically written, as follows:

$$\left.\begin{aligned}i^2=j^2=k^2=ijk&=-1\\jk=-kj&=i\\ki=-ik&=j\\ij=-ji&=k\end{aligned}\right\},\quad . \; . \; . \; . \quad (13.)$$

the operations of transposition and change of sign being independent of the subject of operation.

As it will assist the geometrical interpretation of the operations i, j, k hereafter, to separate each of them into two distinct operations, the formulæ to which such separation gives rise may be properly noticed here. Let then

$$\left.\begin{array}{ll} i'Q=(-x,\ w,\ y,\ z) & i''Q=(w,\ x,\ -z,\ y) \\ j'Q=(-y,\ x,\ w,\ z) & j''Q=(w,\ z,\ y,\ -x) \\ k'Q=(-z,\ x,\ y,\ w) & k''Q=(w,\ -y,\ x,\ z) \end{array}\right\}; \quad (14.)$$

there will then result

$$\left.\begin{array}{l} i=i'\,i''=i''\,i \\ j=j'\,j''=j''\,j' \\ k=k'k''=k''k' \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

to which may be added,

$$\left.\begin{array}{c} i''j'=k'i'',\quad j''k'=i'j'',\quad k''i'=j'k'', \\ j''k''=k''i''=i''j'' \\ \mathrm{P}(ii'i'')=\mathrm{P}(jj'j'')=\mathrm{P}(kk'k'')=-1 \end{array}\right\}, \quad . \quad . \quad (16.)$$

where P represents the symbolical product without reference to order. By means of the above properties of i, j, k, it will be possible to transform the expression of a quaternion (1.) into another of the same form as (10.); for

$$\left.\begin{array}{l} (0,\ x,\ 0,\ 0)=i\,(x,\ 0,\ 0,\ 0)=ix \\ (0,\ 0,\ y,\ 0)=j\,(y,\ 0,\ 0,\ 0)=jy \\ (0,\ 0,\ 0,\ z)=k(z,\ 0,\ 0,\ 0)=kz \end{array}\right\}; \quad . \quad . \quad (17.)$$

so that (8.) may be written thus,

$$Q=w+ix+jy+kz, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

in which i, j, k may be combined according to the laws defined by (13.). It may be observed, that, since by means of the condition (4.) the addition of quaternions is reduced to the addition of ordinary algebraical quantities, the order and clustering of the terms in (18.) is indifferent, so that the *associative principle of addition* among those terms is completely established; the same is obviously the case with respect to the addition of quaternions in general. It may be further remarked, that, since by means of (4.),

$$\left.\begin{array}{lll} i\Sigma Q_n=\Sigma iQ_n, & j\Sigma Q_n=\Sigma jQ_n, & k\Sigma Q_n=\Sigma kQ_n \\ jk\Sigma Q_n=\Sigma jkQ_n, & ki\Sigma Q_n=\Sigma kiQ_n, & ij\Sigma Q_n=\Sigma ijQ_n \\ \quad =j\Sigma kQ_n & \quad =k\Sigma iQ_n, & \quad =i\Sigma jQ_n \end{array}\right\}, \quad (19.)$$

with other like formulæ, *the distributive character of the symbols* i, j, k is also established.

The following verifications, although not essential to the theory, are perhaps worth noticing. If we had taken (18.) as the definition of a quaternion with (13.) as the definitions of i, j, k, we should have found

$$\left.\begin{array}{l} iQ=iw-x+ky-jz \\ jQ=jw-kx-y+iz \\ kQ=kw+jx-iy-z \end{array}\right\}; \quad \ldots \quad (20.)$$

so that the equations

$$Q=Q_1=Q_2=\ldots,$$

which obviously involve also

$$\left.\begin{array}{l} iQ=iQ_1=iQ_2=\ldots \\ jQ=jQ_1=jQ_2=\ldots \\ kQ=kQ_1=kQ_2=\ldots \end{array}\right\}, \quad \ldots \quad (21.)$$

give rise to the equations (3.); for i, j, k being symbolical expressions for $(-)^{\frac{1}{2}}$, render all real terms, to which they are prefixed, imaginary in the ordinary sense of that word. The same definition of a quaternion gives

$$\Sigma Q_n=\Sigma w_n+i\Sigma x_n+j\Sigma y_n+k\Sigma z_n, \quad \ldots \quad (22.)$$

which is in fact identical with (4.).

But the principal advantage of the linear form of the expression for a quaternion is found in the processes of multiplication and division. In the form (1.) it does not seem possible to obtain a complete solution of the problem of multiplication; the following however would be the initial steps to such a solution:

$$Q.Q_1=(w, x, y, z).(w_1, x_1, y_1, z_1)$$

$$=\{w(w_1,x_1,y_1,z_1),\ x(w_1,x_1,y_1,z_1),\ y(w_1,x_1,y_1,z_1),\ z(w_1,x_1,y_1,z_1)\}$$

$$=\{w(w_1, x_1, y_1, z_1), 0, 0, 0\}$$

$$+\{0, x(w_1, x_1, y_1, z_1), 0, 0\}$$

$$+\{0, 0, y(w_1, x_1, y_1, z_1), 0\}$$

$$+\{0, 0, 0, z(w_1, x_1, y_1, z_1)\}$$

$$=(w_1w, w_1x, w_1y, w_1z)$$

$$+\{0, xw_1+x(0, x_1, 0, 0)+x(0, 0, y_1, 0)+x(0, 0, 0, z_1), 0, 0\}$$

$$+\{0, 0, yw_1+y(0, x_1, 0, 0)+y(0, 0, y_1, 0)+y(0, 0, 0, z_1), 0\}$$

$$+\{0, 0, 0, zw_1+z(0, x_1, 0, 0)+z(0, 0, y_1, 0)+z(0, 0, 0, z_1)\}$$

$$= (ww_1,\quad w_1x + x_1w,\quad w_1y + y_1w,\quad w_1z + z_1w)$$
$$+ (0, x, y, z) . (0, x_1, 0, 0)$$
$$+ (0, x, y, z) . (0, 0, y_1, 0)$$
$$+ (0, x, y, z) . (0, 0, 0, z_1)$$
$$= (ww_1, w_1x + x_1w, w_1y + y_1w, w_1z + z_1w) + (0, x, y, z).(0, x_1, y_1, z_1).$$

But if we adopt the form (18.), the constituents of the product of two quaternions are completely determined; in fact, it is found without difficulty that if

$$QQ_1 = Q_2 = w_2 + ix_2 + jy_2 + kz_2. \quad . \quad . \quad . \quad (24.)$$

$$\left.\begin{aligned} w_2 &= ww_1 - xx_1 - yy_1 - zz_1 \\ x_2 &= wx_1 + w_1x + yz_1 - y_1z \\ y_2 &= wy_1 + w_1y + zx_1 - z_1x \\ z_2 &= wz_1 + w_1z + xy_1 - x_1y \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (25.)$$

And also, if

$$Q_1Q = Q_2{}^1 = w_2{}^1 + ix_2{}^1 + jy_2{}^1 + kz_2{}^1, \quad . \quad . \quad . \quad (26.)$$

$$w_2 = w_2{}^1,\ x_2 = -x_2{}^1,\ y_2 = -y_2{}^1,\ z_2 = -z_2{}^1. \quad . \quad (27.)$$

Moreover

$$(ix + jy + kz)^2 = -x^2 - y^2 - z^2 \quad . \quad . \quad . \quad . \quad (28.)$$

$$Q^2 = w^2 - x^2 - y^2 - z^2 + 2w(ix + jy + kz) \quad . \quad (29.)$$

$$\left.\begin{aligned} (w - ix - jy - kz)(w + ix + jy + kz) &= w^2 + x^2 + y^2 + z^2 \\ = (w + ix + jy + kz)(w - ix - jy - kz). & \end{aligned}\right\} \quad (30.)$$

So that the reciprocal of a quaternion is the quotient of the quaternion itself, with the signs of its last three constituents changed divided by the sum of the squares of the constituents.

The constituents of the ratio

$$Q_1 = Q^{-1}Q_2$$

may be found either by solving (25.) with respect to w_1, x_1, y_1, z_1, or by means of the relation (30.), and so reducing the division to multiplication.

§ 2. *Geometrical Interpretation.*

In the general expression

$$Q = (w, x, y, z) = w + ix + jy + kz, \quad . \quad . \quad . \quad (1.)$$

let w, x, y, z represent straight lines drawn in several directions from the origin, and let x, y, z coincide with the three positive axes of coordinates respectively, while the direction of w is arbitrary; x, y, z may then be considered as the coordinates of some point, in general not the extremity of w.

In accordance with the fundamental idea of a quaternion, the position of the line represented by any constituent will be supposed to depend upon the position which that constituent holds in the first expression for Q; so that the directions of the four lines being once chosen, the first, second, third, and fourth constituents will always represent lines drawn in the four directions respectively, whatever changes may have taken place in the order of the constituents as originally given. Now, returning to the equations (13.) and (14.) of the former section, it appears that i'' indicates a change by which the negative axis of z is brought into the old position of the axes of y; and the axis of y into the old position of the axis of z; the positions of w and x remaining unchanged; or i'' may be considered as indicating a change by which w and x are brought into positions such, that they are situated, with respect tot he negative axis of z and the positive axis of y, in the same manner as they were at first with respect to the axes of y and z respectively. When the axes are rectangular, as at present, this change may be represented, either by supposing the plane of yz to revolve in its own plane through half a right angle in the direction from y to z, or by supposing the three axes to remain fixed, and the radius vector w to revolve on the surface of a right cone with a circular base, whose axis is that of x and vertex the origin, through one quarter of a revolution; the direction of rotation being from the axis of z towards that of y. It is easily seen that j'' and k'' may be represented by similar revolutions about the axes of y and z respectively.

Again, i' indicates a change by which the negative axis of x is brought into the old position of w, and w into the old position of the positive axis of x, the positions of y and z remaining unchanged; and if α be the angle between the axis of x and the line w (the vertical angle of the first cone), this change may be represented by bringing the negative axis of the cone into the old position of w, and then opening the vertical angle of the cone through an angle $=\pi-2\alpha$. The changes j' and k' may similarly be represented by supposing the negative axes of the other two cones respectively to take the position of w, and the vertical angle of the cones to be opened through angles $=\pi-2\beta$ and $\pi-2\gamma$ respectively. The above theory becomes much simpler when the position of w is not absolutely determined, but merely restricted to a given plane; in this case its position may be supposed to coincide with the intersection of that plane with one of the coordinate planes; *e. g.* in the case of i, with the intersection with the plane of yz; in that of j, with that of zx; and in that of k, with that of xy; the three cones then become simply the three coordinate planes, and i'', j'', k'' will represent rotations of this line of intersection through angles $=\frac{\pi}{2}$ in the planes of yz, zx, xy respectively; and i', j', k' similar rotations in the planes of wx, wy, wz. In each case the origin of the rotations is determinate.

If the position of w be entirely arbitrary, the positions of the intersections of the planes of wx, wy, wz with those of yz, zx, xy, will be so also; and the only difference arising in the significations of i', j', k', i'', j'', k'', will be that the origin of rotation is restricted only to the three coordinate planes successively, the position in those planes being arbitrary. These considerations will enable us to interpret the various terms in the linear expression for Q; for

$$ix=i(x,0,0,0)=(0,x,0,0) \quad . \quad . \quad . \quad . \quad (2.)$$
$$jy=j(y,0,0,0)=(0,0,y,0) \quad . \quad . \quad . \quad . \quad (3.)$$
$$kz=k(z,0,0,0)=(0,0,0,z) \quad . \quad . \quad . \quad . \quad (4.)$$

Now the first constituent of the quaternions on the right-hand side of the above expression will, according to the principles of interpretation above given, be considered as representing a line coinciding with the intersection of a plane passing through the axis of

x, with the plane of yz, in (2.);
y, ... zx, ... (3.);
z, ... xy, ... (4.);

and consequently ix, jy, kz will represent that the lines whose lengths are represented by x, y, z have revolved through angles each $=\frac{\pi}{2}$ in planes perpendicular to their original directions.

Adopting the above interpretation of the various terms in the expression for a quaternion, the question next arises, in what sense are the lines represented by these terms, and by quaternions generally, said to be added? Now the fundamental formula for the addition of quaternions shows that in whatever way the line Q is formed from the quantities w, x, y, z (with similar expressions for any other quaternions Q_1, Q_2 . .), then the sum ΣQ_n is a new quaternion line formed in the same manner from the quantities Σr_n, Σx_n, Σy_n, Σz_n; and writing

$$\Sigma Q_n=\mathfrak{Q}=(W, X, Y, Z),$$

it appears that W will be the algebraical sum of the lines $w, w_1, \ldots$, supposed, for convenience, to be similarly directed, and X, Y, Z will be the coordinates of the extremity of the diagonal of the parallelopiped formed on the sums of the component coordinates as its edges. From these two facts it appears that straight lines lying in the same straight line are to be added as in ordinary algebraical geometry, while the sum of any other set of straight lines inclined to one another at any angles is the closing side of the polygon formed by placing the beginning of each line at the termination of its predecessor. In fact, lines are to be added as forces are equilibrated in statics. In accordance with this principle, the sum

$$ix+jy+kz$$

will represent the diagonal of the parallelopiped described on the line ix, jy, kz as its edges; and since moreover

$$(ix+jy+kz)^2=-(x^2+y^2+z^2)=-r^2,$$

therefore also

$$ix+jy+kz=(-)^{\frac{1}{2}}r, \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

which, according to the principles of the present calculus, represents not merely a line in a plane perpendicular to r, but a line which has been brought into its position by means of a rotation through an angle $=\frac{\pi}{2}$ in that plane; or, in other words, about an axis whose direction-cosines are $x:r$, $y:r$, $z:r$; and finally, the sum

$$w+ix+jy+kz$$

will represent the diagonal of the square whose sides are

$$w+ix+jy+kz$$

(these two lines being obviously perpendicular). The length of the whole line is consequently

$$(w^2+x^2+y^2+z^2)^{\frac{1}{2}}=\rho, \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

and its direction makes an angle, whose tangent is $=r:w$, with the direction of w; the whole quaternion will therefore represent a line whose length is ρ, which has been turned through an angle $=\tan^{-1}(r:w)$ in the plane, the direction-cosines of whose normal are $x:r$, $y:r$, $z:r$. The expression (1.) may also be written as follows:

$$\{\cos\theta+\sin\theta(il+jm+kn)\}\rho, \quad . \quad . \quad . \quad . \quad (7.)$$

where

$$\left.\begin{array}{c}\theta=\tan^{-1}(r:w)\\ x:l=y:m=z:n=r\end{array}\right\} \quad . \quad . \quad . \quad . \quad (8.)$$

The following cases will exemplify the above interpretation of quaternions.

If ABC be a spherical triangle, the radius being equal to unity, and if Q, Q′, Q″ indicate the rotations of the radius vector from B to C, from C to A, from A to B respectively, it is clear that we must always have

$$Q''Q'Q = QQ''Q' = Q'QQ'' = 1.$$

In order to find the quaternion which will represent the rotation from the line (l, m, n) to the line (l', m', n'), we may construct a quadrantal triangle such that (l, m, n), (l', m', n') pass through the angles opposite to the quadrantal sides; and if Q be the required quaternion,

$$(li+mj+nk)(l'i+m'j+n'k)Q = -1;$$

but since

$$(li+mj+nk)^2 = (l'i+m'j+n'k)^2 = -1,$$

$$\therefore \quad (l'i+m'j+n'k)Q = (li+mj+nk)$$

$$-Q = (l'i+m'j+n'k)(li+mj+nk)$$

$$Q = -(ll'+mm'+nn') + i(mn'-m'n) + j(nl'-n'l) + k(lm'-l'm).$$

To find the quaternions which will represent the rotations from the three coordinate axes to the line (l, m, n), we need only put in the above equation,

$$m=0,\ n=0;\ n=0,\ l=0;\ l=0,\ m=0$$

in succession; hence, dropping the accents,

$$Q_x = -l - jn + km$$

$$Q_y = -m + in - kl$$

$$Q_z = -n - im + jl;$$

to which may be added the following relations:

$$Q_x^2 + Q_y^2 + Q_z^2 = -1$$

$$lQ_x + mQ_y + nQ_z = -1$$

$$iQ_x + jQ_y + kQ_z = -il - jm - kn$$

$$nQ_y - mQ_z = i - l(il+jm+kn)$$

$$lQ_z - nQ_x = j - m(il+jm+kn)$$

$$mQ_x - lQ_y = k - n(il+jm+kn)$$

$$(jm-kn)Q_x + (in-kl)Q_y + (jl-im)Q_z = -2$$

$$= (nQ_y - mQ_z)^2 + (lQ_z - nQ_x)^2 + (mQ_x - lQ_y)^2.$$

If Q, Q′, Q″ be any quaternions, the condition

$$\alpha Q + \beta Q' + \gamma Q'' = 0$$

is equivalent to the system

$$\alpha w + \beta w' + \gamma w'' = 0$$
$$\alpha x + \beta x' + \gamma x'' = 0$$
$$\alpha y + \beta y' + \gamma y'' = 0$$
$$\alpha z + \beta z' + \gamma z'' = 0,$$

from the last three of which may be deduced

$$\begin{vmatrix} x, & x', & x'' \\ y, & y', & y'' \\ z, & z', & z'' \end{vmatrix} = 0$$

which is the condition that the three lines, whose direction-cosines are proportional to x, y, z, ..., lie in the same plane; in other words, the planes of rotation of the three quaternions are all parallel to one straight line.

If Q, Q′, Q″ represent the rotations BC, CA, AB of the spherical triangle ABC, the quaternions

$$\gamma Q' - \beta Q'' = Q_1$$
$$\alpha Q'' - \gamma Q = Q_2$$
$$\beta Q - \alpha Q' = Q_3$$

will represent arcs drawn from the angular points A, B, C, and cutting the opposite sides in points whose segments are in the ratios $\beta : \gamma$, $\gamma : \alpha$, $\alpha : \beta$ respectively, and the resulting condition

$$\alpha Q_1 + \beta Q_2 + \gamma Q_3 = 0$$

shows that the three planes of rotation intersect in a common line, for they all pass through the same point, viz. the centre of the sphere; consequently the three arcs all meet in a point. If

$$\alpha = \beta = \gamma,$$

the points where the arcs Q_1, Q_2, Q_3 meet the sides of the triangle will be the middle points of those sides, and the condition

$$Q_1 + Q_2 + Q_3 = 0$$

will express that the three arcs meet in a point. This theorem includes all the corresponding theorems with respect to plane triangles.