The foregoing examples will illustrate the great fertility of the method employed for deducing identities which are difficult to prove by other means. It may be noticed that, when all the b's are equated to unity, the expression for a_0 vanishes identically. The equation § 1, (4) would lead us to infer that a_1 would also vanish identically on the same supposition, as indeed is obvious from § 1, (13). Similarly, it may be shown that all the a's vanish identically when the b's are equated to unity. Consistently with this fact, it will be then seen that, if in any relation connecting an a-series with a b-series the coefficients of the a's form a convergent series, then the b-series vanishes identically, as in § 2, (9), § 8, Ex. 4, &c.; but, if the b-series does not vanish identically, then the coefficients of the a's form a divergent series, as in § 2, (7), § 8, Ex. 1, 2, 3, &c.

On Regular Difference Terms. By A. B. Kempe, M.A., F.R.S.

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1. Let $\alpha, \beta, \gamma, \ldots$ be a system S_n of n quantities, which may be termed *roots;* and let w differences $\alpha-\beta$, $\alpha-\beta$; $\beta-\gamma$, $\alpha-\gamma$; &c., be formed with these, each root entering into v of the differences. Then the product of these w differences will be called a *regular difference term* of the system S_n, and will be said to be of *degree n*, *order v*, and *weight w*.

2. The expression

$$(\alpha-\beta)^2(\beta-\gamma)(\gamma-\delta)^2(\delta-\alpha)$$

affords an example of a regular difference term of degree 4, order 3, and weight 6.

3. We may have difference terms into which the different roots do not all enter the same number of times; such difference terms are, however, *irregular*. A difference term will be irregular although each of the roots which enters into it enters the same number of times as the others, provided that there are other roots of the system under consideration which do not enter at all. Such a difference term will,

however, be a regular difference term of the reduced system which consists only of the roots which do enter into the term.

4. Where the degree of a regular difference term is even, the order may be as low as unity; but, where the degree is odd, the order cannot be less than 2, for we have

$$vn = 2w,$$

and thus both degree and order cannot be odd.

Regular difference terms of even degree and order 1, or of odd degree and order 2, will be called *elemental terms* of the system of roots considered. Elemental terms of order 1 may be called *linear elements*, and those of order 2 *quadratic elements*.

5. The product of two or more regular difference terms of S_n will, of course, be also a regular difference term of S_n, and its order will be the sum of the orders of the factors. A given regular difference term of S_n may therefore be such as to admit of being expressed as the product of two or more regular difference terms of S_n of lower order; but, on the other hand, it may not be so expressible. Thus

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)(\beta-\zeta)(\gamma-\eta)(\gamma-\theta)(\delta-\iota)(\delta-\kappa) \times(\epsilon-\zeta)^2(\eta-\theta)^2(\iota-\kappa)^2,$$

a regular difference term of degree 10, order 3, and weight 15, of the system of roots

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \theta,\ \iota,\ \kappa,$$

does not admit of being expressed as the product of regular difference terms of the same system of lower order.

6. Regular difference terms which admit of being expressed as the product of others of the same system, of lower orders, may be said to be *decomposable*.

7. Regular difference terms which are not decomposable may be said to be *primitive*. Elemental terms are, of course, primitive.

8. Regular difference terms which are so completely decomposable that they can be expressed as the product of elemental terms may be designated *pure composite terms*.

9. It is known that, for a given system of n roots, the number of

primitive regular difference terms is limited;* so that every regular difference term is either one of this limited number of primitive terms, or is the product of two or more of these, or of their powers. Some progress has also been made towards the specification of the orders and forms of primitive terms.†

10. It does not, however, appear to have been hitherto observed that *every regular difference term*, whether decomposable or primitive, *of a system of roots S_n, can be expressed* as the sum of pure composite terms of S_n, and therefore *as a rational integral function of elemental terms of S_n*.

11. For example, the primitive regular difference term referred to in § 5, viz.:—

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)(\beta-\zeta)(\gamma-\eta)(\gamma-\theta)(\delta-\iota)(\delta-\kappa) \times(\epsilon-\zeta)^2(\eta-\theta)^2(\iota-\kappa)^2,$$

can be expressed in the form

$$\begin{aligned}
&\left[(\alpha-\beta)(\gamma-\theta)(\delta-\iota)(\epsilon-\kappa)(\zeta-\eta)\right]\\
&\qquad\times\left[(\alpha-\gamma)(\beta-\delta)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right]\\
&\qquad\times\left[(\alpha-\delta)(\beta-\gamma)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right]\\
&+\left[(\alpha-\beta)(\gamma-\zeta)(\delta-\iota)(\epsilon-\kappa)(\eta-\theta)\right]\\
&\qquad\times\left[(\alpha-\gamma)(\beta-\delta)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right]\\
&\qquad\times\left[(\alpha-\delta)(\beta-\eta)(\gamma-\theta)(\epsilon-\zeta)(\iota-\kappa)\right]\\
&+\left[(\alpha-\beta)(\gamma-\theta)(\delta-\epsilon)(\zeta-\eta)(\iota-\kappa)\right]\\
&\qquad\times\left[(\alpha-\gamma)(\beta-\kappa)(\delta-\iota)(\epsilon-\zeta)(\eta-\theta)\right]\\
&\qquad\times\left[(\alpha-\delta)(\beta-\gamma)(\epsilon-\zeta)(\eta-\theta)(\iota-\kappa)\right]\\
&+\left[(\alpha-\beta)(\gamma-\zeta)(\delta-\epsilon)(\eta-\theta)(\iota-\kappa)\right]\\
&\qquad\times\left[(\alpha-\gamma)(\beta-\kappa)(\delta-\iota)(\epsilon-\zeta)(\eta-\theta)\right]\\
&\qquad\times\left[(\alpha-\delta)(\beta-\eta)(\gamma-\theta)(\epsilon-\zeta)(\iota-\kappa)\right],
\end{aligned}$$

* See "Ueber die Endlichkeit des Invarientensystems fur binären Grundformen," by D. Hilbert, in the *Mathematische Annalen*, Vol. XXXIII., where the result is obtained by the aid of a theorem of Professor Gordan with regard to a class of diophantine equations.

† See "Die Theorie der Regularen Graphs," by Julius Petersen, in the *Acta Mathematica*, Vol. XV.

that is, as the sum of four pure composite terms, each of which is the product of three linear elemental terms.

12. The object of the present paper is to demonstrate the theorem of § 10. This theorem is one of some importance. Thus, to confine ourselves to one example, let Q_n be a quantic the roots of which are those of the system S_n; then, if T be any regular difference term of S_n, the expression

$$\Sigma T,$$

where the summation extends to all terms derivable from T by transpositions *inter se* of the n roots, is an invariant of Q_n, and every rational integral invariant of Q_n is a rational integral function of invariants of that form. Now, if T be expressible as the sum of pure composite terms of S_n, every rational integral invariant of Q_n is expressible as a rational integral function of invariants, such as

$$\Sigma E_1^{e_1} . E_2^{e_2} . E_3^{e_3}, \ldots,$$

where E_1, E_2, E_3, ... are all elemental terms of S_n, being linear or quadratic according as n is even or odd, and the summation, as before stated, is of all terms obtainable by transposition (not of the elemental terms E_1, E_2, &c., but) of the roots $\alpha, \beta, \gamma, \ldots$. From this result the proof by Hilbert's method (see foot-note § 9) of the finiteness of the number of the invariants of a quantic Q_n in terms of which the whole system of its invariants may be expressed follows immediately.

13. A regular difference term of degree 2 is of the form

$$(\alpha-\beta)^v,$$

and is clearly a pure composite term, for each factor $(\alpha-\beta)$ is a linear elemental term.

14. A regular difference term of degree 3 is of the form

$$(\alpha-\beta)^m (\beta-\gamma)^m (\gamma-\alpha)^m \equiv \left[(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\right]^m,$$

where $2m = v$, and is clearly also a pure composite term, for each factor $\left[(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\right]$ is a quadratic elemental term.

15. We have

$$(\alpha-\beta)(\gamma-\delta) \equiv (\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\gamma-\beta),$$

and similar identities in the case of any other four roots. By the use of these identities any difference term A' of a degree >3 can be expressed in a variety of ways as the sum

$$A_1'+A_2'+A_3'+\ldots$$

of a number of other difference terms. Each of the latter is derived from A' by transpositions of roots, and therefore the number of α's, of β's, &c., in each is the same as the number in A'. If, then, A' be a regular difference term, each of the terms A_1', A_2', ... must also be regular.

16. The transpositions of the roots referred to in the last section are not transpositions such as those referred to in § 12, viz. :—of the whole set of α's with the whole set of β's, and so on; but are transpositions of individual α's with individual β's, and so on. Thus to each α in A' there will correspond a definite α in A_1', in A_2', and so on. Consequently, if for any root α in A' we substitute a new root ξ, not one of those in A', and if we also substitute ξ for the corresponding α in each of the terms A_1', A_2', A_3', ..., the identity of § 15 will be converted into another identity; and, if any term A_s' of the former identity breaks up into factors in any particular way, there will be a corresponding term in the new identity which will break up into factors in the same way, one of those factors containing a root ξ in place of a root α.

17. In precisely the same way, we may substitute ξ's for any number of α's in A', and thus obtain a new term A, and if we also substitute a root ξ for each of the corresponding α's in each of the terms A_1', A_2', A_3', ..., and thus obtain corresponding terms A_1, A_2, A_3, ..., we shall have

$$A \equiv A_1+A_2+A_3+\ldots,$$

and, if A_s' breaks up into factors in any particular way, A_s will break up into factors in the same way, these factors, however, containing in some cases ξ's in lieu of certain of the α's. This result will be found of importance in the sequel (§ 41).

18. In the demonstration which follows it will be shown that every regular difference term of order v of a system S_n may, by the use of the identities of § 15, be expressed as the sum of certain regular difference terms of S_n of order v, designated *uncrossed terms;* that each of these uncrossed terms may, by the same means, be expressed as the sum of certain other regular difference terms of S_n of order v, called *reducible terms;* and that each of these reducible terms may be expressed as the sum of *pure composite terms* of order v, provided

that every regular difference term of the same order v of any system S_{n-2} of $(n-2)$ roots can be expressed as the sum of pure composite terms of S_{n-2}. Since, then, we know that regular difference terms of order v and of degree 2 or 3 can be expressed as the sum of pure composite terms, being, in fact, themselves pure composite terms (§§ 13, 14), it will follow that every regular difference term of order v and degree n of a system S_n can be expressed as the sum of pure composite terms, and therefore as a rational integral function of elemental terms of S_n.

19. For the purpose of the demonstration let the roots of S_n, taken in any order, be represented respectively by the symbols

$$[1],\ [2],\ [3],\ \ldots\ [n-1],\ [n];$$

and let it be supposed that

$$[n+r] = [r],$$

the roots being thus regarded as composing a cycle of period n. The numbers contained in these symbols may be termed the *places* of the roots they respectively represent.

20. In the case of any difference

$$\pm\{[s]-[r]\},$$

the number $(s-r)$ (which may, of course, be negative) may be termed the *distance* between the roots composing the difference.

21. A difference $\{[q]-[p]\}$

may be thrown into the equivalent forms

$$-\{[p]-[q]\},$$
$$\{[n+q]-[p]\},$$
$$-\{[m+p]-[q]\}.$$

Of these four forms, that will always be supposed to be employed in which the distance between the roots is positive and a minimum. Thus, where a difference

$$\{[s]-[r]\}$$

is considered, it is to be understood that

$$s>r,$$

and $$s-r \overline{\overline{<}} (n+r)-s.$$

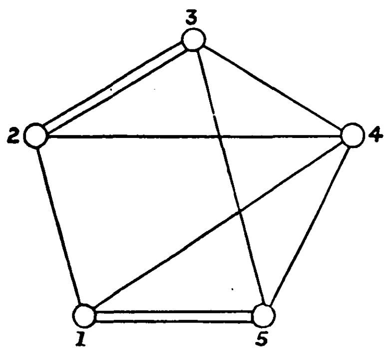
22. We may graphically represent the n roots of a regular difference term by small circular nuclei arranged at the angular points of a regular polygon of n sides, and numbered successively 1, 2, 3, ... n; and any difference

$$\{[s]-[r]\}$$

which is a factor of the term may then be represented by a line lying along a side or diagonal of the polygon connecting the angular points numbered s and r. Thus the regular difference term

$$\{[5]-[4]\}\{[4]-[3]\}\{[3]-[2]\}^2\{[2]-[1]\}\{[5]-[1]\}^2$$
$$\times\{[5]-[3]\}\{[4]-[2]\}\{[4]-[1]\}$$

may be represented by the regular graph



23. If p, q, r, s be four numbers such that

$$s > r > q > p,$$

the two differences

$$\{[s]-[q]\} \quad \text{and} \quad \{[r]-[p]\}$$

may be said to *cross*, or to be *a crossed pair*; other pairs of differences,

e.g., $\{[s]-[r]\}$ and $\{[q]-[p]\}$,

or $\{[s]-[p]\}$ and $\{[r]-[q]\}$,

or $\{[s]-[r]\}$ and $\{[s]-[p]\}$,

being said to be *uncrossed*.

24. In the graphical representation two differences which cross will be represented by two lines lying along diagonals which intersect.

25. A difference term may be said to be *crossed* or *uncrossed* according as it does or does not contain any crossed pairs of factor differences.

26. By the aid of the identity of § 15, we may express the product

$$\{[s]-[q]\}\cdot\{[r]-[p]\}$$

in the form

$$\{[s]-[r]\}\cdot\{[q]-[p]\}+\{[s]-[p]\}\cdot\{[r]-[q]\};$$

i.e., we may express the product of a crossed pair as the sum of two products of uncrossed pairs.

27. Consider now the identity

$$\begin{aligned}&\{[s]-[q]\}\cdot\{[r]-[p]\}\cdot\{[y]-[x]\}\\ \equiv\ &\{[s]-[r]\}\cdot\{[q]-[p]\}\cdot\{[y]-[x]\}\\ &+\{[s]-[p]\}\cdot\{[r]-[q]\}\cdot\{[y]-[x]\}.\end{aligned}$$

Here, if $\{[s]-[r]\}$ and $\{[y]-[x]\}$ are a crossed pair, we have either

$$y>s>x>r>q>p,$$

or $$s>y>r>x>q>p,$$

or $$s>y>r>q \mathrel{\overline{\overline{>}}} x>p,$$

or $$s>y>r>q>p \mathrel{\overline{\overline{>}}} x;$$

and therefore either

$$\{[s]-[q]\} \text{ and } \{[y]-[x]\}$$

are a crossed pair, or

$$\{[r]-[p]\} \text{ and } \{[y]-[x]\}$$

are so.

So, if $\{[q]-[p]\}$ and $\{[y]-[x]\}$ are a crossed pair, we have either

$$y \mathrel{\overline{\overline{>}}} s>r>q>x>p,$$

or $$s>y \mathrel{\overline{\overline{>}}} r>q>x>p,$$

or $$s>r>y>q>x>p,$$

or $$s>r>q>y>p>x;$$

and therefore, again, either

$$\{[s]-[q]\} \text{ and } \{[y]-[x]\}$$

are a crossed pair, or

$$\{[r]-[p]\} \text{ and } \{[y]-[x]\}$$

are so.

Furthermore, if both

$$\{[s]-[r]\} \text{ and } \{[y]-[x]\},$$

and

$$\{[q]-[p]\} \text{ and } \{[y]-[x]\},$$

are crossed pairs, we have

$$s > y > r > q > x > p,$$

and therefore both $\{[s]-[q]\}$ and $\{[y]-[x]\}$,

and $\{[r]-[p]\}$ and $\{[y]-[x]\}$,

are crossed pairs.

Since, then, the pair

$$\{[s]-[q]\} \text{ and } \{[r]-[p]\}$$

are a crossed pair, and the pair

$$\{[s]-[r]\} \text{ and } \{[q]-[p]\}$$

are not, we see that the number of crossed pairs formed by the three factor differences of

$$\{[s]-[q]\}\cdot\{[r]-[p]\}\cdot\{[y]-[x]\}$$

must be greater than the number of crossed pairs formed by the three factor differences of

$$\{[s]-[r]\}\cdot\{[q]-[p]\}\cdot\{[y]-[x]\}.$$

In precisely the same way we may prove that the number of crossed pairs formed by the three factor differences of

$$\{[s]-[q]\}\cdot\{[r]-[p]\}\cdot\{[y]-[x]\}$$

must be greater than the number of crossed pairs formed by the three factor differences of

$$\{[s]-[p]\}\cdot\{[r]-[q]\}\cdot\{[y]-[x]\}.$$

28. Suppose now that the regular difference term T of the system S_n contains as a factor a crossed pair

$$\{[s]-[q]\}\cdot\{[r]-[p]\},$$

so that we may put

$$\begin{aligned} T = & \; L\{[s]-[q]\}\cdot\{[r]-[p]\} \\ = & \; L\{[s]-[r]\}\cdot\{[q]-[p]\} \quad (=T_\lambda) \\ & +L\{[s]-[p]\}\cdot\{[r]-[q]\} \quad (=T_\mu) \\ = & \; T_\lambda+T_\mu; \end{aligned}$$

then, since L consists of factor differences, such as $\{[y]-[x]\}$, it follows immediately from § 27 that both T_λ and T_μ must contain a smaller number of crossed pairs than T.

Taking any crossed pair in T_λ, we may by the same process put

$$T_\lambda = T_\nu + T_\rho,$$

and, similarly, we may put $T_\mu = T_\sigma + T_\tau$,

and therefore $$T = T_\nu + T_\rho + T_\sigma + T_\tau,$$

where T_ν and T_ρ contain a smaller number of crossed pairs than T_λ, and T_σ and T_τ a smaller number than T_μ.

Proceeding to deal with T_ν, T_ρ, T_σ, and T_τ in the same way, and continuing the process on the derived terms, we shall at each stage obtain terms containing a smaller number of crossed pairs than are contained in the terms from which they are derived; and we can continue the process on the derived terms so long as we obtain terms which contain any crossed pairs. In this way we shall ultimately be able to put

$$T \equiv U_1 + U_2 + U_3 + \dots,$$

where the terms on the right-hand side of the identity are all uncrossed regular difference terms of S_n, none of them containing any crossed pairs.

29. We proceed next to consider a special property possessed by any uncrossed regular difference term U of S_n; and to show that, by means of the identities of § 15, U can be expressed as the sum of certain other regular difference terms of S_n, designated *reducible terms.*

30. The differences under consideration may be divided nto two classes, viz.:—we have differences of the form

$$\{[r+1]-[r]\},$$

in which the distance between the roots is unity, and differences in which the distance is greater than unity.

In the graphical representation of a regular difference term, made in accordance with § 22, differences of the former description will be represented by lines lying along the sides of the polygon. Such differences may accordingly be called *side differences.*

Differences of the other sort will similarly be represented by lines lying along the diagonals of the polygon, and may therefore be called *diagonal differences.*

31. The special property of U which we have to consider is this—there must be one root of the system S_n which enters only into factor differences of U which are side differences, and does not enter into any which are diagonal. There must, in fact, be two such roots; but the existence of one is sufficient for our purposes.

In other words, there must be a root $[r]$ which enters only into differences

$$\{[r+1]-[r]\} \quad \text{and} \quad \{[r]-[r-1]\},$$

and consequently U contains a factor

$$\{[r+1]-[r]\}^k \cdot \{[r]-[r-1]\}^{v-k},$$

where the sum of the indices is v, the order of U.

32. The proof of this presents no difficulty. If there are any diagonal differences which are factors of U, there must be one or more in which the roots are at a minimum distance apart d, where

$$d \geqq 2.$$

Let

$$\{[s+d]-[s]\}$$

be one of these. Then each of the roots

$$[s+1],\ [s+2],\ \dots\ [s+d-1],$$

enters only into side differences. For, since U is uncrossed, any root $[s+c]$, where $c < d$ and >0, cannot enter into differences which cross

$$\{[s+d]-[s]\};$$

i.e., $[s+c]$ can only enter into differences such as

$$\{[s+c]-[s]\},$$

where $$\overline{\overline{>}}\ s,$$

or such as $$\{[f]-[s+c]\},$$

where $$f \overline{\overline{<}}\ s+d.$$

In the former case the distance between the roots is

$$s+c-e,$$

which, since $c>d$ and $e \overline{\overline{>}}\ s$, must be $<d$.

In the latter case, the distance between the roots is

$$f-s-c,$$

which, since $$f \overline{\overline{<}}\ s+d,$$

must also be $<d$.

Thus $[s+c]$ can only enter into differences in which the distance between the roots is $<d$.

But the minimum distance between the roots in the case of the diagonal differences of U is d. Thus $[s+c]$ cannot enter into any diagonal differences, but only into side differences.

33. The result arrived at might also have been obtained from a consideration of the graphical representation of U, in which there will be no intersecting diagonals (§ 24); and from a recognition of the fact that, in a regular polygon in which there are no intersecting diagonals, there must be at least two summits from which no diagonals proceed.

34. Since, then, U contains a factor

$$\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k},$$

we may put

$$U=C\{[r+1]-[r-1]\}^h\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k},$$

where C has no factors containing $[r]$, and no factor

$$\{[r+1]-[r-1]\}.$$

35. Now, each root of S_n enters v times into U; thus there will be $(v-h-k)$ factor differences in C containing $[r+1]$, and $(k-h)$ containing $[r-1]$, and these differences will be distinct from each other.

Let the product of the former be denoted by C_{v-h-k}, and of the latter by C_{k-h}. Then we may put

$$U = D.C_{v-h-k}C_{k-h}\{[r+1]-[r-1]\}^h\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-h},$$

where D is the product of factor differences which do not contain either $[r+1]$, $[r]$, or $[r-1]$. The number of these will be

$$\begin{aligned} & w-(v-h-k)-(k-h)-h-k-(v-k) \\ &= w-2v+h \\ &= \frac{vn}{2}-2v+h \quad (\S\,4) \\ &= \frac{v}{2}(n-4)+h. \end{aligned}$$

Now, in cases where $n>3$, and it is with such that we are now dealing (§ 15), this last number must be at least h, *i.e.*, there are as many factors in D as in

$$\{[r+1]-[r-1]\}^h.$$

36. Let

$$\{[b]-[a]\}$$

be any factor of D; then, by the identity of § 15, we have

$$\begin{aligned} & \{[r+1]-[r-1]\}\,.\,\{[b]-[a]\} \\ \equiv\; & \{[r+1]-[b]\}\,.\,\{[r-1]-[a]\} \\ +\; & \{[r+1]-[a]\}\,.\,\{[b]-[r-1]\}; \end{aligned}$$

i.e., since $[a]$ and $[b]$ are both different from $[r+1]$ and $[r-1]$, D being the product of factor differences which contain neither of the latter roots (§ 35), we can express

$$\{[r+1]-[r-1]\}\{[b]-[a]\}$$

as the sum of two terms which do not contain

$$\{[r+1]-[r-1]\}$$

as a factor.

In the same way, by taking h factor differences of D, we can express each of the h factors of

$$\{[r+1]-[r-1]\}^h,$$

when multiplied by one of those factors of D, as the sum of two terms which do not contain

$$\{[r+1]-[r-1]\}$$

as a factor, and thus we can express

$$D\{[r+1]-[r-1]\}^{k}$$

as the sum of terms which do not contain

$$\{[r+1]-[r-1]\}$$

as a factor.

37. We may therefore put

$$U \equiv \{[r+1]-[r]\}^{k}\{[r]-[r-1]\}^{v-k}\{R_1+R_2+R_3+\ldots\},$$

where none of the terms $R_1, R_2, R_3, \ldots$ contain $[r]$ or the factor

$$\{[r+1]-[r-1]\}.$$

38. Any regular difference term

$$\{[r+1]-[r]\}^{k}\{[r]-[r-1]\}^{v-k}R,$$

where R does not contain

$$\{[r+1]-[r-1]\}$$

as a factor, and does not contain $[r]$, may be said to be a *reducible* regular difference term of the system of roots S_n.

39. Each of the roots of S_n other than $[r-1]$, $[r]$, and $[r+1]$ enters v times into R; the root $[r]$ does not enter at all; the root $[r-1]$ enters k times, and the root $[r+1]$ enters $(v-k)$ times. If, then, we were in R to put the root $[r+1]$ in place of the root $[r-1]$, $[r+1]$ would enter v times, and R would become a regular difference term R', of order v and degree $(n-2)$, of the system of roots S_{n-2} obtained by withdrawing the roots $[r]$ and $[r-1]$ from the system S_n.

40. Consequently the difference term R may be obtained by substituting for k properly selected roots $[r+1]$ in R', k roots $[r-1]$, where $[r-1]$ is not a root which enters into R'.

41. Suppose, then, that any regular difference term R' of order v of a system S_{n-2} of $(n-2)$ roots can be expressed as the sum

$$R'_1+R'_2+R'_3+\ldots$$

of a number of pure composite terms of that system. It follows immediately from § 17 that we can put

$$R \equiv R_1+R_2+R_3+\ldots,$$

where R is obtained by substituting k roots $[r-1]$ for k properly selected roots $[r+1]$ in R', and $R_1, R_2, R_3, \ldots$ are obtained by making corresponding substitutions in $R'_1, R'_2, R'_3, \ldots$. It also follows that $R_1, R_2, R_3, \ldots$ break up into factors corresponding to the elemental factors of the pure composite terms $R'_1, R'_2, R'_3, \ldots$.

42. Taking first the case where n is even, and therefore $n-2$ is even, and consequently the elemental factors of any term R'_s are linear (§ 4) and v in number, we see that R_s, the corresponding term, breaks up into v factors, into each of which each of the roots of S_{n-2} other than $[r+1]$ enters once and once only. Into k of these factors $[r+1]$ will not enter, but $[r-1]$ will do so in the case of each once and once only; while $[r-1]$ will not enter into the remaining $v-k$, but $[r+1]$ will do so in the case of each once and once only.

Now, considering the term

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s,$$

we see that, corresponding to each of the k factors of R_s into which $[r-1]$ enters, there is a factor

$$\{[r+1]-[r]\},$$

and, corresponding to each of the $v-k$ factors of R_s into which $[r+1]$ enters, there is a factor

$$\{[r]-[r-1]\},$$

and in each case the product of the two corresponding factors gives a term into which each of the n roots of S_n enters once and once only, *i.e.*, gives a linear elemental term of S_n.

Thus $$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s$$

is the product of v linear elemental terms of S_n.

Hence, provided that R'_s is a pure composite term of the system S_{n-2},

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s$$

will be a pure composite term of the system S_n; and, consequently, provided that R' can be expressed as the sum of terms such as R'_s, which are pure composite terms of S_{n-2}, R can be expressed as the sum of terms such as

$$\{[r+1]-[r]\}^k \{[r]-[r-1]\}^{v-k} R_s,$$

which are pure composite terms of S_n.

43. We can arrive at the same result in the case where n, and therefore also $n-2$, is odd. Here the elemental factors of R'_{r} are quadratic, and are $\frac{v}{2}$ in number, v being necessarily even, (§ 4). Consequently R_{r} breaks up into $\frac{v}{2}$ factors, into each of which each of the roots of S_{n-2} other than $[r+1]$ enters twice and twice only. Into f of these factors (where f is zero or some integer not $> \frac{v}{2}$, and such that $k-f$, and therefore also $v-k-f$, is even) the roots $[r-1]$ and $[r+1]$ both enter once and once only; into $\frac{k-f}{2}$ of the remaining factors $[r-1]$ enters twice and twice only, and $[r+1]$ does not enter at all; and into the rest of the factors, $\frac{(v-k-f)}{2}$ in number, $[r+1]$ enters twice, and twice only, and $[r-1]$ does not enter at all.

Hence, considering the term

$$\{[r+1]-[r]\}^{k}\{[r]-[r-1]\}^{v-k}R_{r},$$

we see that, corresponding to each of the f factors of the first description, we may take a factor

$$\{[r+1]-[r]\}.\{[r]-[r-1]\};$$

corresponding to each of the $\frac{(k-f)}{2}$ factors of the second description, we may take a factor

$$\{[r+1]-[r]\}^{2};$$

and, corresponding to each of the $\frac{(v-k-f)}{2}$ factors of the third description, we may take a factor

$$\{[r]-[r-1]\}^{2};$$

and in so doing we shall exactly take all the factors of

$$\{[r+1]-[r]^{k}\{[r]-[r-1]\}^{v-k}.$$

In each case the product of the two corresponding factors gives a term into which each of the n roots of S_n enters twice and twice only, *i.e.*, gives a quadratic elemental term of S_n.

Thus $$\{[r+1]-[r]\}^{k}\{[r]-[r-1]\}^{v-k}R_{r}$$

is the product of $\frac{v}{2}$ quadratic elemental terms of S_n, and is therefore a pure composite term of S_n.

Consequently, as in the preceding section, we see that, provided that R'_s is a pure composite term of the system S_{n-2},

$$\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k}R_s$$

will be a pure composite term of S_n, and therefore, provided that R' can be expressed as the sum of terms such as R'_s, which are pure composite terms of S_{n-2}, R can be expressed as the sum of terms such as

$$\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k}R_s,$$

which are pure composite terms of S_n.

44. Whether, then, n be even or odd,

$$\{[r+1]-[r]\}^k\{[r]-[r-1]\}^{v-k}R,$$

i.e., any reducible term of S_n, can be expressed as the sum of terms which are pure composite terms of S_n, provided that R' can be expressed as the sum of terms which are pure composite terms of S_{n-2}.

45. Hence the general regular difference term T of S_n of order v, being expressible as the sum of uncrossed terms of S_n (§ 28), which uncrossed terms are expressible as the sum of reducible terms of S_n (§ 37), can be expressed as the sum of pure composite terms of S_n, provided that any regular difference term of S_{n-2} of order v can be expressed as the sum of pure composite terms of S_{n-2}.

46. In other words, any regular difference term of order v and degree n can be expressed as a rational integral function of elemental terms of the system of roots to which it belongs, provided that any regular difference term of order v and degree $n-2$ can be so expressed. But we have seen (§§ 13, 14) that regular difference terms of order v and of degrees 2 and 3 can be so expressed. Therefore every regular difference term can be expressed as a rational integral function of the elemental terms of the system of roots to which it belongs.

47. We may carry the matter a step further. If we represent the roots, as hitherto, by the symbols [1], [2], [3], ... [n], the elemental terms may be divided into crossed and uncrossed terms. Now, every crossed elemental term may be expressed as the sum of uncrossed elemental terms (§ 28). Hence *every regular difference term can be expressed as a rational integral function of the uncrossed elemental terms of the system of roots*

$$[1],\ [2],\ \dots\ [n]$$

to which it belongs.

Thursday, May 10th, 1894.

Prof. GREENHILL, F.R.S., Vice-President, in the Chair.

The following communications were made:—

On the Kinematical Discrimination of the Euclidean and non-Euclidean Geometries: Mr. A. E. H. Love.

Permutations on a Regular Polygon: Major MacMahon.

The Stability of a Tube: Professor Greenhill (Dr. J. Larmor in the Chair).

Researches in the Calculus of Variations—Part V., The Discrimination of Maxima and Minima Values of Integrals with Arbitrary Values of the Limiting Variations; Part VI., The Theory of Discontinuous or Compounded Solutions: Mr. E. P. Culverwell.

The following present was made for the Album:—cabinet likeness of Mr. E. P. Culverwell.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LV., No. 332.

"Philosophical Transactions of the Royal Society," Vols. 180–184, 1889–1893, and a list of Fellows of the Society, dated November 30th, 1893.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 4; Leipzig, 1894.

"Seventh Annual Report of the Canadian Institute"; Toronto, 1894.

"Bulletin of the New York Mathematical Society," Vol. III., No. 7.

"Bulletin des Sciences Mathématiques," Tome XVIII., Fév. and Mars, 1894; Paris, 1894.

Macfarlane, Alex.—"Principles of Elliptic and Hyperbolic Analysis," 8vo; Boston.

"Transactions of the Russian Mathematical Society," Tome XV.; Odessa, 1893.

"Transactions of the Canadian Institute," No. 7, Vol. IV., Pt. I., March, 1894; Toronto.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 7, 1 Sem.; Roma, 1894.

"Annali di Matematico," Serie 2, Tomo XXII., Fasc. 1 and 2, April, 1894; Milano.

"Educational Times," May, 1894.

"Annales de la Faculté des Sciences de Toulouse," Tome VIII., Fasc. 1; Paris, 1894.

"Journal für die reine und angewandte Mathematik," Bd. CXIII., Heft 2; Berlin, 1894.

"Annals of Mathematics," Vol. VIII., No. 4; Virginia University, May, 1894.

"Indian Engineering," Vol. xv., Nos. 12–15.
"Trigonometrical Survey of India," Vol. xv.; Dehra Dun, 1893.
Wright, J. M. F.—"Commentary on Newton's 'Principia,'" 2 vols., 8vo; London, 1828.
Brougham, Henry Lord, and E. J. Routh.—"Analytical View of Newton's 'Principia,'" 8vo; London, 1855.
"American Journal of Mathematics," Vol. xvi., No. 2; Baltimore, April, 1894.
Byerly, W. E.—"Fourier's Series and Spherical Harmonics," 8vo; Boston, 1893.

Researches in the Calculus of Variations—Part V., The Discrimination of Maxima and Minima Values of Integrals with Arbitrary Values of the Limiting Variations. By E. P. Culverwell, M.A., F.T.C.D. Received May 8th, 1894. Read May 10th, 1894.

1. Discussions of true maxima and minima of integrals with variable limits, as distinguished from merely stationary solutions, are rare in the standard text-books. Moigno has none; Jellett, Todhunter, and Carll have each obtained different and erroneous results in the one example they all give, that of the maximum solid of revolution for given superficial area (see Jellett, *Cal. of Var.*, pp. 161–165; Todhunter, *History of Cal. of Var.*, p. 408; Carll, *Cal. of Var.*, pp. 122 and 129). The only other problem with variable limits I can find attempted in those text-books is one selected by Mr. Todhunter in his *History*, p. 328, in order to show that the ordinary method is insufficient when the limits themselves enter into the quantity to be integrated. Mr. Carll adopts Mr. Todhunter's view, insisting even more strongly on the inadequacy of the ordinary method. But the ordinary method, though clumsy, is in every case adequate.

The absence of examples is doubtless due to the fact that writers on the calculus of variations have considered the variability of the constants as introducing only a problem of the differential calculus, and have contented themselves by saying that, if the stationary value of the integral be expressed in terms of the arbitrary constants, the rule for ascertaining whether the solution is a maximum or a mini-