PAPERS

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ON THE ROOTS OF THE EQUATION $\dfrac{1}{\Gamma(x+1)} = c$

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1. The present paper forms part of some investigations concerning the roots of a number of transcendental equations of particular forms, which I have undertaken in the hope of throwing some light on the exceedingly difficult and important general question of the relations subsisting between the roots of the equations comprised in the form $F(x) = \phi(x)$, where $F(x)$ is a given integral function, and $\phi(x)$ a constant polynomial, or integral function whose increase (*croissance*) is less than that of $F(x)$. I need hardly say that our present knowledge about this question is almost entirely limited to the *moduli* of the roots; what we know about the arguments is practically *nil*: and I think that the results which I have obtained may be of some interest, in spite of their very special character, as indicating to some extent the various kinds of cases which may occur.

I have considered particularly the equations

$$(1)\quad \sin x = P(x),$$

$$(2)\quad e^{-ix} = P(x),$$

$$(3)\quad e^{ax} \sin bx = P(x),$$

(a and b being real and positive).

$$(4)\quad \Pi_\rho(x) \equiv \prod_1^\infty (1+x/n^\rho) = P(x)$$

—$\Pi_\rho(x)$ being one of the functions considered by Mr. E. W. Barnes in his memoir "On Integral Functions,"* and $\rho > 1$—and

$$(5) \quad \frac{1}{\Gamma(x+1)} = P(x).$$

In all of these $P(x)$ is an arbitrary polynomial. The question to which I have given especial attention is whether there is in each case any form of the polynomial for which the nature of the zeroes is abnormal. The asymptotic solutions of (1) and (2) were given in two papers in the *Messenger*.† In the case of (1) there is no abnormal case; in the case of (2) the only abnormal case is that of $P(x) \equiv 0$, the familiar Picard case of exception in which there are no roots at all. In the case of (3), which I have investigated in a recent paper in the *Quarterly Journal*,‡ there is again one abnormal case, that of $P(x) \equiv 0$, but it is abnormal in quite a different way, which essentially involves the arguments of the roots: and the same is true of (4) if $1 < \rho < 2$; but, if $2 < \rho$, there is no abnormal case. I hope on some future occasion to make a further communication concerning these equations, with especial reference to the case of $\rho = 2$. At present I shall confine myself to the equation (5). I may, however, remark that in every case the following proposition is true:—*if* $a_n(c)$ *is the* n-th *root of* $F(x) = c$, and one particular value of c is excluded from consideration, *then the roots can be arranged in a finite number of groups, such that within each group*

$$\lim_{n=\infty} \frac{a_n(c')}{a_n(c'')} = 1.$$

It is, of course, supposed that in each group the roots are arranged according to ascending order of moduli. For the constants c, c', c'' we may substitute polynomials. That any such theorem is true *in general* I do not for one moment suppose, even if we confine ourselves to the moduli of the roots§; but it is certainly true for large classes of the most important functions. I may add that it may be shown that the exceptional case in equations (3) and (4) is exceptional in the same way with whole classes of functions.

2. I come now to the equation (5). The function $\Pi(x) \equiv \dfrac{1}{\Gamma(x+1)}$ is

* *Phil. Trans.* (A), Vol. CXCIX., p. 411.

† Vol. XXXI., p. 161, and Vol. XXXII., p. 36.

‡ Vol. XXXV., p. 261.

§ That it is true in this sense is suggested by Borel, *Fonctions entières*, p. 100.

an integral function whose apparent and real orders* are each unity. Now M. Borel has proved the two following theorems, the second being a generalization of Picard's theorem :—

(i.) If the apparent order of $F(x)$ is finite and not integral, the real order is equal to it :

(ii.) If the apparent order is an integer p, then among the functions $\phi(x) F(x) - \psi(x)$, where ϕ, ψ are integral functions whose apparent order $< p$, there is one at most whose real order $< p$.

From the second theorem it follows that the real order of all the functions $\Pi(x) - c$ is the same, with possibly one exception. Is there such an exception? The answer seems to me very interesting. It is that there is not, and yet that the case $c = 0$ is abnormal not merely as regards the distribution of the zeroes in the plane, but also as regards the increase of their moduli. In fact, the increase of the zeroes for $c \neq 0$ is like that of $n/\log n$, whereas that of the zeroes for $c = 0$ is, of course, like that of n. This shows the possibility of cases of exception of a nature too subtle to be indicated by any alteration of the real order of the function.†

The result is also interesting in connection with the apparent paradox originally noted by M. Borel, that the increases of the functions

$$(6) \quad \Pi(x), \quad \sin \tfrac{1}{2}\pi x$$

are different, being those (roughly) of

$$(7) \quad e^{r \log r}, \quad e^{r}$$

while the increases of the moduli of their zeroes are the same. One is tempted to say that, if we substitute for $\Pi(x)$

$$(8) \quad \Pi(x) - c,$$

* See *Leçons sur les Fonctions entières*. The *real order* ρ of a function is the index of convergence of the reciprocals of the moduli of its zeroes ; the *apparent order* is the least number ρ' such that, however small be ϵ, the maximum of $|F(x)|$ on a circle of radius r is less than $\exp(r^{\rho'+\epsilon})$ for all values of r greater than a certain value. It follows from the first theorem that the Picard case can only occur if ρ is an integer ; but cases in which the behaviour of the zeroes is abnormal for a particular value of c certainly can, as is shown by the example of the function $\Pi_\rho(x)$.

† Two very important memoirs on integral functions have appeared in the last few years— P. Boutroux, "Sur quelques propriétés des fonctions entières (*Acta Mathematica*, Vol. XXVIII.) ; E. Lindelöf, "Mémoire sur la théorie des fonctions entières de genre fini" (*Acta Soc. Fennicæ*, Vol. XXXI.). Each of these writers has introduced greater precision into the known theorems concerning functions of non-integral order and has proved interesting results concerning functions of integral order. I refer further in the text to some of M. Boutroux's results. I am indebted to M. Boutroux for a number of suggestions concerning the results proved and referred to in this paper.

the apparent paradox disappears, as we have then for the increases of the moduli of the zeroes

$$(9)\quad n/\log n,\ n,$$

which correspond naturally enough to (7). But this, as M. Boutroux pointed out to me, is not quite a sufficient account of the matter. M. Boutroux has in fact shown that in the case of a function of integral real order there are *two* typical laws of increase of its modulus. These typical laws may be said roughly to correspond to the cases in which all the roots are real and (*a*) all positive, (*b*) equal and opposite in pairs. The two laws for the function corresponding to the law n for the zeroes would be e^r, $e^{r\log r}$, and corresponding to the law $n/\log n$ would be $e^{r\log r}$, $e^{r(\log r)^2}$. This sufficiently elucidates the behaviour of the functions (6); but seems at first sight to raise a precisely similar difficulty with reference to the function (8), since its increase is the same whether $c=0$ or not, while the increase of its zeroes differs in the two cases. This difficulty, however, disappears when we note (what will be obvious later on) that what we may call the two *principal* sets of zeroes of (8), *i.e.* those whose increase is *least*, are arranged on the "equal and opposite" type (*b*), approximately of course. M. Boutroux when writing his memoir was of course not aware of the nature of the zeroes of (8). I proceed now to the proof of the assertions which I have made about them.

2. It has been shown by Mellin* that, if $-\pi+\epsilon < am\,.\,x < \pi-\epsilon$,

$$\Pi(x) = \frac{1}{\sqrt{(2\pi)}} \exp\left\{-(x+\tfrac{1}{2})\log x + x - \sum_1^k (-)^{\nu-1}\frac{B_\nu}{2\nu(2\nu-1)}x^{1-2\nu} + I(x,\kappa)\right\}, \tag{A}$$

where
$$I(x,\kappa) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin\pi t}\,\frac{x^t}{t}\,\zeta(t)\,dt$$
$$(-2k-1<\kappa<-2k+1),$$

k being a positive integer, $\zeta(t)$ the Riemann ζ function, and $\log n$ having its principal value; and that $|I(x,\kappa)| < C|x|^\kappa$ where C depends only on κ and on ϵ. From this and the formula

$$\Pi(x)\,\Pi(-x-1) = -\pi^{-1}\sin\pi x$$

we easily deduce that

$$\Pi(x) = -\frac{\sin\pi x}{\sqrt{(\frac{1}{2}\pi)}}\exp\{-(x+\tfrac{1}{2})\log(-x-1)+x+1+\dots\},$$

an asymptotic expression for $\Pi(x)$ valid throughout a region of the plane

* *Acta Societatis Fennicæ*, Vol. XXIX.

which includes the part hitherto excluded; and it is easy to see that the two expressions are equivalent in the domain common to their regions of validity. Moreover, these expressions hold *uniformly* for all values of the amplitude of x; *i.e.* the ratio of $\Pi(x)$ to one or other (or to either) of them differs from unity by a quantity numerically less than $C|x|^{\kappa}$, C being independent of the amplitude of x.

Now, if $x = re^{i\theta}$,

$$|\exp\{-(x+\tfrac{1}{2})\log x + x\}| = \exp\{-(r\log r - r)\cos\theta + r\theta\sin\theta - \tfrac{1}{2}\log r\}.$$

This tends to ∞ with r if $\frac{1}{2}\pi \leqslant \theta \leqslant \pi-\epsilon$ or $-\pi+\epsilon \leqslant \theta \leqslant -\frac{1}{2}\pi$, to 0 if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. If $\pi-\epsilon < \theta < \pi$ or $-\pi < \theta < -\pi+\epsilon$, we see by the help of the other asymptotic expression that $|\Pi(x)|$ becomes on the whole exceedingly large, except in the immediate neighbourhood of its zeroes.* Thus we see that the large roots of $\Pi(x) = c$ must be sought either in the direction of the negative real axis, or in the direction of either part of the imaginary axis; and that the roots (if any) in the latter directions will lie to the right of the imaginary axis.†

The first series of roots does not particularly interest us here. It is not difficult to show that, if we draw the curves $|\Pi(x)| = |c|$, those of them which lie in the direction of the negative real axis at a great distance from the origin are small closed curves surrounding the points $-n$; and hence that the roots of $\Pi(x) = c$ tend asymptotically to the points $-n$, one and only one being associated with each point, and passing continuously into it for $c = 0$. I proceed to consider the other sets of zeroes. Suppose that

$$\Pi(x) = c = \gamma e^{i\mu}, \quad x = \xi + i\eta,$$

ξ, η being positive and η large. If we make $k = 1$ in (A), we find that

$$\log\Pi(x) = -\log\sqrt{(2\pi)} - (x+\tfrac{1}{2})\log x + x$$
$$-\frac{B_1}{2x} + \frac{1}{2\pi i}\int_{\kappa-i\infty}^{-\kappa+i\infty}\frac{\pi}{\sin\pi t}\frac{x^t}{t}\zeta(t)\,dt \quad (-3 < \kappa < -1).$$

If we vary κ so that the line $(\kappa-i\infty,\ \kappa+i\infty)$ lies to the right of the point -1, as by taking $\kappa = -\frac{1}{2}$, we obtain

* As do simpler functions, such as $e^x \sin x$ when x is large and positive.

† It is essential to the truth of these statements that we should know that $|\Pi(x)|$ tends *uniformly* to ∞ with $|x|$ throughout the region $\pi-\epsilon < \theta < -\pi+\epsilon$. It is perfectly possible for the modulus of a function to tend to ∞ with $|x|$ in *every* direction, and yet for it to have an infinity of roots near ∞; but in such a case it must tend to ∞ non-uniformly. See a recent note in the *Comptes Rendus* by Prof. Mittag-Leffler.

$$\log \Pi(x) = -\log \sqrt{(2\pi)} - (x+\tfrac{1}{2}) \log x + x + \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi}{\sin \pi t} \frac{x^t}{t} \zeta(t)\, dt,$$

since the residue at $t = -1$ is

$$\frac{1}{x} \zeta(-1) = -\frac{B_1}{2x}.$$

Now, if
$$\phi(x) = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi}{\sin \pi t} \frac{x^t}{t} \zeta(t)\, dt,$$

$$\frac{d\phi}{dx} = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi}{\sin \pi t}\, x^{t-1} \zeta(t)\, dt,$$

and so *
$$|\phi(x)| < C\, |x|^{-\frac{1}{2}}, \qquad \left|\frac{d\phi}{dx}\right| < C\, |x|^{-\frac{3}{2}},$$

and, if $\phi(x) = u + iv$, the moduli of $\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}$ and $\frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}$ are each numerically less than a constant multiple of $|x|^{-\frac{3}{2}}$.

Now
$$\exp\{-(x+\tfrac{1}{2}) \log x + x + \phi(x)\} = c\sqrt{(2\pi)},$$

and therefore

$$-(x+\tfrac{1}{2}) \log x + x + \phi(x) = \log \gamma \sqrt{(2\pi)} - i(2p\pi - \mu)$$

where p is an integer. That is

$$(1) \quad -\tfrac{1}{2}(\xi+\tfrac{1}{2}) \log(\xi^2+\eta^2) + \eta \tan^{-1}\frac{\eta}{\xi} + \xi + u = \log \gamma \sqrt{(2\pi)}$$

$$(2) \quad -(\xi+\tfrac{1}{2}) \tan^{-1}\frac{\eta}{\xi} - \tfrac{1}{2}\eta \log(\xi^2+\eta^2) + \eta + v = -2p\pi + \mu,$$

$\tan^{-1}\frac{\eta}{\xi}$ being a positive angle $< \frac{1}{2}\pi$ and, since η is large compared with ξ, nearly $= \frac{1}{2}\pi$.

Let us consider the curves (1), (2). Since η is large, and large compared with ξ, it is clear that p is large and positive and that a first approximation to the form of the curves (2) in the part of the plane under consideration is given by $\eta \log \eta = 2p\pi$. It is easy to see † that a region can be defined by inequalities of the form

$$\frac{\pi}{2} - \delta \leqslant \theta \leqslant \tfrac{1}{2}\pi, \quad r \geqslant R$$

* See Mellin, *loc. cit.* This follows from the fact that $\lim_{t=-\frac{1}{2}\pm i\infty} e^{-\epsilon|t|}\zeta(t) = 0$ for any $\epsilon > 0$.

† These assertions are easily verified, and there is nothing of interest in the formal proof of them.

(δ being small and R large) within which to each value of ξ corresponds one and only one value of η for each of the curves (2), that the values of η increase with p, and that $d\eta/d\xi$ is small and negative along each of the curves. Moreover, the curve (1) consists of a single branch whose equation may be put in the form $\xi = \frac{1}{2}\pi\eta(1+\epsilon)/\log\eta$ where ϵ is very small; and along this curve $d\eta/d\xi$ is large and positive. From these facts it follows that the curve (1) cuts each of the curves (2) in one and only one point in the region in question. Each of these points is a root of $\Pi(x) = c$, and it is clear that at such a point

$$\eta = 2p\pi(1+\epsilon)/\log p, \qquad \xi = p^2\pi(1+\epsilon)/(\log p)^2.$$

Therefore *the equation* $\Pi(x) = c$ $(c \neq 0)$ *has an infinity of roots lying in the direction of the imaginary axis, and given by the formula*

$$x_p = \frac{\pi^2 p}{(\log p)^2}(1+\epsilon) + \frac{2\pi p i}{\log p}(1+\epsilon)$$

where in each bracket ϵ *is a quantity whose limit for* $p = \infty$ *is* 0. Also

$$|x_p| = r_p = 2\pi p(1+\epsilon)/\log p;$$

so that the increase of these roots is that of $p/\log p$.

It is obvious that there is a corresponding set of roots in the negative direction of the imaginary axis. The equation $\Pi(x) = c$ has therefore three sets of roots whose increases are p, $p/\log p$, $p/\log p$; and so the increase of its roots is on the whole $p/\log p$.

3. An investigation only very slightly more complicated shows that the increase of the roots of the equation

$$\Pi(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

is also $p/\log p$. In fact, we obtain exactly the same first approximation to them as in the simpler case. I have not been successful in an attempt to approximate to the roots with sufficient accuracy to distinguish between the nature of the roots for different values of n. In the case of the other equations referred to at the beginning of this paper a more accurate approximation is possible.