



LVI. Quaternions as a practical instrument of physical research

Alexander McAulay M.A.

To cite this article: Alexander McAulay M.A. (1892) LVI. Quaternions as a practical instrument of physical research , Philosophical Magazine Series 5, 33:205, 477-495, DOI: [10.1080/14786449208620289](https://doi.org/10.1080/14786449208620289)

To link to this article: <http://dx.doi.org/10.1080/14786449208620289>



Published online: 08 May 2009.



Submit your article to this journal [↗](#)



Article views: 4



View related articles [↗](#)

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[FIFTH SERIES.]

JUNE 1892.

LVI. *Quaternions as a practical Instrument of Physical Research.* By ALEXANDER M^CAULAY, M.A., Ormond College, Melbourne*.

IN writing the history of the Mathematics of the 19th century the historian will be brought face to face with a phenomenon hard to account for.

The inventor of quaternions was one of the greatest, perhaps the greatest, mathematician of this century. His work was varied and far reaching in its effects, but his name was always associated in the mind of the hearer with one well-defined group of his works—his quaternion researches. Thus the subject was brought forth in the full light of day, and has by no means passed into the limbo of forgotten things. Indeed the word “Quaternions” is fully as familiar even with non-mathematicians as the phrase “Cartesian Geometry.”

But in spite of this he has left scarcely a successor. His two huge volumes form far more than half the whole original work that has been done in the subject. And this notwithstanding the fact that he deliberately did not attempt to apply the subject to Physics, although expressing his belief that in other hands it would prove a powerful instrument of research.

Can any cause be assigned for this extraordinary case of arrested development? The answer that by far the majority of physicists would give as to the physical aspect of the subject is, that the instrument is an admirable one for

* Communicated by the Author; having been read before the Australian Association for the Advancement of Science, January 1892.

expressing the results obtained by the old-fashioned and artificial methods—but there their admiration would stop. The stern work, say they, of advancement into the unknown regions must be made by the clumsier but stronger means. When the country has been reduced to order and civilization, let quaternions be introduced as a luxury conducing to the recreation of the exhausted toilers. Nevertheless there is a small minority among these workers that venture to deny this slur on the character of their favourite implement. They assert that it has only to be used to be appreciated, and this to such an extent that all other geometrical implements are, or ought to be, cast aside in favour of it. I confess myself to be one of the extremest partizans of this sect. When directly I proceed, however, to state exactly what I believe to be the mission of quaternions in the domain of Physics, it will appear that in a short paper it is impossible to establish the contentions. The chief object of the present paper is to *shake* the belief of mathematical physicists. It is too much to hope to *overturn* that belief.

The first question to answer is the one already put—Can the apathy of physicists with regard to quaternions be accounted for if it be not that quaternions are unsuitable for their purposes? I confess that the more I think of this apathy the more extraordinary does it appear, and, as already hinted, it will probably prove an insoluble problem to the future historian of Mathematics. But reasons can be given, though not perfectly satisfactory—at least to my mind—for this almost criminal negligence.

Let us state the case against us. It can be put in very few words as follows—*Not much advance in Physics has been made by the aid of Quaternions.* I do not for a moment question this. It is too evident even for an advocate to blink. In all sincerity, however, I believe that this is all that can be assigned as reason for believing that quaternions do not provide means for extending the bounds of the known in the domain of Physics. Probably through the minds of more than one who listen to this is passing the thought—“But *I* have tried quaternions and *I* have found them wanting. I came to the subject unprejudiced, but I could produce nothing out of it.” Perhaps it is to the point to say that the present writer has passed through this phase. He, too, found that though quaternions were a fascinating study, they seemed to fail, for some inscrutable reason, to furnish the means for any real advance. Looking back, I think I can see the reason for this.

Maxwell, I fear, is responsible to a large extent for the

discredit into which quaternions have fallen among physicists. In his 'Electricity and Magnetism' (2nd edit. § 10) he remarks:—"I am convinced that the introduction of the ideas, *as distinguished from the operations and methods*, of Quaternions will be of great use to us in all parts of our subject." Now though very many study quaternions in a sort of dilettante way, very few who have not read and been struck with this passage do so before their mathematical ideas and methods are nearly or completely crystallized. Workers naturally find themselves, when still inexperienced in the use of quaternions, incapable of clearly thinking through them and of making them do the work of Cartesian Geometry, and they conclude that quaternions do not provide suitable treatment for what they have in hand. They then grow rather disgusted with these vexatious quaternions, and consoling themselves with the reflection that Maxwell, before penning the above extract, had had more experience than themselves, decide that the subject only requires a superficial study to be rendered of as great utility as it is capable.

The fact is that the subject requires a slight development before being applicable to many important physical questions, and these physicists do not continue their interest or enthusiasm for the subject sufficiently long to enable them to furnish that development.

Quaternions differ in an important respect from other branches of mathematics that are studied by mathematicians after they have in the course of years of hard labour laid the foundation of all their future work. In nearly all cases these branches are very properly so called. They each grow out of a definite spot of the main tree of mathematics, and derive their sustenance from the sap of the trunk as a whole. But not so with quaternions. To let these grow in the brain of a mathematician, he must start from the seed as with the rest of his mathematics regarded as a whole. He cannot graft them on his already flourishing tree, for they will die there. They are independent plants that require separate sowing and the consequent careful tending.

These are the explanations that can be given of the arrest in the development of quaternions that followed on the death of Hamilton.

It is now well to describe what I believe quaternions can do, and what should not be demanded of them in the researches of Physics. It is quite certain that the views about to be enunciated will be voted, to say the least, extreme, and it will not be possible to justify them in a short paper like the present. Still it seems proper to give them in all their nakedness.

I believe that Physics would advance with both more rapid and surer strides were quaternions introduced to serious study to the almost complete exclusion of Cartesian Geometry, except in an insignificant way, as a particular case of the former. All the geometrical processes occurring in physical *theories* and *general* physical problems are much simpler and more graceful in their quaternion than in their Cartesian garb. To illustrate the meaning here to be attached to "theory" and "general problem," take the case of elasticity. What is meant by the general theory of elasticity is well enough known. What I mean by a general problem is illustrated by St. Venant's torsion problem for *any* cylinder. The same problem for a cylinder of particular form would be called a *particular* problem. For such particular problems we require of course the theories specially constructed for the solution of particular problems, such as Fourier's theories, complex variables, spherical and ellipsoidal harmonics, &c. It will thus be seen that I do not propose to banish these theories, but merely Cartesian Geometry.

To establish these views it would be necessary to make good the following two statements:—

(1) *Quaternions are already in such a state of development as to justify the practically complete banishment of Cartesian Geometry from physical questions of a general nature.*

(2) *Quaternions will in Physics produce many new results that cannot be obtained by the rival and older geometrical method at all.*

To establish completely the first of these statements, it would be necessary to go over the whole ground covered by general physical questions. This would require a treatise of no small dimensions.

It is the second statement that must be considered of the greater importance. Unfortunately this, too, cannot be justified here. It has already been conceded that the subject requires a slight development, and this of course would necessitate a rather lengthy introduction. It is only after this development that I believe startling physical progress will be made by help of quaternions.

To bear out in part the assertions, however, a few examples of the application of quaternions to a variety of physical questions will be given. Some of the results below are of interest in themselves. They have not been chosen mainly on this account, however, but to illustrate as widely as is possible in a short paper the variety of the questions in which the subject may be expected to prove useful.

The following are the examples chosen:—(1). A theorem in

potentials illustrated by applying it to a general electrical problem. (2) Two examples in curvilinear coordinates. (3) A quaternion proof of a well-known theorem of Jacobi's of great utility in Physics. (4) A generalization of one of the well-known integrals of fluid motion. (5) The well-known particular solution of the differential equation expressing the conditions of equilibrium of an isotropic elastic solid subject to arbitrary bodily forces. (6) A short criticism of Prof. Poynting's theory of the transference of energy through an electric field.

In the proofs below, so far as quaternion knowledge is concerned, an acquaintance with Tait's 'Quaternions,' 3rd edit., only will be assumed. The following two equations from §§ 498, 499 of that treatise will be frequently required below.

$$\int d\rho q = \iint \nabla U \nu \nabla_1 q \, ds, \quad . \quad . \quad . \quad . \quad (1)$$

$$-\iint U \nu q \, ds = \iiint \nabla q \, ds. \quad . \quad . \quad . \quad . \quad (2)$$

In equation (1) ds is an element of a surface, $U\nu$ the unit normal at ds , $d\rho$ a vector element of the boundary, and q a quaternion function of a point in space. In equation (2) ds is an element of volume, ds an element of the bounding surface, $U\nu$ the unit normal at ds pointing away from the region bounded. If the surface in (1) contain lines of discontinuity, or the volume in (2) surfaces of discontinuity in q , the equations are still true if such lines and surfaces of discontinuity are included in the boundary of the region. In such a case, of course, the elements $d\rho$ and ds will each occur twice in the integrals (1) and (2) respectively, namely once for each of the two regions bounded by the element.

I. Potentials.

In the volume and surface integrals that are now required it is necessary to pay attention to the following convention. Let ρ_a be the vector coordinate of a certain point under consideration, and let ρ_b be the vector coordinate of the element of volume ds . It will frequently happen that we have to deal with integrals of the form $\iiint \phi(r) \, ds$, where ϕ is any quaternion function of a quaternion r . [In all the applications below, ϕ will be a linear function, but this is not necessary.] *The form of ϕ is a function of ρ_b only, and r is a function of $\rho_b - \rho_a$ only.* Thus in the expression $\nabla \iiint \phi(r) \, ds$ the only meaning that can be given to the differentiations implied by ∇ is such that these differentiations require ρ_a in

the vector $\rho - \rho_a$ to vary. On the other hand, in the expression $\iiint \nabla_1 \phi(r_1) d\mathbf{s}$ (where the numerical suffixes imply, as throughout this paper they will imply, that the ∇ with a suffix only operates on the symbols which have the same suffix) either end of $\rho_b - \rho_a$ might be considered the variable one. Since ϕ will in general involve other ∇ 's which of necessity must presuppose ρ_b to be the variable, it is convenient to lay down the rule that for all ∇ 's under the integral sign ρ_b is supposed to be the variable. Thus when ∇ crosses the integral sign its sign must be changed, or

$$\nabla \iiint \phi(r) d\mathbf{s} = - \iiint \nabla_1 \phi(r_1) d\mathbf{s}. \quad . \quad . \quad . \quad (3)$$

With one exception the only value of r of equation (3) that will be required below is the scalar u defined by

$$u = T^{-1}(\rho_b - \rho_a). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is well known that if q be any quaternion function of the position of a point,

$$4\pi q = \nabla^2 \iiint u q d\mathbf{s} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$4\pi q = - \nabla \iiint \nabla u q d\mathbf{s},$$

which gives by means of equation (2)

$$4\pi q = - \nabla \iiint u U v q d\mathbf{s} + \nabla \iiint u \nabla q d\mathbf{s}. \quad . \quad . \quad . \quad . \quad (6)$$

Here q may be discontinuous at specified surfaces.

This is the theorem in potentials spoken of. To show that it is really useful let us apply it to an electrical problem.

Maxwell's theory of the electromagnetic field is well enough known. Let us denote by the term "the ordinary theory" what is now to be described. In the ordinary theory there is a certain vector connected with an electromagnetic field, called the vector potential. This vector consists of two parts, one depending solely on the magnetism of the field and the other depending solely on the currents of the field. The vector magnetic force at a point also consists of two such parts. On the ordinary theory, the magnetic part of the vector potential \mathbf{A} and the magnetic part of the magnetic force \mathbf{H} are given in terms of the magnetic moment \mathbf{I} per volume by certain equations investigated in the 3rd Part (*Magnetism*) of Maxwell's 'Electricity and Magnetism.' *On the ordinary theory, the second part of each is obtained by assuming that each (closed) elementary current produces terms in \mathbf{A} and \mathbf{H} that would be produced by the corresponding magnetic shell.*

Maxwell makes no such assumptions as these, and does not show that they are on his theory true—as, indeed, he was not called upon to do. It is of interest, then, to inquire whether they are true on his theory. At a surface of discontinuity in any physical quantity let the two regions bounded be denoted by the suffixes a and b , and let us for brevity write $[\]_{a+b}$ instead of $[\]_a + [\]_b$. Thus, for instance, $U\nu_a$ will be a unit normal pointing away from the region a , *i. e.* into the region b , and $[U\nu]_{a+b}=0$, or $U\nu_a = -U\nu_b$. In place of the above assumptions Maxwell's theory gives

$$4\pi\mathbf{C} = \nabla\mathbf{H} (7)$$

$$[\nabla U\nu\mathbf{H}]_{a+b}=0 (8)$$

$$\nabla\mathbf{A} = \mathbf{B} = \mathbf{H} + 4\pi\mathbf{I} (9)$$

$$[\mathbf{S}U\nu\mathbf{B}]_{a+b}=0 (10)$$

It appears rather a formidable problem to deduce the ordinary theory from these equations, and to do it directly by Cartesians would require rather a bewildering array of symbols.

Before proceeding to see what expressions the ordinary theory gives for \mathbf{A} and \mathbf{H} in terms of \mathbf{I} and \mathbf{C} , it is convenient to deduce from equation (10) the equation

$$[\nabla U\nu\mathbf{A}]_{a+b} = \nabla U\nu_a W, (11)$$

where W is some scalar function of the position of a point. Equation (10) may be written

$$\mathbf{S}U\nu_a \mathbf{B}_{a-b} = 0, \text{ or } \mathbf{S}U\nu_a \nabla \mathbf{A}_{a-b} = 0.$$

\therefore by equation (1) above $\int S d\rho_a \mathbf{A}_{a-b}$ is zero for any closed curve drawn on the surface. $\therefore \int S d\rho_a \mathbf{A}_{a-b}$ has the same value for any two reconcilable paths on the surface from one definite point to another; *i. e.* the resolved part of \mathbf{A}_{a-b} parallel to the surface is the resolved part of ∇W parallel to the surface, where W is some scalar. Thus

$$U\nu_a \nabla U\nu_a \mathbf{A}_{a-b} = U\nu_a \nabla U\nu_a \nabla W.$$

Multiplying by $U\nu_a$ we get equation (11).

The ordinary theory gives the following equations. Defining \mathbf{A}_0 and Ω by the equations

$$\mathbf{A}_0 = \iiint u \mathbf{C} d\mathbf{s}, (12)$$

$$\Omega = -\iiint \mathbf{S} \mathbf{I} \nabla u d\mathbf{s}, (13)$$

it will follow that

$$\mathbf{A} = \mathbf{A}_0 + \iiint \nabla \mathbf{I} \nabla u d\mathbf{s}, (14)$$

$$\mathbf{H} = -\nabla \Omega + \nabla \mathbf{A}_0; (15)$$

i. e. Ω is what is called the magnetic potential, \mathbf{A}_0 is the part of the vector potential due to currents, and $\nabla\mathbf{A}_0$ the part of the magnetic force due to the same cause. The statements that the parts of \mathbf{H} and \mathbf{A} due to magnetism are $\nabla\iiint\mathbf{S}\mathbf{I}\nabla u d\varsigma$ and $\iiint\mathbf{V}\mathbf{I}\nabla u d\varsigma$ respectively are taken directly from the third part of Maxwell's 'Electricity and Magnetism.' It remains to prove that the parts of \mathbf{H} and \mathbf{A} due to \mathbf{C} are $\nabla\iiint u\mathbf{C} d\varsigma$ and $\iiint u\mathbf{C} d\varsigma$ respectively. Since the * part of \mathbf{H} due to magnetism is $\nabla\iiint\mathbf{S}\mathbf{I}\nabla u d\varsigma$, the part of \mathbf{H} contributed by a shell of strength c is

$$\begin{aligned} c\nabla\iiint\mathbf{S}\mathbf{U}\nu\nabla u d\varsigma &= -c\iiint\mathbf{S}\mathbf{U}\nu\nabla\cdot\nabla u d\varsigma = c\iiint\mathbf{V}\mathbf{U}\nu\nabla\cdot\nabla u d\varsigma [\because \nabla^2 u = 0] \\ &= -c\iiint\nabla\nabla\mathbf{U}\nu\nabla u = c\nabla\iiint\mathbf{V}\mathbf{U}\nu\nabla u d\varsigma = c\nabla\int u d\rho [\text{equation (1) above}]. \end{aligned}$$

Hence the part of \mathbf{H} contributed by \mathbf{C} is $\nabla\iiint u\mathbf{C} d\varsigma$. This proves equation (15).

Since the part contributed to \mathbf{A} by \mathbf{I} is $\iiint\mathbf{V}\mathbf{I}\nabla u d\varsigma$, the part contributed by the shell is $c\iiint\mathbf{V}\mathbf{U}\nu\nabla u d\varsigma = c\int u d\rho$. Hence the part contributed by \mathbf{C} is $\iiint u\mathbf{C} d\varsigma$. This proves equation (14).

It remains to see how far Maxwell's theory agrees with equations (12) to (15). Since $\mathbf{S}\nabla(\mathbf{V}\nabla\mathbf{A})=0$, we have from equation (9) $\mathbf{S}\nabla\mathbf{H}=-4\pi\mathbf{S}\nabla\mathbf{I}$. Hence from equation (7)

$$\nabla\mathbf{H}=4\pi(\mathbf{C}-\mathbf{S}\nabla\mathbf{I}).$$

Again, from equation (10), $[\mathbf{S}\mathbf{U}\nu\mathbf{H}]_{a+b}=-4\pi[\mathbf{S}\mathbf{U}\nu\mathbf{I}]_{a+b}$. Hence from equation (8),

$$[\mathbf{U}\nu\mathbf{H}]_{a+b}=-4\pi[\mathbf{S}\mathbf{U}\nu\mathbf{I}]_{a+b}.$$

Putting then in equation (6) $q=\mathbf{H}$,

$$\begin{aligned} \mathbf{H} &= \nabla\iiint u\mathbf{S}\mathbf{U}\nu\mathbf{I} d\varsigma + \nabla\iiint u(\mathbf{C}-\mathbf{S}\nabla\mathbf{I}) d\varsigma \\ &= \nabla\iiint (u\mathbf{C} + \mathbf{S}\mathbf{I}\nabla u) d\varsigma \end{aligned}$$

by equation (2). This is equation (15).

Again, substituting \mathbf{A} for q , and utilizing equations (9), (11),

$$\begin{aligned} 4\pi\mathbf{A} &= \nabla\iiint u(\mathbf{H} + 4\pi\mathbf{I}) d\varsigma - \nabla\iiint u\mathbf{V}\mathbf{U}\nu_a\nabla\mathbf{W} d\varsigma \\ &\quad + \nabla(-\iiint u\mathbf{S}\mathbf{U}\nu\mathbf{A} d\varsigma + \iiint u\mathbf{S}\nabla\mathbf{A} d\varsigma). \end{aligned}$$

* In what follows I deliberately give in their quaternion form certain results that might be quoted from Maxwell's 'Electricity and Magnetism,' in order to show that the problem when thus worked out at full is by no means a long one when treated by means of quaternions.

Here, as usual, the element ds is taken twice, viz., once for each region bounded, except in the expression

$$-\nabla\iiint u \nabla U \nu_a \nabla W ds,$$

where it is taken only once.

This last expression requires transformation. If the surface of discontinuity in \mathbf{B} or $\nabla \nabla \mathbf{A}$ is not closed, we see by equation (11) that the component of ∇W parallel to the surface is zero at the edges; i. e., W is constant at the edges. Let W_0 be this constant value. In this case

$$\begin{aligned} -\nabla\iiint u \nabla U \nu_a \nabla W ds &= -\nabla\iiint u \nabla U \nu_a \nabla (W - W_0) ds \\ &= \nabla \{ \iiint (W - W_0) \nabla U \nu_a \nabla u ds - \int u (W - W_0) d\rho \} \text{ [equation (1)]}. \end{aligned}$$

The line integral is zero $\because W = W_0$ at the edges. The surface integral gives

$$\begin{aligned} -\iiint (W - W_0) \nabla_1 \nabla U \nu_a \nabla_1 u_1 ds &= \iiint (W - W_0) \nabla U \nu_a \nabla \cdot \nabla u ds \\ &= -\iiint (W - W_0) S U \nu_a \nabla \cdot \nabla u ds \text{ [} \because \nabla^2 u = 0 \text{]} = \nabla \iiint (W - W_0) S U \nu_a \nabla u ds. \end{aligned}$$

If the surface is a closed one we may regard any point on it as the bounding curve, or we may proceed thus:—

$$\begin{aligned} -\iiint u \nabla U \nu_a \nabla W ds &= -\iiint \nabla \nabla (u \nabla W) d\mathbf{s} \text{ [equation (2)]} \\ &= -\iiint \nabla \nabla u \nabla W ds = \iiint \nabla \nabla (W \nabla u) d\mathbf{s} = \iiint W \nabla \nabla U \nu_a \nabla u ds. \end{aligned}$$

Hence

$$-\nabla\iiint u \nabla U \nu_a \nabla W ds = \nabla \iiint W \nabla \nabla U \nu_a \nabla u ds = \nabla \iiint W S U \nu_a \nabla u ds$$

by the same transformation as for

$$\nabla \iiint (W - W_0) \nabla U \nu_a \nabla u ds.$$

Defining then the scalar w by the equation

$$4\pi w = \iiint (W - W_0) S U \nu_a \nabla u ds - \iiint u S U \nu_a \nabla W ds + \iiint u S \nabla \nabla \mathbf{A} d\mathbf{s},$$

we have

$$4\pi(\mathbf{A} - \nabla w) = 4\pi \iiint \nabla \mathbf{I} \nabla u ds + \iiint \nabla \mathbf{H} \nabla u ds$$

and

$$\begin{aligned} \iiint \nabla \mathbf{H} \nabla u ds &= \iiint u \nabla \nabla \mathbf{H} d\mathbf{s} + \iiint u \nabla \mathbf{H} U \nu_a ds \text{ [equation (2)]} \\ &= 4\pi \iiint u \mathbf{C} d\mathbf{s} \text{ [equations (7), (8)].} \end{aligned}$$

Hence

$$\mathbf{A} = \mathbf{A}_0 + \iiint \nabla \mathbf{I} \nabla u ds + \nabla w, \quad . \quad . \quad . \quad (16)$$

so that Maxwell's theory differs from the ordinary theory solely by having an arbitrary vector of the form ∇w in \mathbf{A} .

II. Curvilinear Coordinates.

Let ϕ be the stress-function, so that the equation of motion of an elastic solid is

$$\phi\Delta + D\mathbf{F} = D\ddot{\epsilon}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where D is the density, \mathbf{F} is the external force per unit mass, ϵ is the (small) displacement of a point, and $\phi\Delta$ is defined by the equation

$$\phi\Delta \equiv \frac{d\phi i}{dx} + \frac{d\phi j}{dy} + \frac{d\phi k}{dz}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Required to deduce from equation (17) Lamé's transformation into curvilinear coordinates. Changing the $\rho, S, T, U, \partial/\partial s_1, \partial/\partial s_2, \partial/\partial s_3$, of equations (43), § 237 of Ibbetson's 'Elasticity,' into $D, L, M, N, D_\xi, D_\eta, D_\zeta$ respectively, those equations become

$$\begin{aligned} D_\xi P + D_\eta N + D_\zeta M + D(\Xi - \ddot{u}) \\ = (P - Q)_\xi \varpi_\zeta + (P - R)_\xi \varpi_\eta + N(2 \cdot \eta \varpi_\zeta + \eta \varpi_\xi) + M(2 \cdot \zeta \varpi_\eta + \zeta \varpi_\xi) \end{aligned} \quad (19)$$

and two similar equations. The notation is as follows:—

$$\xi = \text{constant}, \quad \eta = \text{constant}, \quad \zeta = \text{constant}, \quad . \quad . \quad (20)$$

are three families of surfaces cutting everywhere orthogonally. ${}_\xi \varpi_\eta, {}_\xi \varpi_\zeta$ are defined as the curvatures of the normal sections of the ξ surface through the tangents to the $\xi\eta$ and $\xi\zeta$ curves respectively. If I, J, K be unit vectors normal to the three surfaces ξ, η, ζ respectively, then $P, Q, R, L, M, N, \Xi, H, Z, u, v, w$, are defined by the equations

$$\phi I = PI + NJ + MK, \text{ \&c., \&c.} \quad . \quad . \quad . \quad (21)$$

$$\mathbf{F} = \Xi I + HJ + ZK \quad . \quad . \quad . \quad . \quad (22)$$

$$\epsilon = uI + vJ + wK. \quad . \quad . \quad . \quad . \quad (23)$$

Lastly, D_ξ, D_η, D_ζ denote, not differentiations with regard to ξ, η, ζ , but differentiations *per unit length* in the directions of I, J, K respectively. So much for the rather formidable

* It will be noticed that while giving the same definition as Ibbetson ('Elasticity,' § 232) of ${}_\xi \varpi_\eta, {}_\xi \varpi_\zeta$ equation (19) is not the same with reference to these symbols as his equation (43) § 237. This is because his definition is inconsistent with the meanings he assigns to the symbols. To obtain those meanings he ought to give the inconvenient definition that ${}_\xi \varpi_\eta, {}_\xi \varpi_\zeta$ are the curvatures of the normal sections of ξ through the tangents to $\xi\zeta, \xi\eta$ respectively. This is simply illustrated by equation (66) § 243, where with θ for colatitude and ω for longitude he asserts that ${}_\theta \varpi_\omega = -\cot \theta/r$, whereas with his definitions, which I adopt, this should clearly be ${}_\theta \varpi_r = -\cot \theta/r$.

notation. Now compare the following proof with the Cartesian one given in Ibbetson's 'Elasticity,' § 237. We have

$$\nabla = I D_{\xi} + J D_{\eta} + K D_{\zeta} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

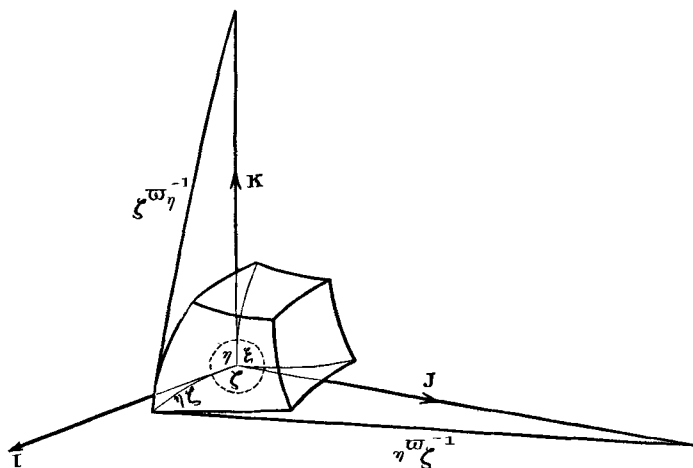
Hence

$$\phi \Delta = \Sigma D_{\xi} \phi \cdot I = \Sigma \{ D_{\xi} (\phi I) - \phi D_{\xi} I \},$$

or in full

$$\begin{aligned} \phi \Delta = & D_{\xi} (PI + NJ + MK) + D_{\eta} (NI + QJ + LK) \\ & + D_{\zeta} (MI + LJ + RK) - \phi (D_{\xi} I + D_{\eta} J + D_{\zeta} K) \quad . \quad . \quad (25) \end{aligned}$$

Since the curve $\eta \zeta$ is a principal line of curvature on each of



the surfaces η and ζ (Dupin's theorem) it is obvious (see figure) that if a point move with unit velocity along it, carrying the system I, J, K of vectors, this system will be rotating with (vector) angular velocity

$$-J_{\zeta} \varpi_{\eta} + K_{\eta} \varpi_{\zeta}.$$

Hence

$$D_{\xi} I = \eta \varpi_{\zeta} J + \zeta \varpi_{\eta} K, \quad D_{\xi} J = -\eta \varpi_{\zeta} I, \quad D_{\xi} K = -\zeta \varpi_{\eta} I \quad . \quad (26)$$

$\therefore D_{\xi} I + D_{\eta} J + D_{\zeta} K = I(\xi \varpi_{\eta} + \xi \varpi_{\zeta}) + J(\eta \varpi_{\zeta} + \eta \varpi_{\xi}) + K(\zeta \varpi_{\xi} + \zeta \varpi_{\eta});$
and therefore

$$\begin{aligned} \phi (D_{\xi} I + D_{\eta} J + D_{\zeta} K) \\ = I \{ P(\xi \varpi_{\eta} + \xi \varpi_{\zeta}) + N(\eta \varpi_{\zeta} + \eta \varpi_{\xi}) + M(\zeta \varpi_{\xi} + \zeta \varpi_{\eta}) \} + J \{ \} + K \{ \}. \quad (27) \end{aligned}$$

Hence

$$\begin{aligned} \phi \Delta = & I \{ (D_{\xi} - \xi \varpi_{\eta} - \xi \varpi_{\zeta}) P + (D_{\eta} - \eta \varpi_{\zeta} - \eta \varpi_{\xi}) N + (D_{\zeta} - \zeta \varpi_{\xi} - \zeta \varpi_{\eta}) M \} \\ & + I \{ -\eta \varpi_{\zeta} N - \zeta \varpi_{\eta} M + \xi \varpi_{\zeta} Q + \xi \varpi_{\eta} R \} + J \{ \} + K \{ \}; \end{aligned}$$

or

$$\phi\Delta = I\{(D_\xi P + D_\eta N + D_\zeta M) - (P - Q)_\xi \varpi_\zeta - (P - R)_\xi \varpi_\eta \\ - N(2_\eta \varpi_\zeta + \eta \varpi_\xi) - M(2_\zeta \varpi_\eta + \zeta \varpi_\xi)\} + J\{\} + K\{\} \quad (28)$$

Substituting this value of $\phi\Delta$ and the values of \mathbf{F} and ϵ given by equations (22) and (23) in equation (17) we get equation (19).

As a second example in curvilinears let us find the strain in this notation. Let ψ be the pure strain due to ϵ , *i. e.* let

$$2\psi\omega = -S\omega\nabla.\epsilon - \nabla_1 S\omega\epsilon_1 \quad \dots \quad (29)$$

With the usual notation (e, f, g, a, b, c) for the coordinates of pure strain, we have

$$2\psi I = 2eI + cJ + bK, \text{ \&c., \&c.} \quad \dots \quad (30)$$

Substituting in equation (29) for ∇ from equation (24), we get

$$2\psi I = D_\xi \epsilon - ISID_\xi \epsilon - JSID_\eta \epsilon - KSID_\zeta \epsilon.$$

But [equation (23)]

$$D_\xi \epsilon = ID_\xi u + JD_\xi v + KD_\xi w + uD_\xi I + vD_\xi J + wD_\xi K \\ = I(D_\xi u - \eta \varpi_\zeta v - \zeta \varpi_\eta w) + J(D_\xi v + \eta \varpi_\zeta u) + K(D_\xi w + \zeta \varpi_\eta u) \quad [\text{eq. (26)}]$$

Similarly,

$$-SID_\eta \epsilon = D_\eta u + \xi \varpi_\zeta v, \quad -SID_\zeta \epsilon = D_\zeta u + \xi \varpi_\eta w.$$

Hence

$$2\psi I = I(2D_\xi u - \eta \varpi_\zeta v - \zeta \varpi_\eta w) + J\{(D_\xi + \xi \varpi_\zeta)v + (D_\eta + \eta \varpi_\zeta)u\} \\ + K\{(D_\zeta + \zeta \varpi_\eta)u + (D_\xi + \xi \varpi_\eta)w\},$$

whence from equation (30)

$$\left. \begin{aligned} 2e &= 2D_\xi u - \eta \varpi_\zeta v - \zeta \varpi_\eta w, & 2f &= \dots, & 2g &= \dots \\ 2a &= (D_\eta + \eta \varpi_\xi)w + (D_\zeta + \zeta \varpi_\xi)v, & 2b &= \dots, & 2c &= \dots \end{aligned} \right\} \dots \quad (31)$$

Compare this with § 234 of Ibbetson's 'Elasticity.'

III. *Jacobi's Theorem.*

In Todhunter's 'History of the Calculus of Variations' (§ 323), this theorem is thus enunciated.

Let v be any function of x, y, z , and G any function of $x, y, z, v, \partial v/\partial x, \partial v/\partial y, \partial v/\partial z$. Let x, y, z be three functions of any three other variables λ, μ, ν . When expressed in terms of λ, μ, ν , let v be denoted by ϕ ; and when expressed in terms of $\lambda, \mu, \nu, \phi, \partial\phi/\partial\lambda, \partial\phi/\partial\mu, \partial\phi/\partial\nu$, let G be denoted by Γ . Lastly, let

$$\Pi = \frac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)}.$$

Then shall

$$\begin{aligned} & \Pi \left\{ \frac{\partial G}{\partial} - \frac{\partial}{\partial x} \left(\frac{\partial G}{\partial \frac{\partial v}{\partial x}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial G}{\partial \frac{\partial v}{\partial y}} \right) - \frac{\partial}{\partial z} \left(\frac{\partial G}{\partial \frac{\partial v}{\partial z}} \right) \right\} \\ &= \Pi \frac{\partial \Gamma}{\partial \phi} - \frac{\partial}{\partial \lambda} \left(\frac{\partial (\Pi \Gamma)}{\partial \frac{\partial \phi}{\partial \lambda}} \right) - \frac{\partial}{\partial \mu} \left(\frac{\partial (\Pi \Gamma)}{\partial \frac{\partial \phi}{\partial \mu}} \right) - \frac{\partial}{\partial \nu} \left(\frac{\partial (\Pi \Gamma)}{\partial \frac{\partial \phi}{\partial \nu}} \right) \end{aligned}$$

We may omit the first term on each side since it is easy to see that $\partial G / \partial v = \partial \Gamma / \partial \phi$.

Expressed in quaternion language this may be put thus:—

Let v be any scalar function of ρ a vector, and G any scalar function of ρ , v , and ∇v . Let ρ be a function of ρ' . When expressed in terms of ρ' , let v be denoted by v' ; and when expressed in terms of ρ' , v' , and $\nabla' v'$ [∇' standing towards ρ' as ∇ towards ρ], let G be denoted by G' . Lastly, let

$$m = \frac{S d\rho'_a d\rho'_b d\rho'_c}{S d\rho_a d\rho_b d\rho_c},$$

where $d\rho_a, d\rho_b, d\rho_c$ are three arbitrary increments of ρ and $d\rho'_a, d\rho'_b, d\rho'_c$, the consequent increments in ρ' . Then shall

$$m^{-1} S \nabla \nabla_v G = S \nabla' \nabla'_v (m^{-1} G'),$$

where v, v' stand for $\nabla v, \nabla' v'$, and ∇_v, ∇'_v stand towards v, v' as ∇, ∇' towards ρ, ρ' .

We have

$$d\rho' = -S d\rho \nabla_1 \rho' = \chi d\rho$$

say. Also

$$S d\rho \nabla v = S d\rho' \nabla' v' = S d\rho \chi' \nabla' v',$$

where as usual χ' stands for the conjugate of χ . Hence

$$v = \chi' v',$$

or generally

$$\nabla_1 = \chi' \nabla'_1.$$

Again,

$$S d v' \nabla'_v G' = S d v \nabla_v G = S d v' \chi \nabla_v G.$$

Hence

$$\nabla'_v G' = \chi \nabla_v G.$$

Hence

$$\begin{aligned} S \nabla \nabla_v G &= S \chi' \nabla'_1 \chi_1^{-1} \nabla'_v G'_1 \\ &= S \Delta' \chi \chi^{-1} \nabla'_v G'_1 + S \chi_1'^{-1} \chi' \nabla'_1 \nabla'_v G' \\ &= S \nabla' \nabla'_v G' + S \chi_1'^{-1} \chi' \nabla'_1 \nabla'_v G'. \end{aligned}$$

Now

$$\chi_1'^{-1}\chi'\nabla_1' = -\chi'^{-1}\chi_1'\nabla_1' [\cdot\cdot\chi_1'^{-1}\chi_1'\nabla_1' = \nabla 1 = 0];$$

and, as will be proved directly, $m_1^{-1}\chi_1'\nabla_1' = 0$, so that

$$\chi_1'\nabla_1' = -m\chi'\nabla'(m^{-1}).$$

Hence

$$m^{-1}S\nabla\nabla_v G = m^{-1}S\nabla'\nabla_v'G' + S\nabla'(m^{-1})\nabla_v'G' = S\nabla'\nabla_v'(m^{-1}G').$$

This theorem is a particular case of one of several allied quaternion theorems that I have found very useful in Physics, but which have not yet been published. The proof just given is not the simplest or most natural quaternion proof, but is the simplest I can furnish when the theorem is divorced from what may be called its natural surroundings.

It still remains to prove that $m_1^{-1}\chi_1'\nabla_1' = 0$. If it were assumed, which is true, and which ought to be thoroughly familiar to every mathematical physicist, that

$$2m^{-1}\chi'\omega = -V\rho_1\rho_2S\omega\nabla_1'\nabla_2',$$

the statement would be obvious. As unhappily, however, this is not generally known, the following indirect method of proof may be adopted. Take ρ and ρ' as the coordinate vectors of any point in two different positions. Then

$$0 = \iiint U\nu ds = \iiint m^{-1}\chi'U\nu' ds' = \iiint m_1^{-1}\chi_1'\nabla_1' ds' [\text{equation (2)}].$$

This being true for any space, is true for a single element ds' , and therefore $m_1^{-1}\chi_1'\nabla_1' = 0$. The assumption that

$$U\nu ds = m^{-1}\chi'U\nu' ds'$$

is easily seen to follow from the definition of m . For $U\nu ds$ may be taken as $Vd\rho_b d\rho_c$ and $U\nu' ds'$ as $Vd\rho_b' d\rho_c'$, so that the definition of m gives

$$Sd\rho_a U\nu ds = m^{-1}Sd\rho_a' U\nu' ds' = m^{-1}Sd\rho_a \chi' U\nu' ds'.$$

IV. An Integral of the Equations of Fluid Motion.

The integral I am about to give I do not propose to prove; because what seems to me the best quaternion proof requires properties of quaternions not proved in Tait's 'Quaternions.' It has been said over and over again that quaternions in Physics are only useful for expressing the results obtained by other processes, and, perhaps, occasionally for furnishing a neater proof of a truth when it has been discovered by other means. To persons holding such views, of course there can be no difficulty in furnishing a Cartesian proof of a quaternion theorem. I will enter very fully into detail in the statement of the theorem, to leave no doubt as to the meaning.

Let p be the pressure, σ the (vector) velocity, and v the force potential at any point of a fluid. Let $\int dp/(\text{density})$ be put as usual = P . Let d/dt denote differentiation with regard to time *which follows the motion of matter*. The fluid may be finite or infinite, and there may be any possible kind of discontinuity in the motion. \iint_b denotes a surface integral taken over the true boundary of the fluid (including the surface at infinity if the fluid is infinite). \iint denotes a surface integral taken over this true boundary, and also over both sides of any surface of discontinuity, which is thus supposed to bound the region on both sides. The surface to which \iint_b refers may not always contain the same fluid particles, since for instance fissures may form leading to new parts of the true boundary, or such fissures may close up. When a fissure is in process of formation \iint_b refers to the boundary thus created from the instant when the portions of fluid begin to move asunder. Let τ be twice the (vector) spin; *i. e.* $\tau = V \nabla \sigma$. Let n be the convergence; *i. e.* $n = S \nabla \sigma$. Let N be* the surface expansion; *i. e.* $N = -S U \nu \sigma$. Let u be defined by equation (4) above. Then

$$4\pi(v+P) = \iint_b (v+P) S U \nu \nabla u ds - \left\{ \iint u \frac{d(Nds)}{dt} + \iiint u \frac{d(nds)}{dt} \right. \\ \left. - \iint S \{ V \sigma \tau - n \sigma + \nabla(\sigma^2/2) \} \nabla u ds. \right. \quad (32)$$

This may be put also in the form

$$4\pi(v+P-\sigma^2/2) = \iint_b (v+P) S U \nu \nabla u ds - \iint (\sigma^2/2) S U \nu \nabla u ds \\ - \left\{ \iint u \frac{d(Nds)}{dt} + \iiint u \frac{d(nds)}{dt} \right\} - \iiint S (V \sigma \tau - n \sigma) \nabla u ds. \quad (33)$$

The usual integral for an infinite irrotational continuously moving fluid,

$$v + P - \sigma^2/2 + \partial \phi / \partial t = H,$$

can easily be shown to be a particular case. It may be noticed that

$$V \sigma \tau - n \sigma = -V \nabla \sigma \sigma,$$

and therefore

$$-S \nabla u (V \sigma \tau - n \sigma) = S \nabla u \nabla \sigma \sigma.$$

* It might be thought that in analogy to n it would be better to put $N = S U \nu \sigma$, but by considering the analogy between ordinary convergence and a contracting bubble it will be seen that the definition of N given is perhaps better. It must be remembered that $U \nu$ points *from the fluid bounded* and therefore *into the bubble*.

I sought and found the integral in attempting to consider, from a fresh point of view, Sir William Thomson's vortex-atom theory and Prof. Hicks's proposed modification of it; and though I have not made any serious attempt to apply it in this direction, I still think that it can be made of use in discussing how large groups of vortices act on other large groups.

There would be apparent attractions or repulsions of a definite kind between vortices if the acceleration $d\sigma/dt$ could be expressed as a function of the vortices. In both the theories just mentioned the fluid is assumed to be incompressible and subject to no external forces, so that $v=0$. In Sir William Thomson's theory the fluid is unbounded, and in Prof. Hicks's it is infinite but bounded at certain places by what may be called bubbles. In these bubbles the pressure is zero. We have then from equation (32) for both cases

$$4\pi\dot{\sigma} = \nabla \left\{ \iint u \frac{d(Nds)}{dt} + \iiint S \{ V\sigma\tau - \nabla(\sigma^2/2) \} \nabla u ds \right\}; \quad (34)$$

so that the vortices will move as though subject to a force potential w given by

$$4\pi w = - \iint u \frac{d(Nds)}{dt} - \iiint S \{ V\sigma\tau - \nabla(\sigma^2/2) \} \nabla u ds. \quad (35)$$

I do not propose to discuss the bearing of these results here. I merely give them to indicate that the integral just given may, notwithstanding its apparent complexity when stated perfectly generally, prove of great utility.

A very different form may be given to the above integral. It will probably hint to one familiar with quaternion methods one way of proving the result. It is convenient to change the notation. Instead of the former σ , Uv , ds , d_s , and ∇ , write now σ' , Uv' , ds' , $d_{s'}$, and ∇' . Let ρ' , the vector coordinate of any point, be supposed a function of an independent variable vector ρ (say the coordinate of the point's initial position), and t the time. Let u now stand for the reciprocal of the distance between two points in the ρ space, instead of as just now in the ρ' space. Similarly let Uv , ds , d_s , and ∇ refer to the ρ space. Finally, let $\sigma = -\nabla_1 S \sigma' \rho_1'$. Then instead of equation (32) we may write

$$4\pi(v+P) = \iint_b (v+P) S U v \nabla u ds + \iiint S \nabla u \nabla(\sigma'^2/2) d_s \\ + \iint u \frac{dS U v \sigma}{dt} ds - \iiint u \frac{dS \nabla \sigma}{dt} d_s. \quad (36)$$

V. *A particular Integral in the Theory of Elasticity.*

Consider an elastically isotropic solid in equilibrium under arbitrary external bodily forces. Let ϵ be the (small vector) displacement at any point. Also let \mathbf{F} be the external force per unit volume. Then [Thomson and Tait's 'Natural Philosophy,' § 698, equation (7)] the equation of equilibrium is

$$N\nabla^2\epsilon + M\nabla S\nabla\epsilon \equiv \psi\epsilon = \mathbf{F}; \quad . \quad . \quad . \quad (37)$$

where M, N have been put for Thomson and Tait's m, n respectively, and where ψ is a symbolic self-conjugate linear vector function of a vector,

$$\therefore \psi\omega = -T^2\nabla(N\omega - MU\nabla S\omega U\nabla),$$

we have

$$\psi^{-1}\omega = -T^{-2}\nabla\{N^{-1}\omega + (N^{-1} - [N + M]^{-1})U\nabla S\omega U\nabla\}.$$

[Here it is assumed that if

$$A\omega - BiSi\omega \equiv \psi\omega = \tau,$$

then

$$\tau = \psi^{-1}\omega \equiv A^{-1}\omega + (A^{-1} - [A + B]^{-1})iSi\omega,$$

which can be easily verified, if not obvious, by operating on the last equation by ψ]. Therefore by equation (37),

$$\epsilon = \psi^{-1}\mathbf{F} = N^{-1}\nabla^{-2}\mathbf{F} - MN^{-1}(M + N)^{-1}\nabla^{-1}S\nabla^{-1}\mathbf{F}. \quad . \quad (38)$$

By equation (5) a particular value of $\nabla^{-2}\mathbf{F}$ is given by

$$4\pi\nabla^{-2}\mathbf{F} = \iiint u\mathbf{F} d\mathbf{s};$$

and the integration may be supposed extended over the given region, or, in addition, over any external region where \mathbf{F} may be given any convenient values. Again,

$$4\pi\nabla^{-1}S\nabla^{-1}\mathbf{F} = 4\pi\nabla^{-1}S\nabla\nabla^{-2}\mathbf{F} = \nabla^{-1}S\nabla\iiint u\mathbf{F}d\mathbf{s} = \iiint S\mathbf{F}\nabla \cdot \nabla^{-1}u$$

Since we are seeking only a particular solution any particular value of $\nabla^{-1}u$ will serve. Putting σ for the $\rho_b - \rho_a$ of equation (4), it is known that

$$\nabla U\sigma = -2u,$$

so that

$$\nabla^{-1}u = -U\sigma/2.$$

Hence

$$8\pi N(M + N)\epsilon = \iiint \{2(M + N)u\mathbf{F} + M\mathbf{S}\mathbf{F}\nabla \cdot U\sigma\} d\mathbf{s}. \quad (39)$$

is a particular solution of equation (37).

If these symbolic methods be objected to—though they are just as legitimate as the ordinary symbolic methods adopted for discovering particular solutions—they may be regarded as

Phil. Mag. S. 5. Vol. 33. No. 205. June 1892. 2 L

furnishing only a hint of a particular solution, and the very easy verification that equation (39) really is a particular solution of equation (37) may follow.

This form of the particular solution is not the same as Thomson and Tait's ['Natural Philosophy,' § 731, equation (19)], and, for aught that appears above, may be a particular solution different from theirs. It is, however, the same as theirs. To verify this, express $\mathbf{SF}\nabla \cdot \mathbf{U}\sigma$ in terms of \mathbf{F} and $u^{-2}\mathbf{SF}\nabla \cdot \nabla u$, thus :—

$$\mathbf{SF}\nabla \cdot \mathbf{U}\sigma = \mathbf{SF}\nabla \cdot (u\sigma) = -u\mathbf{F} + \sigma\mathbf{SF}\nabla u,$$

$$\mathbf{SF}\nabla \cdot \nabla u = -\mathbf{SF}\nabla \cdot (u^3\sigma) = u^3\mathbf{F} - 3u^2\sigma\mathbf{SF}\nabla u.$$

Eliminating $\mathbf{SF}\nabla u$,

$$\mathbf{SF}\nabla \cdot \mathbf{U}\sigma = -\frac{2}{3}u\mathbf{F} - \frac{1}{3}u^{-2}\mathbf{SF}\nabla \cdot \nabla u.$$

Equation (39) thus becomes

$$24\pi N(M+N)\epsilon = \iiint \{2(2M+3N)u\mathbf{F} - Mu^{-2}\mathbf{SF}\nabla \cdot \nabla u\} d\tau, \quad (40)$$

the form given by Thomson and Tait.

Thomson and Tait regard this particular solution as the solution of the statical problem for an infinite solid. In this case some law of convergence must apply to \mathbf{F} to make these integrals convergent. Thomson and Tait ('Natural Philosophy,' § 730) say that this law is that $\mathbf{F}r$ (where r is the distance from some arbitrary origin at a finite distance) converges to zero at infinity. This, I think, can be disproved by a particular case. Put, from $r=0$ to $r=a$, $\mathbf{F}=0$; and from $r=a$ to $r=\infty$, $\mathbf{F}r=r^{-n}\alpha$; where α is a constant vector, and n is a constant positive scalar less than unity. Equation (39) then gives for the displacement at the origin, due to the part of the integral extending throughout a sphere whose centre is the origin and radius $R(>a)$,

$$\epsilon = \frac{M+3N}{3N(M+N)} \frac{R^{1-n}-a^{1-n}}{1-n} \alpha.$$

Putting $R=\infty$, we get $\epsilon=\infty$. The real law of convergence does not seem to be worth seeking, as the practical utility of equations (39), (40) is owing to the fact that either of them is a particular integral for a finite body.

VI. *The Transference of Energy through an Electric Field.*

What follows is generalized in a paper about to be published. It is given here, as it will probably, even in the

particular form, interest such physicists as listen to the paper.

Prof. Poynting's theories of the transference of energy through an electromagnetic field are to-day universally known. In the Philosophical Transactions, 1884, part ii. pp. 343 to 349, he attempts to prove that the energy in the field has a time-flux τ given by

$$4\pi\tau = \mathbf{V}\mathbf{E}\mathbf{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where \mathbf{E} and \mathbf{H} are the electromotive and magnetic forces respectively. In the Brit. Assoc. Reports, 1885, pp. 151, 152, Prof. J. J. Thomson points out that Prof. Poynting's reasoning is open to criticism; and that ν , given by $\nu = \tau + \mathbf{V}\nabla\sigma$ (where, if σ is discontinuous at any surface, the tangential component is not so) will serve equally well, if assumed to be the time-flux of energy, according to Prof. Poynting's reasoning, to explain the known facts. Put $4\pi\sigma = \Psi\mathbf{H}$, where Ψ is the electrostatic potential. [Note that, in order that this value of σ may satisfy the surface-conditions just mentioned, Ψ must be continuous.] We get as the time-flux of energy,

$$4\pi\nu = 4\pi\tau + \mathbf{V}\nabla(\Psi\mathbf{H}) = \mathbf{V}(\mathbf{E} + \nabla\Psi)\mathbf{H} + 4\pi\Psi\mathbf{C},$$

where \mathbf{C} , the current, is put in place of $\mathbf{V}\nabla\mathbf{H}/4\pi$. Substituting from equation (10) § 599 of Maxwell's 'Electricity and Magnetism,' viz.

$$\mathbf{E} = \mathbf{V}\mathbf{G}\mathbf{B} - \partial\mathbf{A}/\partial t - \nabla\Psi,$$

where \mathbf{G} is the velocity of matter at the point in question, we see that for a *steady* field,

$$\nu = \Psi\mathbf{C}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

so that, assuming ν instead of τ to be the true time-flux of energy, in this matter of the transference of energy through the field, as in so many other respects, \mathbf{C} the current and Ψ the potential are the exact analogues of a liquid current and the pressure. Compare this with the very different conclusion of Prof. Poynting (Phil. Trans. 1884, part ii. p. 361:)—“I think it is necessary that we should realize thoroughly that, if we accept Maxwell's theory of energy residing in the medium, we must no longer consider a current as something conveying energy along the conductor.” According to the present result, in a steady field the sole means of conveyance of energy would be precisely the means Prof. Poynting warns us against, namely the electricity itself.