The Theory of Cauchy's Principal Values. (Third Paper: *Differentiation and Integration of Principal Values.*) *By* G. H. HARDY. Received February 15th, 1902. Read March 13th, 1902. Revised November, 1902.

1. In my second paper* I stated the general conditions under which

$$P\int_a^A f(x, \alpha)\, dx \tag{1}$$

is a continuous function of α. In this paper I shall deal with two of the most important special cases of this general problem, which lead to theorems corresponding to the ordinary rules for differentiation and integration under the integral sign. That (1) is convergent for all values of α in question will be presupposed in all that follows.

Differentiation under the Sign of the Principal Value.

2. The first question which will engage us is that of finding sufficient conditions for the truth of the equation

$$\frac{d}{d\alpha} P\int_a^A f(x, \alpha)\, dx = P\int_a^A \frac{\partial f(x, \alpha)}{\partial \alpha}\, dx, \tag{1}$$

which is a generalization of Leibniz's theorem. This equation asserts that, if

$$\Delta_h f(x, \alpha) = \frac{1}{h}\{f'(x, \alpha+h) - f(x, \alpha)\},$$

$$\lim_{h=0} P\int_a^A \Delta_h f(x, \alpha)\, dx = P\int_a^A \lim_{h=0} \Delta_h f(x, \alpha)\, dx,$$

that is to say that

$$P\int_a^A \Delta_h f(x, \alpha)\, dx \tag{2}$$

is a continuous function of h for $h = 0$. Hence sufficient conditions for (1) are (II., § 25) (i.) that $f(x, \alpha)$ is continuous except on certain curves, and (ii.) that (2) is uniformly convergent in an interval $(-H, H)$.

This general criterion is, however, not very easy to apply.

* In this paper the first and second papers (*Proceedings*, Vol. XXXIV., pp, 16, 55) will be referred to as I., II.

3. I shall assume at present that A is finite, and that we can find a positive value of H such that $f(x, \alpha)$ and its derivates $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \alpha}$ are continuous functions of both variables in the rectangle

$$(a, A, \alpha_0 - H, \alpha_0 + H),$$

except on a finite number of curves $x = X_i(\alpha)$. We may without loss of generality suppose that there is only one such curve. It is also convenient to take $a + A > 2X$ for $\alpha = \alpha_0$; we can then choose H so small that this condition is satisfied throughout $(\alpha_0 - H, \alpha_0 + H)$.

Finally, we suppose *either* that

$$\lim_{\epsilon=0} \{f(X-\epsilon, \alpha) - f(X+\epsilon, \alpha)\} = 0$$

uniformly for all values of α in question; *or* that $\frac{dX}{d\alpha}$ is identically zero, *i.e.*, X independent of α. As X is independent of ϵ, both alternatives are included in the single condition that

$$\lim_{\epsilon=0} \frac{dX}{d\alpha} \{f(X-\epsilon, \alpha) - f(X+\epsilon, \alpha)\} = 0$$

uniformly for all values of α in question. Then

$$P\int_a^A f\,dx = P\int_a^{2X-a} f\,dx + P\int_{2X-a}^A f\,dx,$$

and

$$P\int_a^{2X-a} f\,dx = \lim_{\epsilon=0}\int_\epsilon^{X-a} \{f(X+y, \alpha) + (X-y, \alpha)\}\,dy$$

$$= \int_0^{X-a} \phi(y, \alpha)\,dy,$$

if

$$\phi(y, \alpha) = f(X+y, \alpha) + f(X-y, \alpha).$$

Now ϕ is continuous and has continuous first derivates except for $y = 0$. And

$$\int_\epsilon^{X-a} \frac{\partial\phi}{\partial\alpha}\,dy = \int_\epsilon^{X-a} \left\{ \left[\frac{\partial f(t, \alpha)}{\partial\alpha} + \frac{dX}{d\alpha}\frac{\partial f(t, \alpha)}{\partial t}\right]_{t=X+y} + \left[\frac{\partial f(t, \alpha)}{\partial\alpha} + \frac{dX}{d\alpha}\frac{\partial f(t, \alpha)}{\partial t}\right]_{t=X-y} \right\} dy$$

$$= \left(\int_a^{X-\epsilon} + \int_{X+\epsilon}^{2X-a}\right) \left\{\frac{\partial f(x, \alpha)}{\partial\alpha} + \frac{dX}{d\alpha}\frac{\partial f(x, \alpha)}{\partial x}\right\} dx$$

$$= \left(\int_a^{X-\epsilon} + \int_{X+\epsilon}^{2X-a}\right) \frac{\partial f}{\partial\alpha}\,dx + \frac{dX}{d\alpha}\{f(2X-a) - f(X+\epsilon) + f(X-\epsilon) - f(a)\}.$$

The right-hand side tends uniformly to the limit

$$P\int_a^{2X-a}\frac{\partial f}{\partial a}dx+\frac{dX}{da}\{f(2X-a)-f(a)\}.$$

when ϵ tends to zero. Hence

$$\int_0^{X-a}\frac{\partial \phi}{\partial a}dy$$

is uniformly convergent; and therefore

$$\frac{d}{da}\int_0^{X-a}\phi\,dy=\int_0^{X-a}\frac{\partial\phi}{\partial a}dy+\frac{dX}{da}\phi(X-a),$$

by the ordinary theorem of differentiation under the integral sign. Therefore

$$\begin{aligned}\frac{d}{da}P\int_a^A f\,dx&=\frac{d}{da}\int_0^{X-a}\phi\,dy+\frac{d}{da}\int_{2X-a}^A f\,dx\\&=P\int_a^{2X-a}\frac{\partial f}{\partial a}dx+2\frac{dX}{da}f(2X-a)\\&\quad+\int_{2X-a}^A\frac{\partial f}{\partial a}dx-2\frac{dX}{da}f(2X-a)\\&=P\int_a^A\frac{\partial f}{\partial a}\partial x.\end{aligned}$$

THEOREM 1.—*If* $f(x, a)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial a}$ *are continuous in*

$$(a,\ A,\ a_0-H,\ a_0+H),$$

except along a finite number of curves $x=X_i(a)$, *which do not meet* $x=a$, *or* $x=A$, *or one another, and have at every point a definite direction never parallel to* x; *if, moreover,*

$$\lim_{\epsilon=0}\frac{dX}{da}\{f(X_i-\epsilon,\ a)-f(X_i+\epsilon,\ a)\}=0$$

uniformly for all values of a *in* $(a_0-H,\ a_0+H)$, *and*

$$P\int_a^A\frac{\partial f}{\partial a}dx$$

is uniformly convergent in $(a_0-H,\ a_0+H)$; *then*

$$\frac{d}{da}P\int_a^A f\,dx=P\int_a^A\frac{\partial f}{\partial a}dx.$$

We have supposed A finite. But no new difficulty arises from supposing the upper limit infinite, if there are only a finite number of curves across which the integrals are not unconditionally convergent. For then we can choose A so that

$$X_i(a) < A, \quad a_0 - H \leq a \leq a_0 + H;$$

and

$$\frac{d}{da} P \int_{a}^{\infty} f\, dx = \frac{d}{da} P \int_{a}^{A} + \frac{d}{da} \int_{A}^{\infty}.$$

The first term is

$$P \int_{a}^{A} \frac{\partial f}{\partial a}\, dx,$$

by Theorem 1; and the second is

$$\int_{A}^{\infty} \frac{\partial f}{\partial a}\, dx,$$

if this integral is uniformly convergent; so that

$$\frac{d}{da} P \int_{a}^{\infty} f\, dx = P \int_{a} \frac{\partial f}{\partial a}\, dx.$$

Case 1 (*X_i independent of* a).

4. The simplest way to satisfy the conditions of the theorem is to suppose that*

$$f(x, a) = \Omega_\nu (x - X)\, \Theta (x, a),$$

Θ being a function which has continuous derivates $\frac{\partial \Theta}{\partial a}$, $\frac{\partial \Theta}{\partial x}$, and X independent of a.

5. (i.) If

$$I(a) = P \int_0^p \log\left(1 + \frac{a}{x}\right) \frac{dx}{x - t} \quad (0 < t < p,\ 0 < a),$$

$$\frac{dI}{da} = P \int_0^p \frac{dx}{(x + a)(x - t)} = \frac{1}{a + t} \log\left(\frac{p - t}{t}\ \frac{a}{p + a}\right).$$

Integrating from $a = 0$ to $a = q$ and putting $q = p$, we find

$$P \int_0^p \log\left(1 + \frac{p}{x}\right) \frac{x\, dx}{x^2 - t^2} = \tfrac{1}{2} \log\left(\frac{p}{t} + 1\right) \log\left(\frac{p}{t} - 1\right).$$

(ii.) It is easy to prove by differentiation that

$$P \int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right) \frac{dx}{x^2 - a^2} = -\frac{\pi}{a} \tan^{-1} \frac{a}{a}.$$

(iii.) If $0 < a < 1$,

$$\int_0^\infty \frac{x^{a-1} \log x}{1 - x}\, dx = \frac{d}{da} P \int_0^\infty \frac{x^{a-1}\, dx}{1 - x} = -\left(\frac{\pi}{\sin a\pi}\right)^2.$$

In the latter case the symbol of the principal value is unnecessary in the derived integral. In general, however, when X is independent of a, both integrals are only principal values

(iv.)

$$P \int_0^\infty \frac{x^{a-1} - x^{\beta-1}}{1 - x} \frac{dx}{\log x} = \log\left(\frac{\sin a\pi}{\sin \beta\pi}\right).$$

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* The functions Ω_ν, ψ_ν are defined in I. (§§ 8, 9).

Case 2 (X_i dependent on α).

6. The more interesting case of the theorem is, however, that in which X_i depends upon α.

THEOREM 2.—*If $f(x,\alpha)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial \alpha}$ are continuous except upon the curves $x = X_i(\alpha)$, and $f(x,\alpha)$ can, near any of these curves, be expressed in the form*

$$\psi_\nu\{x-X_i(\alpha)\}\,\Theta(x,\alpha),$$

where ψ_ν is a product of logarithmic factors only, and Θ, $\frac{\partial \Theta}{\partial x}$, $\frac{\partial \Theta}{\partial \alpha}$ are continuous without exception, then

$$\frac{d}{d\alpha}\int_a^A f(x,\alpha)\,dx = P\int_a^A \frac{\partial f}{\partial \alpha}\,dx.$$

For
$$\psi_\nu(x-X) = \prod_1^\nu |\, l^i(x-X)\,|^{s_i};$$

in this product some or all of the signs of the absolute value may possibly be omitted. And

$$\lim_{\epsilon=0}\{f(X-\epsilon)-f(X+\epsilon)\} = \lim_{\epsilon=0}\psi_\nu(\epsilon)\{\Theta(X-\epsilon)-\Theta(X+\epsilon)\} = 0$$

uniformly for all values of α.

Moreover

$$\frac{\partial f}{\partial \alpha} = \psi_\nu(x-X)\frac{\partial \Theta}{\partial \alpha} + \frac{\Theta}{x-X}\sum_{i=1}^{\nu} s_i \prod_{\kappa=1}^{i} |\, l^\kappa(x-X)\,|^{s_\kappa-1} \prod_{i+1}^{\nu} |\, l^\kappa(x-X)\,|^{s_\kappa}.$$

The first term is unconditionally integrable. The rest consists of a finite number of terms of the form

$$\Omega_\nu(x-X)\,\Theta(x,\alpha).$$

Hence (II., § 15)
$$P\int_a^A \frac{\partial f}{\partial \alpha}\,dx$$

is uniformly convergent, and so the conditions of Theorem 1 are satisfied. In this case it is always an ordinary integral whose derivate is expressed as a principal value.

7. (i.) It is easily verified that

$$\int_a^A l(x-\alpha)\,dx = \tfrac{1}{2}\int_a^A \log(x-\alpha)^2\,dx = (A-\alpha)\,l(A-\alpha)-(a-\alpha)\,l(a-\alpha)-(A-a)$$
$$(0<\alpha<A);\text{*}$$

and
$$\frac{d}{d\alpha}\int_a^A l(x-\alpha)\,dx = l(a-\alpha)-l(A-\alpha) = -P\int_a^A \frac{dx}{x-\alpha}.$$

* By lx I denote $\log|x|$. (I., § 8).

(ii.) It is easy to prove that, if n is integral and $0 < a < \pi$,

$$\int_0^\pi l\,(\cos x - \cos a)\cos nx\,dx = -\frac{\pi}{n}\cos na,$$

or $= 0$, according as $n >$ or $= -\pi \log 2$. Hence

$$P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos a} = \frac{\pi \sin na}{\sin a} \quad \text{or} \quad 0.$$

Similarly, from

$$\int_0^\pi l\,(\cos x - \cos a)\, l\,(\cos x - \cos \beta)\,dx = \pi\,(\log 2)^2 + \tfrac{1}{2}\pi\,(a^2 + \beta^2) - \pi^2 a + \tfrac{1}{3}\pi^3$$

$$(0 < \beta < a < \pi),$$

we deduce
$$P\int_0^\pi \frac{l\,(\cos x - \cos a)}{\cos x - \cos \beta}\,dx = \frac{\pi\,(a-\pi)}{\sin a} \quad \text{or} \quad \frac{\pi a}{\sin a},$$

according as $\beta <$ or $> a$. The integral is discontinuous for $\beta = a$ (II., § 36), and its value for $\beta = a$ is the mean of its values for $\beta = a \pm 0$.*

(iii.) We find [*cf.* § 5 (i.)] that

$$\int_q^p \log\left(\frac{p-x}{x-q}\right)^2 \frac{dx}{x} = \left\{\log\left(\frac{p}{q}\right)\right\}^2 \quad (p > q).$$

(iv.) $$P\int_0^\infty \frac{\log\{1 + a\,(x-a)\}^2}{(x-a)^2}\,dx = -\frac{1}{2a}\left\{aa \log(aa)^2 + (1 - aa)\log(1 - aa)^2\right\},$$

unless $aa = 0$ or 1. This example is instructive, as (according to the values of a and a) it may be an example of the use of Leibniz's theorem, or of either or both of Theorems 1, 2.

(v.) The following formulæ afford further illustrations; in all a, β, γ are positive:

$$\int_0^\infty \log\left(1 - \frac{a^2}{x^2}\right)^2 \log\left(1 - \frac{\beta^2}{x^2}\right)^2 dx = 2\pi^2 a\ (a < \beta),\ \ 2\pi^2\beta\ (a > \beta);$$

$$P\int_0^\infty \log\left(1 - \frac{\beta^2}{x^2}\right)^2 \log\left(1 - \frac{\gamma^2}{x^2}\right) \frac{dx}{a^2 - x^2}$$

$$= \frac{\pi^2}{a}\left\{l\left(\frac{\gamma^2}{a^2} - 1\right) + 2l\left(1 + \frac{\beta}{a}\right)\right\},\ \ \frac{\pi^2}{a}\left\{l\left(\frac{\beta^2}{a^2} - 1\right) + 2l\left(1 + \frac{\gamma}{a}\right)\right\},$$

$$\frac{2\pi^2}{a}\, l\left(1 + \frac{\beta}{a}\right),\ \ \frac{\pi^2}{a}\, l\,\frac{a+\beta}{a-\beta},\ \ \frac{2\pi^2}{a}\, l\left(1 + \frac{\gamma}{a}\right),\ \ \frac{\pi^2}{a}\, l\,\frac{a+\gamma}{a-\gamma},$$

according as $a < \beta < \gamma$, $a < \gamma < \beta$, $\beta < a < \gamma$, $\beta < \gamma < a$, $\gamma < a < \beta$, or $\gamma < \beta < a$;

$$\int_0^\infty \log\left(1 - \frac{a^2}{x^2}\right)^2 \log\left(1 - \frac{\beta^2}{x^2}\right)^2 \log\left(1 - \frac{\gamma^2}{x^2}\right)^2 dx$$

$$= 4\pi^2\left\{a \log\left(\frac{\gamma^2}{a^2} - 1\right) + \gamma \log\frac{\gamma + a}{\gamma - a} + 2a \log\left(1 + \frac{\beta}{a}\right) + 2\beta \log\left(1 + \frac{a}{\beta}\right)\right\}$$

$$(a \leqq \beta \leqq \gamma).$$

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* In II., § 36, the discontinuity is assigned only half its true value.

8. THEOREM 3.—*If* $f(x, a) = \Omega_\nu(x-a)\Theta(x, a)$,

Θ being a function whose derivates of the first two orders are continuous in (a, A, a_0-H, a_0+H), *where* $a < a_0-H < a_0+H < A$, *then*

$$I(a) = P\int_a^A f(x, a)\,dx$$

will have a continuous derivate equal to

$$P\int_a^A \Omega_\nu(x-a)\left\{\frac{\partial\Theta}{\partial x}+\frac{\partial\Theta}{\partial a}\right\}dx+f(a, a)-f(A, a).$$

It is assumed that f and its derivates are continuous except for $x = a$. A product of logarithmic factors such as occurs in $\Omega_\nu(u)$ may become infinite for values of u other than 0, such as 1, e, ... We suppose that such a contingency is avoided by a suitable choice of the range (a, A) and the exponents s_i.

Suppose $2a-a<A$; then, by the transformation used in § 3,

$$I(a) = \int_0^{a-a}\phi(y, a)\,dy+\int_{2a-a}^A f\,dx,$$

where $$\phi(y, a) = f(a+y, a)+f(a-y, a).$$

Now, for values of y other than zero,

$$\begin{aligned}\frac{\partial\phi}{\partial a} &= \left[\frac{\partial f(t, a)}{\partial a}+\frac{\partial f(t, a)}{\partial t}\right]_{t=a+y}+\left[\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\right]_{t=a-y}\\ &= \Omega_\nu(y)\left\{\left[\frac{\partial\Theta}{\partial a}+\frac{\partial\Theta}{\partial t}\right]_{t=a+y}-\left[\frac{\partial\Theta}{\partial a}+\frac{\partial\Theta}{\partial t}\right]_{t=a-y}\right\}\\ &= 2y\Omega_\nu(y)\left[\frac{\partial^2\Theta}{\partial a\,\partial t}+\frac{\partial^2\Theta}{\partial t^2}\right]_{t=a+\theta y}\quad(-1\leqq\theta\leqq1).\end{aligned}$$

This is only logarithmically infinite for $y = 0$, and

$$\int_0^{a-a}\frac{\partial\phi}{\partial a}\,dy$$

is uniformly convergent. Hence

$$\frac{d}{da}\int_0^{a-a}\phi\,dy = \int_0^{a-a}\frac{\partial\phi}{\partial a}\,dy+\{f(2a-a, a)+f(a, a)\}.$$

Also $$\frac{d}{da}\int_{2a-a}^A f\,dx = \frac{d}{da}\int_{a-a}^{A-a}f(a+y, a)\,dy$$

$$= \int_{a-a}^{A-a}\left[\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\right]_{t=a+y}dy-\{f(A, a)+f(2a-a, a)\}.$$

Therefore

$$\frac{dI}{da} = \int_0^{A-a}\left[\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\right]_{t=a+y} dy+\int_0^{a-a}\left[\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\right]_{t=a-y} dy+f(a,a)-f(A,a).$$

Finally, $\int_\epsilon^{A-a}\left[\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\right]_{t=a+y} dy+\int_\epsilon^{a-a}\left[\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\right]_{t=a-y} dy$

$$= \left(\int_{a+\epsilon}^{A}+\int_{a}^{a-\epsilon}\right)\left\{\frac{\partial f}{\partial a}+\frac{\partial f}{\partial x}\right\}dx;$$

and, as the left hand tends uniformly to its limit for $\epsilon = 0$, the right hand tends uniformly to its limit

$$P\int_a^A\left(\frac{\partial f}{\partial a}+\frac{\partial f}{\partial x}\right)dx.$$

This principal value is therefore uniformly convergent and continuous, and the theorem follows. If we had taken f in the more general form

$$\Omega_\nu\{x-X(a)\}\,\Theta(x,a),$$

the derivate would have been

$$P\int_a^A\Omega_\nu(x-X)\left\{\frac{\partial\Theta}{\partial a}+\frac{dX}{da}\frac{\partial\Theta}{\partial x}\right\}dx+\frac{dX}{da}\{f(a,a)-f(A,a)\}.$$

9. (i.) If $$I(a) = P\int_a^A\frac{dx}{x-a} = \log\frac{A-a}{a-a}\quad (a<a<A),$$

$$\frac{dI}{da} = \frac{1}{a-a}-\frac{1}{A-a},$$

which is evidently correct.

(ii.) From $$P\int_0^\pi\frac{dx}{\cos x-\cos a} = 0\quad (0<a<\pi)$$

we deduce by successive differentiation

$$P\int_0^\pi\frac{\sin x-\sin a}{(\cos x-\cos a)^2}dx = -\frac{2}{\sin^2 a},\quad P\int_0^\pi\frac{(\sin x-\sin a)^2}{(\cos x-\cos a)^3}dx = \frac{4\cos a}{\sin^3 a}.$$

(iii.) From $$P\int_0^\infty\frac{dx}{x^2-a^2} = 0\quad (a>0)$$

we deduce $$P\int_0^\infty\frac{dx}{(x-a)(x+a)^{n+1}} = -\frac{1}{2^n a^{n+1}}\left(1+\frac{2}{2}+\frac{2^2}{3}+\dots+\frac{2^{n-1}}{n}\right).$$

(iv.) If $$I = P\int_{-\infty}^\infty\frac{\sin px}{x-a}dx\quad (p,\ a>0),$$

we find $$\frac{d^2I}{da^2}+p^2I=0,\quad I=\pi\cos pa,$$

and so $$P\int_0^\infty \frac{x\sin px}{x^2-a^2}\,dx=\tfrac{1}{2}\pi\cos ap,\quad P\int_0^\infty \frac{\cos px}{x^2-a^2}\,dx=-\frac{\pi}{2a}\sin ap.$$

Infinite Limits; the general case.

10. None of the preceding theorems cover the case in which the number of singular curves is infinite.*

THEOREM 4.—*If* $f(x,\alpha)$, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial \alpha}$ *are continuous throughout any finite part of*

$$(a,\ \infty,\ \alpha_0-H,\ \alpha_0+H),$$

except upon a finite number of curves $x=X_i(\alpha)$, *which satisfy the same conditions as in* 1, *and*

$$\lim_{\epsilon=0}\frac{dX_i}{d\alpha}\{f(X_i-\epsilon,\ \alpha)-f(X_i+\epsilon,\ \alpha)\}=0$$

uniformly for all values of α *in* $(\alpha_0-H,\ \alpha_0+H)$, *and*

$$P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx$$

is uniformly convergent, then will

$$\frac{d}{d\alpha}P\int_a^\infty f\,dx=P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx.$$

For let $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ be any series of descending positive quantities whose limit is 0. We can choose a value of a_1 such that

$$P\int_a^{a_1}\frac{\partial f}{\partial \alpha}\,dx$$

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* For the purposes of these theorems I shall alter slightly the meaning of the expressions *uniformly convergent*, *regularly convergent*, defined in my last paper for principal values whose upper limit is ∞.

In the definition of uniform convergence (II., § 19) condition (ii.) is to read "however small be the positive quantity σ, and however great be H, we can find a value of $A>H$, such that ..."

In the definition of regular convergence (II., § 21) condition (iii.) is to read "however small be the positive quantity σ, and however great be H, we can find (1) a value of $A>H$, (2) ....

The definitions, in their original form, suffice for the deductions drawn from them in my former paper. But we have not been and shall not be concerned with any principal values which satisfy the original and not the modified form of them.

is uniformly convergent, and

$$\left| P\int_{a_1}^{\infty} \right| < \sigma_1$$

for all values of a in $(a_0-H,\ a_0+H)$; a value of a_2, such that $P\int_{a_1}^{a_2}$ is uniformly convergent, and $\left| P\int_{a_2}^{\infty} \right| < \sigma_2$; and so on. We can suppose $a_1 < a_2 < \ldots$, $\lim a_n = \infty$. Then none of the curves $x = X_i$ can meet any of the lines $x = a_1, a_2, \ldots$; and the conditions of I. are satisfied in each of the regions

$$(a_n,\ a_{n+1},\ a_0-H,\ a_0+H),$$

and
$$\frac{d}{da} P\int_{a_n}^{a_{n+1}} f\,dx = P\int_{a_n}^{a_{n+1}} \frac{\partial f}{\partial a}\,dx.$$

Moreover the series
$$\sum_0^{\infty} \frac{d}{da} P\int_{a_n}^{a_{n+1}} f\,dx$$

$(a_0 = a)$ is uniformly convergent. Therefore

$$\frac{d}{da} P\int_a^{\infty} f\,dx = \frac{d}{da}\sum_0^{\infty} P\int_{a_n}^{a_{n+1}} f\,dx = \sum_0^{\infty} \frac{d}{da} P\int_{a_n}^{a_{n+1}} f\,dx$$

$$= \sum_0^{\infty} P\int_{a_n}^{a_{n+1}} \frac{\partial f}{\partial a}\,dx = P\int_a^{\infty} \frac{\partial f}{\partial a}\,dx.$$

11. (i.) Thus, if
$$I(a) = P\int_0^{\infty} \sin ax \tan x \frac{dx}{x^2},$$

$$\frac{dI}{da} = P\int_0^{\infty} \cos ax \tan x \frac{dx}{x} = \tfrac{1}{2}\pi.$$

And, as $I(a)$ is continuous for $a = 0$,

$$I(a) = \tfrac{1}{2}\pi a.$$

Similarly
$$P\int_0^{\infty} \frac{\cos ax - \cos \beta x}{x^3} \tan x\,dx = \tfrac{1}{4}\pi (a^2 - \beta^2).$$

Similarly we can establish the following results:—(ii.) If $a > 0$,

$$P\int_0^{\infty} \log\left(1+\frac{a^2}{x^2}\right) \frac{dx}{\cos x} = 2\pi \tan^{-1} \tanh \tfrac{1}{2}a,$$

$$P\int_0^{\infty} \log\left(1-\frac{a^2}{x^2}\right)^2 \frac{dx}{\cos x} = 0,\ \pi^2,\ 2\pi^2,$$

according as $(2n-\frac{1}{2})\pi < a < (2n+\frac{1}{2})\pi$, $a = (n+\frac{1}{2})\pi$, or $(2n+\frac{1}{2})\pi < a < (2n+\frac{3}{2})\pi$. In the latter case we prove first that the integral is constant where continuous, and evaluate its discontinuities by II., § 36.

(iii.) If $$0 < (m+\tfrac{1}{2})\pi < \beta < \alpha < (m+\tfrac{3}{2})\pi,$$

$$P\int_0^\infty \frac{\log\left(1-\frac{\alpha^2}{x^2}\right)^2}{\beta^2-x^2}\frac{dx}{\cos x} = 2\pi^2\left[\frac{1}{2\beta\cos\beta}+\sum_0^m \frac{(-)^\kappa}{\beta^2-\{(\kappa+\frac{1}{2})\pi\}^2}\right].$$

(iv.) $$P\int_0^\infty \frac{1}{\cos x-\cos\alpha}\frac{dx}{a^2+x^2} = \frac{\pi}{2a(\cosh a-\cos\alpha)}.$$

12. THEOREM 5.—*If the conditions of* 4 *are satisfied, except that*

$$P\int_a^\infty \frac{\partial f}{\partial \alpha}dx$$

is only regularly convergent,

$$\frac{d}{d\alpha}P\int_a^\infty f\,dx = P\int_a^\infty \frac{\partial f}{\partial \alpha}dx.$$

For let σ_1, σ_2, ... be a series of descending positive quantities whose limit is 0. We can choose a value of a_1, a set of positive quantities $p_{1,i}$, each less than some fixed quantity p_0, and a division of (α_0-H, α_0+H) into two sets of partial intervals $\theta_{1,i}$, $\eta_{1,i}$, such that

$$P\int_a^{a_1}\frac{\partial f}{\partial \alpha}dx$$

is uniformly convergent in $\theta_{1,i}$ and

$$P\int_a^{a_1-p_{1,i}}\frac{\partial f}{\partial \alpha}dx$$

in $\eta_{1,i}$; and, moreover, $\left|P\int_a^{a_1}\right| < \sigma_1$ in $\theta_{1,i}$, and $\left|P\int_a^{a_1-p_{1,i}}\right| < \sigma$ in $\eta_{1,i}$. If then we define $a_1'(\alpha)$ as being $= a_1$ in θ_1 and $a_1-p_{1,i}$ in $\eta_{1,i}$, $P\int_a^{a_1'}$ is uniformly convergent, and

$$\left|P\int_a^{a_1'}\right| < \sigma_1$$

for all values of α in (α_0-H, α_0+H). We next choose a_2, $p_{2,i}$, $\theta_{2,i}$, $\eta_{2,i}$, similarly corresponding to σ_2, and define $a_2'(\alpha)$ in the same way; and so on. And we can suppose that the least value of $a_{n+1}'(\alpha)$ is greater than the greatest of $a_n'(\alpha)$, and that $\lim_{n=\infty} a_n'(\alpha) = \infty$ uniformly for all values of α. Then, if $a_0' = a$,

$$P\int_a^\infty f\,dx = \sum_0^\infty P\int_{a_n'}^{a_{n+1}'},\quad P\int_a^\infty \frac{\partial f}{\partial \alpha}dx = \sum_0^\infty P\int_{a_n'}^{a_{n+1}'};$$

and the second series is uniformly convergent. So the theorem follows as before.*

13. If, *e.g.*,
$$I(a) = \int_0^\infty \frac{\log\cos^2 ax}{x^2}\,dx \quad (a > 0),$$
$$\frac{dI}{da} = -2P\int_0^\infty \frac{\tan ax}{x}\,dx = -\pi;$$
$$I(a) = -a\pi.$$

Again, it is easy to prove that
$$\int_0^\infty \cos ax \log\cos^2 ax \frac{dx}{1+x^2} = \pi\cosh a\log(1+e^{-2a}) - \pi e^{-a}\log 2,$$
$$P\int_0^\infty \cos ax\log\cos^2 ax\frac{dx}{1-x^2} = \pi a\cos a - \pi\sin a\log 2,$$
if $0 < a < 2a$, and in the second formula $2a < \pi$. Differentiating,
$$P\int_0^\infty \cos ax\tan ax\frac{x\,dx}{1+x^2} = \frac{\pi\cosh a}{e^{2a}+1},$$
$$P\int_0^\infty \cos ax\tan ax\frac{x\,dx}{1-x^2} = -\tfrac{1}{2}\pi\cos a.$$

If
$$I(a) = \int_0^\infty \log\left(1-\frac{a^2}{x^2}\right)^2\log\cos^2 ax\,dx,$$
$$\frac{dI}{da} = -2P\int_0^\infty \log\left(1-\frac{a^2}{x^2}\right)^2\tan ax\,dx,$$
$$\frac{d^2I}{da\,da} = -8aP\int_0^\infty \frac{x\tan ax}{a^2-x^2}\,dx = 4a\pi \quad \left(0 < a,\ 0 < a < \frac{\pi}{2a}\right).$$

Hence
$$\frac{dI}{da} = 2a^2\pi, \quad I = 2a^2a\pi.$$

14. A good deal of the substance of §§ 2–13 appeared in a paper "On Differentiation and Integration under the Integral Sign" (*Quarterly Journal of Mathematics*, No. 125, 1900, p. 66).

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* There is a difficulty here which should be expressly mentioned. If
$$\int_{a'_n}^{a'_{n+1}} = u_n, \quad P\int_{a'_n}^{a'_{n+1}} = u'_n,$$
$\frac{du_n}{da} = u'_n$ for $a = a_0$, and indeed throughout an interval $(a_0 - H_n, a_0 + H_n)$, which may, however, decrease indefinitely as n increases. Since a'_n is not continuous throughout $(a_0 - H, a_0 + H)$, neither u_n nor u'_n is as a rule continuous throughout that interval for all values of n; and for certain values of a, which may approach indefinitely near to a_0 as n increases, $\frac{du_n}{da}$ ceases to be determinate at all. The series Σu_n may none the less be differentiated for $a = a_0$. I omit the formal proof of this statement, as it is detailed and perhaps tedious.

Integration under the sign of the Principal Value.

15. If $f(x, y)$ is a continuous function of both variables in the rectangle (a, A, b, B),

$$\int_b^B dy \int_a^A f(x, y)\, dx = \int_a^A dx \int_b^B f(x, y)\, dy.$$

But this equation is true under much more general conditions, which have been studied by many writers, among whom I may particularly mention M. Ch. de la Vallée-Poussin. It is sufficient, for instance, (when the limits are finite), that $f(x, y)$ be finite throughout any part of (a, A, b, B) which does not contain any point situated on a set of curves satisfying certain conditions, and

$$\int_a^A f(x, y)\, dx, \quad \int_b^B f(x, y)\, dy$$

be uniformly convergent.

I shall not, however, enter into any discussion of these general conditions in this paper. The question which concerns us now is: *Under what circumstances is* (1) *true when some or all of the integrals contained in it are only principal values?* And I have already pointed out that it is not worth while to attempt to state theorems connected with the principal value with all the generality we can give them. What is worth doing is to distinguish and examine the various simple cases which occur when we are dealing with functions which present themselves naturally in analysis.

We shall suppose then that $f(x, y)$ behaves in a normal manner throughout (a, A, b, B), except in the immediate neighbourhood of a finite number of simple curves, along which it becomes infinite in such a way that its integral with respect to x or y is not unconditionally convergent across them.

16. The simplest case is that in which these curves are straight lines parallel to the axes.

Let us suppose, in the first place, that $f(x, y)$ is continuous except for $x = \alpha$. Then, however small be the positive quantity ϵ,

$$\left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) dx \int_b^B f\, dy = \int_b^B dy \left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) f\, dx.$$

Further, let us suppose that

$$P\int_a^A f\, dx$$

is uniformly convergent. Then, however small be σ, we can choose ϵ so that

$$\left| P\int_{a-\epsilon}^{a+\epsilon} f\,dx \right| < \sigma$$

throughout (b, B). Thus the limit of the right hand is

$$\int_b^B dy\, P\int_a^A f\,dx\,;$$

and so the left hand also tends to a limit which is, by definition,

$$P\int_a^A dx \int_b^B f\,dy.$$

THEOREM 6.—*If $f(x, y)$ is a continuous function of both variables throughout any part of (a, A, b, B) which does not include any point of the line $x = a$, and*

$$P\int_a^A f\,dx$$

is uniformly convergent, then

$$P\int_a^A dx \int_b^B f\,dy = \int_b^B dy\, P\int_a^A f\,dx.$$

17. If $a = b = 0$, $A = B = c$, and

$$f(x, y) = \frac{1}{(x-\alpha)(x+y+\beta)} \quad (0<\beta,\ 0<\alpha<c),$$

we find on integration that

$$P\int_0^c \log\left(1+\frac{c}{x+\beta}\right)\left(\frac{1}{x-\alpha}+\frac{1}{x+\alpha+\beta}\right) dx = \log\frac{c-\alpha}{\alpha}\log\frac{c+\alpha+\beta}{\alpha+\beta}.$$

18. Let us suppose now that B is infinite, and that, for any finite value of $B' > b$,

$$P\int_a^A dx \int_b^{B'} f\,dy = \int_b^{B'} dy\, P\int_a^A f\,dx\,;$$

then, if $\int_b^\infty f\,dy$ is convergent, except for $x = a$, and

$$\lim_{B'=\infty} P\int_a^A dx \int_{B'}^\infty f\,dy = 0,$$

this equation passes over in the limit into

$$P\int_a^A dx \int_b^\infty f\,dy = \int_b^\infty dy\, P\int_a^A f\,dx.$$

Suppose, for instance, that

$$f = \frac{\psi(x, y)}{x-a},$$

ψ being a function whose derivate $\dfrac{\partial\psi}{\partial x}$ is continuous throughout (a, A, b, ∞), and that

$$\int_b^\infty \psi\, dy, \quad \int_b^\infty \frac{\partial\psi}{\partial x}\, dy$$

are uniformly convergent in (a, A); then

$$\Psi(x) = \int_{B'}^{\infty} \psi\, dy$$

is continuous, and has a continuous derivate represented by

$$\int_{B'}^{\infty} \frac{\partial\psi}{\partial x}\, dy.$$

Also

$$P\int_a^A dx \int_{B'}^{\infty} f\, dy$$

is determinate, and equal to

$$\left(\int_a^{a-\epsilon} + \int_{a+\epsilon}^{A}\right)\int_{B'}^{\infty} + P\int_{a-\epsilon}^{a+\epsilon}\int_{B'}^{\infty}$$

for any small positive value of ϵ. The last term

$$= 2\epsilon\Psi'(a+\theta\epsilon) \quad (-1 \leqq \theta \leqq 1);$$

and this is numerically $< \epsilon K$, where K is a quantity independent of ϵ and of B'. We can therefore make it less than any assigned positive quantity $\frac{1}{2}\sigma$ by choice of a value of ϵ independent of B'. We can then choose B' so that

$$\left|\int_{B_1}^{\infty} \psi\, dy\right| < \frac{\sigma\epsilon}{2(A-a)}$$

for all values of x in $(a, a-\epsilon)$ and $(a+\epsilon, A)$, and all values of $B_1 \geqq B'$; and therefore

$$\left|\left(\int_a^{a-\epsilon} + \int_{a+\epsilon}^{A}\right)\frac{dx}{x-a}\int_{B_1}^{\infty} \psi\, dy\right| < \tfrac{1}{2}\sigma.$$

Hence

$$\left|P\int_a^A dx \int_{B_1}^{\infty} f\, dy\right| < \sigma;$$

so that

$$\lim_{B'=\infty} P\int_a^A dx \int_{B'}^{\infty} f\, dy = 0.$$

This is equally true if for $\frac{1}{x-a}$ we substitute the more general factor $\Omega_\nu(x-a)$.

19. The extension of §§ 16, 18 to the case in which $A=\infty$ does not present any fresh difficulty which is particularly interesting to us now. For we have only to combine the equations

$$P\int_a^A dx \int_b^B f\,dy = \int_b^B dy\, P\int_a^A f\,dx$$

and
$$\int_A^\infty dx \int_b^B f\,dy = \int_b^B dy \int_A^\infty f\,dx;$$

and the difficulties which may meet us when we try to prove this last are not those with which this paper is concerned.

20. We can prove, for instance, without much difficulty, that the theorem holds if $a=b=0$, $A=B=\infty$, and

$$f(x,y) = \frac{\cos xy}{(x^2-a^2)(1+y^2)} \quad (a>0).$$

We deduce
$$\int_0^\infty \frac{\sin ay\,dy}{1+y^2} = -\frac{2a}{\pi}\int_0^\infty \frac{dy}{1+y^2}\, P\int_0^\infty \frac{\cos xy}{x^2-a^2}\,dx$$

$$= -\frac{2a}{\pi}\, P\int_0^\infty \frac{dy}{x^2-a^2}\int_0^\infty \frac{\cos xy}{1+y^2}\,dy$$

$$= \tfrac{1}{2}\left\{e^{-a}\,li\,(e^a) - e^a\,li\,(e^{-a})\right\},$$

a result proved otherwise by Schlömilch and Kronecker.

21. It is to be observed that we need not insist on the condition laid down in § 18, that

$$\int_b^\infty \frac{\partial\psi}{\partial x}\,dy$$

should be uniformly convergent, if we can assure ourselves that

$$\Psi(x) = \int_{B'}^\infty \psi\,dy$$

has a continuous derivate always numerically less than some quantity independent of x and B'.

This case occurs, *e.g.*, if

$$f(x,y) = \frac{\cos xy}{(x^2-a^2)\,y^\mu} \quad (0<\mu<1),$$

when we deduce
$$P\int_0^\infty \frac{t^{\nu-1}dt}{1-t} = \pi\cot\nu\pi \quad (0<\nu<1).$$

22. Let us suppose next that the limits are finite, and f continuous except along $x = a$ and $y = \beta$, where $a < a < A$, $b < \beta < B$; and that

$$P\int_a^A f\,dx, \quad P\int_b^B f\,dy$$

are uniformly convergent, except for $y = \beta$ and $x = a$ respectively. Then

$$P\int_a^A dx\left(\int_b^{\beta-\theta}+\int_{\beta+\theta}^B\right) f\,dy = \left(\int_b^{\beta-\theta}+\int_{\beta+\theta}^B\right) dy\,P\int_a^A f\,dx,$$

however small be θ. This equation will pass over in the limit into

$$(1) \quad P\int_a^A dx\,P\int_b^B f\,dy = P\int_b^B dy\,P\int_a^A f\,dx,$$

provided only $\quad P\int_a^A dx\,P\int_{\beta-\theta}^{\beta+\theta} f\,dx$

be determinate, and tend to zero for $\theta = 0$. And it is clear that this will be the case if the same is true of

$$P\int_{a-\epsilon}^{a+\epsilon} P\int_{\beta-\theta}^{\beta+\theta} = P\int_{-\epsilon}^{\epsilon} d\xi\,P\int_{-\theta}^{\theta} f(a+\xi,\,\beta+\eta)\,d\eta.$$

Now let us suppose that

$$f(x, y) = \frac{\psi(x, y)}{(x-a)(y-\beta)},$$

where ψ is a function which has continuous first derivates throughout (a, A, b, B). To save unnecessary discussion we shall assume that $\psi(x, y)$ is capable of expansion in a Taylor's series in the neighbourhood of (a, β). Then

$$\psi(a+\xi,\,\beta+\eta) = \psi(a, \beta) + \xi\psi_1(\xi, \eta) + \eta\psi_2(\xi, \eta),$$

where ψ_1, ψ_2 are functions which also behave regularly near (a, β). We consider the integrals arising from these three terms separately.

(i.) Since $\quad P\int_{-\theta}^{\theta} \frac{d\eta}{\xi\eta} = 0,$

the first term contributes nothing.

(ii.) The second contributes

$$\int_{-\epsilon}^{\epsilon} d\xi\,P\int_{-\theta}^{\theta} \frac{\psi_1(\xi, \eta)}{\eta}\,d\eta = 2\theta\int_{-\epsilon}^{\epsilon} \frac{\partial\psi_1(\xi, \eta')}{\partial\eta'}\,d\xi \quad (-\epsilon \leq \eta' \leq \epsilon),*$$

which is determinate and tends to zero for $\theta = 0$.

* I., § 12.

(iii.) The third contributes

$$P\int_{-\epsilon}^{\epsilon}\frac{d\xi}{\xi}\int_{-\theta}^{\theta}\psi_2(\xi,\eta)\,d\eta.$$

This is also determinate and tends to zero for $\theta = 0$; for

$$\int_{-\theta}^{\theta}\psi_2\,d\eta$$

is a function of ξ which possesses a continuous derivate

$$\int_{-\theta}^{\theta}\frac{\partial\psi_2}{\partial\xi}\,d\eta,$$

and $$P\int_{-\epsilon}^{\epsilon}\frac{d\xi}{\xi}\int_{-\theta}^{\theta}\psi_2\,d\eta = 2\epsilon\left[\int_{-\theta}^{\theta}\frac{\partial\psi_2}{\partial\xi}\,d\eta\right]_{\xi=\xi'},$$

where $$-\epsilon \leqq \xi' \leqq \epsilon.$$

Hence in this case

$$P\int_a^A dx\,P\int_b^B f\,dy = P\int_b^B dy\,P\int_a^A f\,dx.$$

And it is easy to see that the same conclusion holds in the more general case in which

$$f(x,y) = \Omega_\mu(x-a)\,\Omega_\nu(y-\beta)\,\psi(x,y).$$

23. So long as there is but a finite number of singular lines parallel to either axis, the extension of (1) of § 22 to the case in which A or B or both are infinite does not present any fresh difficulty, as the rectangle (a, ∞, b, ∞) can be divided up into a finite number of partial rectangles each of which satisfies the conditions of one or other of the preceding sections.

24. If

and

we obtain, after some reduction,

$$P\int_0^\infty\frac{\log u\,du}{(u+\alpha)(u-\beta)} = \frac{1}{\alpha+\beta}\left\{\pi^2+(\log\alpha)^2-(\log\beta)^2\right\}.$$

25. It sometimes happens that the formula

$$\int_a^A dx\int_b^B f\,dy = \int_b^B dy\int_a^A f\,dx$$

is only true if we introduce the sign of the principal value before one of the outer integral signs. We may have

$$(1)\quad \left(\int_a^{a-\epsilon}+\int_{a+\epsilon}^{A}\right)\int_b^B=\int_b^B\left(\int_a^{a-\epsilon}+\int_{a+\epsilon}^{A}\right),$$

and $\int_a^A$, $\int_b^B\int_a^A$ convergent, while $\int_a^A\int_b^B$ is not so. If, however,

$$\lim \int_b^B\int_{a-\epsilon}^{a+\epsilon}=0,$$

equation (1) will pass over in the limit into

$$P\int_a^A dx\int_b^B f\,dy=\int_b^B dy\int_a^A f\,dx.$$

This case is not of the same type as those which we have been discussing.

26. Suppose $$b=0,\quad B=\infty,$$

and $$f(x,y)=e^{-y(1+\cos x)}\sin(x-y\sin x)\,\phi(x),$$

$\phi(x)$ being a function of x which possesses a continuous derivate. Then, so long as x is not an odd multiple of π,

$$\int_0^\infty f\,dy=\frac{\sin x\,(1+\cos x)-\cos x\sin x}{(1+\cos x)^2+\sin^2 x}\,\phi(x)=\tfrac{1}{2}\tan\tfrac{1}{2}x\,\phi(x).$$

Hence, if A is not an odd integer,

$$P\int_a^A\tfrac{1}{2}\tan\tfrac{1}{2}x\,\phi(x)\,dx=\int_0^\infty e^{-y}\,dy\int_a^A e^{-y\cos x}\sin(x-y\sin x)\,\phi(x)\,dx;$$

provided the condition of the preceding section is satisfied.

Now $$e^{-y\cos x}\sin(x-y\sin x)=\sum_1^\infty(-)^{n-1}\frac{y^n\sin nx}{n!},$$

and so
$$\begin{aligned}P\int_a^A\tfrac{1}{2}\tan\tfrac{1}{2}x\,\phi(x)\,dx&=\int_0^\infty e^{-y}\,dy\int_a^A\phi(x)\sum_1^\infty(-)^{n-1}\frac{y^n\sin nx}{n!}\,dx\\&=\int_0^\infty e^{-y}\,dy\sum_0^\infty(-)^{n-1}\frac{y^n}{n!}\int_a^A\sin nx\,\phi(x)\,dx\\&=\sum_0^\infty(-)^{n-1}\frac{1}{n!}\int_0^\infty e^{-y}y^n\,dy\int_a^A\sin nx\,\phi(x)\,dx\\&=\sum_0^\infty(-)^{n-1}\int_a^A\sin nx\,\phi(x)\,dx,\end{aligned}$$

provided this series be convergent. As this result has already been obtained by another method in II. (§ 28), I shall not delay over the proof that the inversion of the order of integration is as a matter of fact legitimate.

27. So far I have supposed that the singular lines are all straight lines parallel to an axis. I shall now consider the case in which they are continuous curves never parallel to either axis. The

simplest such curve is the line

$$x = y,$$

and I shall begin by supposing that this is the only singular curve. It will not be difficult to generalize the results.

28. Suppose, then, that $f(x, y)$ is a function whose derivates $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous throughout any part of the square

$$(a, A, a, A)$$

which does not include any point of the line $x = y$; and that, near $x = y$, $f(x, y)$ may be expressed in the form

$$\Omega_\nu (x-y)\, \Theta(x, y),$$

Θ being a function whose derivates $\frac{\partial \Theta}{\partial x}, \frac{\partial \Theta}{\partial y}$ are continuous without exception.

Then $P\int_a^A f\,dy$ is uniformly convergent in $x = (a, A)$, except for $x = a, A$; and $P\int_a^A f\,dx$ in $y = (a, A)$, except for $y = a, A$. And the function

$$\phi(x, y) = P\int_a^y f\,dy$$

is a continuous function of both variables except along the line $x = y$, and as y approaches x becomes infinite (if at all) in such a way that

$$\lim_{\eta=0} \eta^\mu \phi(x, x-\eta) = 0$$

for any positive value of μ. And

$$\int_a^A \phi(x, y)\,dx$$

is uniformly convergent in (a, A).

Similarly, if $$\psi(x, y) = P\int_a^x f\,dx,$$

$\int_a^A \psi(x, y)\,dy$ is uniformly convergent in (a, A). We exclude $x = y$ from the field of integration by the two lines $x-y = \pm\epsilon$. Applying the ordinary theorem, that the order of integration is indifferent, to that part of the remainder of the field which lies between

$$y = a+\epsilon, \quad y = A-\epsilon,$$

we find (Fig. 1)

$$(1)\quad \int_{a+\epsilon}^{A-\epsilon} dy \left(\int_a^{y-\epsilon} + \int_{y+\epsilon}^{A}\right) f\,dx = \int_a^{A-2\epsilon} dx \int_{x+\epsilon}^{A-\epsilon} f\,dy + \int_{a+2\epsilon}^{A} dx \int_{a+\epsilon}^{x-\epsilon} f\,dy.$$

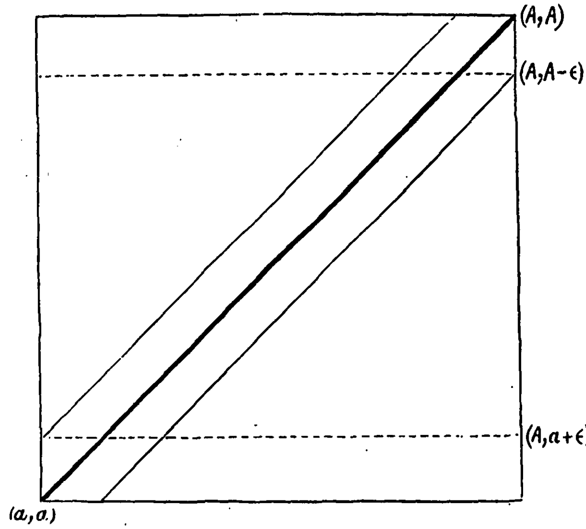

FIG. 1.

Now, if $$a+\epsilon \leqq y \leqq A-\epsilon,$$

$$P\int_{y-\epsilon}^{y+\epsilon} f\,dx = \left[\frac{\partial\Theta}{\partial x}\right]_{x=y+\theta\epsilon} \int_{y-\epsilon}^{y+\epsilon} (x-y)\,\Omega_\nu(x-y)\,dx,$$

where $$-1 \leqq \theta \leqq 1.$$

We can therefore choose ϵ_0 so that $P\int_{y-\epsilon}^{y+\epsilon}$ is numerically less than any assigned positive quantity $\dfrac{\sigma}{2(A-a)}$ for all values of $\epsilon \leqq \epsilon_0$. Then the left-hand side of (1) differs from

$$\int_{a+\epsilon}^{A-\epsilon} dy\, P\int_a^A f\,dx$$

by less than $\frac{1}{2}\sigma$. But we can also suppose ϵ_0 so small that

$$\int_a^{a+\epsilon} P\int_a^A, \quad \int_{A-\epsilon}^{A} P\int_a^A$$

are also numerically less than $\frac{1}{4}\sigma$. Then the left-hand side of (1) differs from

$$\int_a^A dy\, P\int_a^A f\,dx$$

by less than σ. And the right-hand side of (1) is

$$\int_{a+2\epsilon}^{A-2\epsilon} dx \left(\int_{x+\epsilon}^{x-\epsilon} + \int_{x+\epsilon}^{A-\epsilon} \right) f\,dy + \Delta,$$

where
$$\Delta = \int_{a}^{a+2\epsilon} dx \int_{x+\epsilon}^{A-\epsilon} f\,dy + \int_{A-2\epsilon}^{A} dx \int_{a+\epsilon}^{x-\epsilon} f\,dy.$$

And an argument similar to that used above shows that ϵ_0 can be chosen so small that this differs from

$$\int_a^A dx\, P \int_a^A f\,dy$$

by less than σ. Hence

$$\int_a^A dy\, P \int_a^A f\,dx = \int_a^A dx\, P \int_a^A f\,dy.$$

28. Now, let $y = X(u)$, $X(u)$ being a function which has continuous derivates $\frac{dX}{du}$, $\frac{d^2X}{du^2}$, the first of which does not vanish between $y = a, A$. And suppose

$$X(b) = a, \quad X(B) = A.$$

Then
$$F(x, u) = f\{x, X(u)\}$$

may be expressed in the form

$$\bar{F} + \sum_{i=1}^{n} \Omega_{\nu_i} \{x - X(u)\}\, \Phi_i(x, u),$$

where $\bar{F}$ is a function which is continuous throughout

$$(a, A, b, B),$$

except on $x = X(u)$, and possesses an unconditionally and uniformly convergent integral across it; and Φ_ν is a function whose derivates $\frac{\partial \Phi_\nu}{\partial x}$, $\frac{\partial \Phi_\nu}{\partial u}$ are continuous without exception.

Also
$$P \int_a^A f\,dx = P \int_a^A F(x, u)\,dx,$$

and, by I., § 21,
$$P \int_a^A f\,dy = P \int_b^B F(x, u) \frac{dX}{du}\,du;$$

and so
$$\int_b^B du\, P \int_a^A G\,dx = \int_a^A dx\, P \int_b^B G\,du,$$

where $$G = F(x, u)\frac{dX}{du}$$

is a function of the form

$$\bar{G}+\sum_{i=1}^{n}\Omega_{\nu_i}\{x-X(u)\}\Psi_i.$$

We may suppose, moreover, that G is *any* expression of this kind; for any such expression, when expressed in terms of y, becomes a sum of terms to each of which the argument of the preceding pages may be applied. And the conditions satisfied by $X(u)$ amount to this, that $x = X(u)$ is a curve of continuous curvature which is never parallel to x or u, and passes through two corners of the rectangle (a, A, b, B).

29. Theorem 7. — *If* $f(x, y)$ *is a function whose derivates* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *are continuous throughout*

$$(a, A, b, B),$$

except on a finite number of curves of continuous curvature

$$x = X_i(y), \quad y = Y_i(x),$$

which do not intersect and are nowhere parallel to the axes, and if $f(x, y)$, *in the immediate neighbourhood of any one of these curves, can be expressed in the form*

$$\bar{f}(x, y)+\sum_{1}^{n}\Omega_{\nu_\kappa}\{x-X_i(y)\}\Theta_\kappa(x, y),$$

or in the form $$\bar{g}(x, y)+\sum_{1}^{n}\Omega_{\nu_\lambda}\{y-Y_i(x)\}\Theta_\lambda(x, y),$$

where $\bar{f}(x, y)$ *is a function which becomes at most logarithmically infinite along* $x = X_i(y)$, *and* Θ_κ *a function whose derivates* $\frac{\partial \Theta}{\partial x}, \frac{\partial \Theta}{\partial y}$ *are continuous without exception, then will*

$$\int_a^A dx\left(P\int_b^B f\,dy\right) = \int_b^B dy\left(P\int_a^A f\,dx\right).$$

For (see Fig. 2) we can divide the rectangle (a, A, b, B) into a finite number of rectangles, to each of which we may apply either the equation proved in the last paragraph or the ordinary theorem as to the interchange of two integrations.

30. If $$a = b = -1,\ A = B = 1,\ \alpha, \beta > 1,$$
and $$f(x, y) = \frac{1}{(x-y)(x+\alpha)(y+\beta)},$$
we deduce, after some reduction,*
$$\int_{-1}^{1} \log\left(\frac{1-u}{1+u}\right)\frac{du}{\alpha+u} = \tfrac{1}{2}\left(\log\frac{\alpha+1}{\alpha-1}\right)^2.$$

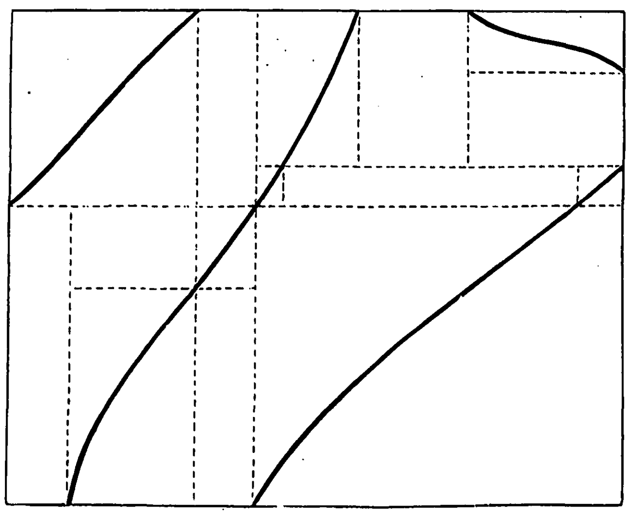

Fig. 2.

31. A considerable variety of different cases may occur when we suppose A or B infinite, but the singular curves still finite in number. If, for instance, $x = y$ is the only singular curve, no new difficulty arises when one of A, B is infinite; but when they are both infinite further discussion is necessary.

32. Let $$a = 0,\ A = \infty,\ b = 0,\ B = H > 1,$$
and $$f(x, y) = \frac{\phi(y)}{x^2-y^2+1},$$
$\phi(y)$ being a function whose derivate is continuous. The singular curve is the hyperbola $x^2-y^2+1 = 0$ (Fig. 3), and satisfies the conditions of 7, except at (0, 1), where it becomes parallel to x. Hence, if $\epsilon > 0$,
$$\int_\epsilon^\infty dx\, P\int_0^H \frac{\phi(y)\,dy}{x^2-y^2+1} = \int_0^H \phi(y)\,dy\, P\int_\epsilon^\infty \frac{dx}{x^2-y^2+1}.$$
But it is not difficult to prove that
$$\lim_{\epsilon=0}\int_0^H \phi(y)\,dy\, P\int_0^\epsilon \frac{dx}{x^2-y^2+1} = 0.$$
Hence $$\int_0^\infty dx\, P\int_0^\infty \frac{\phi(y)\,dy}{x^2-y^2+1} = \int_0^H \phi(y)\,dy\, P\int_0^\infty \frac{dx}{x^2-y^2+1} = \tfrac{1}{2}\pi\int_0^1 \frac{\phi(y)\,dy}{\sqrt{(1-y^2)}},$$
from which we can derive various integrals.

* This example is worked out in detail in the paper in the *Quarterly Journal* already referred to, p. 131.

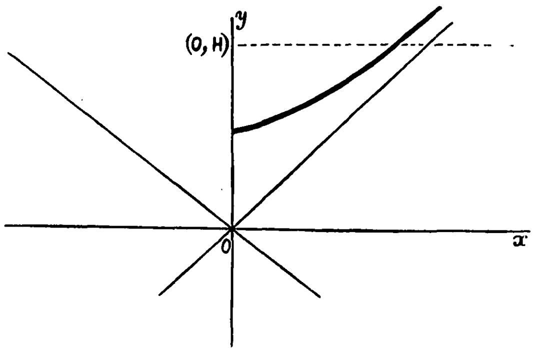

Fig. 3.

33. When A and B are both infinite there is another condition to be attended to. Let us suppose that there is one singular curve (such as $x-y=0$ or $x^2-y^2+1=0$) which extends to infinity in the positive quadrant in a definite direction not parallel to x or y. Then for any finite values of A and B

$$\int_a^A P \int_b^B = \int_b^B P \int_a^A ;$$

and we may suppose one of the upper limits, say A, replaced by ∞, as in the example of § 32. We need not stop to discuss the conditions under which this is legitimate; for, if the singular curve meets $y=B$ in (A', B), we have only to combine the two equations

$$\int_a^{A_1} P \int_b^B = \int_b^B P \int_a^{A_1}, \qquad \int_{A_1}^{\infty} \int_b^B = \int_b^B \int_{A_1}^{\infty}$$

($A_1 > A'$; see Fig. 4). Let us suppose, therefore, that

$$\int_a^{\infty} P \int_b^B = \int_b^B P \int_a^{\infty}.$$

Then, if

$$\int_a^{\infty} P \int_B^{\infty}$$

is determinate, and tends to the limit 0 for $B=\infty$, this equation ultimately passes over into

$$\int_a^{\infty} P \int_b^{\infty} = \int_b^{\infty} P \int_a^{\infty}.$$

34. Let

$$a = -\infty, \quad b = 0,$$

and

$$f(x, y) = \frac{e^{pix}\,\phi(y)}{x-y} \quad (p > 0),$$

where $\phi(y)$ is a function whose derivate is continuous. Then, if the condition of the previous section (or that which results by interchanging x and y) is satisfied,

$$(1)\quad \int_{-\infty}^{\infty} e^{pix}\,dx\,P\int_0^{\infty}\frac{\phi(y)\,dy}{x-y} = \pi i\int_0^{\infty} e^{piy}\,\phi(y)\,dy.$$

It can be shown (I omit the proof) that this equation certainly holds if $\int_0^{\infty}\phi(y)\,dy$ is absolutely convergent. If, *e.g.*,

$$\phi(y) = \frac{1}{1+y^2},$$

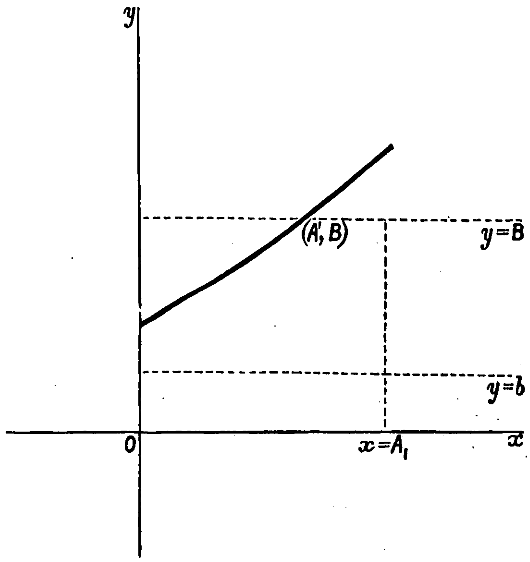

FIG. 4.

gives
$$\int_0^{\infty}\frac{\cos px\log x}{1+x^2}\,dx = \tfrac{1}{4}\left\{e^{p}\,li\,e^{-p} - e^{-p}\,li\,e^{p}\right\}.$$

But (1) also may be true if $\int_0^{\infty}\phi(y)\,dy$ is not absolutely convergent. If, *e.g.*,

$$\phi(y) = \frac{1}{y+a}\quad (a>0),$$

it gives $\displaystyle\int_0^{\infty}\log\left(\frac{x}{a}\right)\frac{\cos px}{x^2-a^2}\,dx = -\frac{\pi}{2a}\left\{\left(\frac{\pi}{2}-Si\,pa\right)\cos pa - Ci\,pa\sin pa\right\}.$

35. Moreover (1) is only a particular case of the more general formula

$$(2)\quad \int_{-\infty}^{\infty} e^{pix}\,dx\,P\int_b^{\infty}\frac{\phi(y)}{x-\theta(y)}\,dy = \pi i\int_b^{\infty}\phi(y)\,e^{pi\theta(y)}\,dy,$$

which holds when $\theta(y)$ satisfies certain conditions.

If, *e.g.*, $\quad b=-\infty,\quad \phi(y) = \dfrac{1}{1+y^2},\quad \theta(y) = ay-\dfrac{b}{y}\quad (a,\,b>0),$

(2) becomes, after a little reduction,

$$\int_0^{\infty}\cos\left(ay-\frac{b}{y}\right)\frac{dy}{1+y^2} = \tfrac{1}{2}\pi e^{-a-b}.$$

36. There are two more questions which we must discuss in order to complete this series of investigations. In the first place, when two singular curves intersect within the field of integration, the formula

$$(1) \quad \int_a^A P \int_b^B = \int_b^B P \int_a^A$$

generally ceases to be true. If, for instance,

$$f(x, y) = \frac{\psi(x, y)}{\lambda(x, y)\,\mu(x, y)},$$

where ψ is a function with continuous derivates, and $\lambda = 0$, $\mu = 0$ are two curves which satisfy the conditions of 7, except that they intersect simply at the one point (α, β), the difference between the two sides of (1) will be

$$\frac{2\pi^2\psi(\alpha, \beta)}{\dfrac{\partial(\lambda, \mu)}{\partial(\alpha, \beta)}}.$$

We shall have to discuss this case, and some other similar cases in which the corresponding "correction" or "residue" can be found.

In the second place, we must attempt to extend the theorems of the latter part of this paper to the case in which not only are the limits infinite, but the singular curves infinite in number.

Types of Perpetuants. *By* J. H. Grace.

Received and read June 12th, 1902.

1. I propose to apply the symbolical method of Aronhold directly to the discovery of the irreducible system of covariants of an indefinite number of binary forms of infinite order.

Suppose the forms are $a_x^n, b_x^n, c_x^n, \ldots,$

when n is indefinitely large; then the problem is to evolve a system of forms of the type

$$(ab)^\lambda (bc)^\mu (cd)^\nu \ldots a_x^p b_x^q \ldots$$

in terms of which all others can be expressed.