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THE NECESSARY POSTULATES OF GEOMETRY.

An Essay on the Foundations of Geometry. By Bertrand A. W. Russell, M.A. Demy 8vo. Pp. xvi + 201. (Cambridge: at the University Press, 1897.)

THE title of this essay suggests a number of distinct problems. We may ask what the postulates of geometry are, or we may seek the source of our knowledge of them; in the latter inquiry, again, we may set out to discover how the fundamental geometrical notions grew up, or it may be our object to ascertain how we can have certainty concerning them. Mr. Russell's essay deals with the last of these questions. It is, on the one hand, a criticism of existing theories of geometry, and, on the other hand, it is constructive, and aims at formulating a new philosophical theory of the foundations of the science.

An abstract geometry, logically arranged, would start from a small number of definitions and postulates, and would proceed deductively. In the process there would occur places in the argument where a choice would be possible among different hypotheses, and at such places ambiguity would be removed by the introduction of fresh postulates. There would thus be different orders of postulates, some being required in order that there might be any geometry at all, and others being adapted to make geometry applicable to the formulation of experience. The problem of separating the postulates into such classes is the problem of transcendental geometry, or, as the author calls it, *Metageometry.*

Mr. Russell gives in his first chapter an outline of the history of metageometry. He shows how it began with attempts to deduce Euclid's parallel axiom from the remaining axioms, and how this attempt issued in the construction of logically consistent geometries which did not adopt the axiom; he describes how Riemann and Helmholtz attempted to classify geometrical axioms as successive determinations of space considered as a particular example of a more general class-conception—that of a manifold or numerical aggregate, and here he does not omit to summarise the extremely important results obtained by Lie in modifying and completing Helmholtz's investigation; lastly he explains how Cayley and Klein connected metageometry with projective geometry, and here he incidentally gives an account of what projective geometry is, and of its independence of the notion of measurement, a notion which was fundamental in Riemann and Helmholtz's methods. The chapter is partly historical and partly critical. It contains *inter alia* an answer to Cayley's challenge demanding that philosophy should either take account of the use of imaginaries in analytical geometry, or show that it has a right to disregard it (pp. 41-46).

The second chapter contains a criticism of philosophical theories of geometry propounded by Kant, Riemann, Helmholtz, Erdmann, Lotze, and others.

In the third chapter we have a discussion of the question what postulates are necessary in order that there may be a geometry at all. The same result is arrived at whether the subject is considered from the projective or the metrical point of view; it is that the necessary postulates are those of homogeneity, and continuity of space, and the existence of the straight line as a unique figure determined by two points. Thus the most general possible geometry includes Euclidean geometry, the hyperbolic geometry of Lobatschewsky, the spherical geometry of Riemann, and the elliptic geometry of Klein, but besides these there is no other. The postulates necessary to this general geometry are declared to be *à priori* axioms, while the parallel axiom and the axiom of three dimensions are found to be empirical.

The fourth chapter deals with some difficulties met with in the previous chapter, and traces some of the philosophical consequences of the theory proposed.

Mathematicians will turn with most interest to Chapter iii., to see what Mr. Russell lays down as the essential postulates of geometry, and how he establishes his conclusions. The chapter is divided into two sections, dealing respectively with the "Axioms of Projective Geometry," and the "Axioms of Metrical Geometry." In projective geometry, as the author points out, the notions of the point, straight line, and plane are presupposed. Technically, the subject starts from these notions, and determines by the methods of projection and section what figures are equivalent to a given figure. Philosophically, the subject has a wider aim, consisting in the determination of all figures which cannot be distinguished by their internal relations when quantity is excluded (p. 133). The kernel of the argument consists in the identification of projective equivalence with qualitative similarity. The author attempts to prove that a *form of externality* (a notion essential to the knowledge of a world of diverse and inter-related things) must possess precisely the properties attributed to space in projective geometry, these properties including homogeneity, and continuity, and the possibility of the straight line, or in other words of a unique figure determined by two points. He seeks, in fact, to deduce these properties of the form from the relativity of position. Without wishing to impugn the correctness of the deduction, or to deny the legitimacy of the conclusion, we cannot help thinking the argument obscure. This is especially the case in all that concerns the notion of the *point.* Thus, in speaking of the infinite divisibility of the form of externality (p. 138) he says:

> "The relation between any two things is infinitely divisible, and may be regarded, consequently, as made up of an infinite number of the would-be elements of our form, or again as the sum of two relations of externality."

He finds in the notion of the *point* "a self-contradictory result of hypostatizing the form of externality." This difficulty he recurs to again and again. Would it be presumptuous for a mere mathematician to suggest that this alleged contradiction may arise from the adoption of an antiquated mode of statement? We are told (p. 188) that the difficulty is extremely ancient. Is it not safe to say that the ancient philosophers had not firmly grasped and completely analysed the concept of the *mathematical continuum?* Mr. Russell says (p. 189):

"Whatever can be divided, and has parts, possesses some thinghood, and must, therefore, contain two ultimate units, the whole namely, and the smallest element possessing thinghood."

The *mathematical continuum* contains no "smallest element," and there is, accordingly, no necessity for a thing which can be divided, and which has parts, to contain such an element. This remark may perhaps offer the key for the solution of the problem set by Mr. Russell, the problem namely of determining the properties of a *form of externality*. It is conceivable that, in arriving at the axioms of projective geometry as constituting a statement of these properties, he has assumed the solution of a problem in the *theory of manifolds* just as Helmholtz, in arriving at the axiom of constant *space-curvature* as necessary to congruence, assumed the solution of a problem in the *theory of groups*. In the latter case the weapon needed to attack the problem was forged at a much later date by Lie. In the case of Mr. Russell's problem the appropriate engine of discovery is still undeveloped, the mathematics of the manifold being at present limited to numerical aggregates. No one has yet done for the science of space what Dedekind did for the science of number.

Mr. Russell is happier in his treatment of the axioms of metrical geometry, and he has done real service to mathematics in pointing out the essential weakness of the Riemann-Helmholtz method. This method started from the consideration of space as a numerical aggregate, whose points are determined by coordinates, and then sought for the condition of the possibility of measurement. This condition was found in the uniformity of the measure of space-curvature, and it was shown, on the one hand, to imply the possibility of the straight line, and, on the other, to be equivalent to the statement that figures which can be brought to congruence are equal. The argument, as Mr. Russell shows, really involved a vicious circle. For space can be regarded as a numerical aggregate only if we have the means of assigning to points coordinates which have some spatial import, and coordinates which have such import presuppose measurement. The conclusion arrived at by Mr. Russell is that the essential postulate of metrical geometry is the *axiom of free mobility*, or the assertion of the possibility of equal figures in different places, and he has shown that the denial of this axiom would lead to logical and philosophical absurdities. In this connection it is only fair to Riemann to remember that his essay "Ueber die Hypothese, welche der Geometrie zu Grunde liegen" remained unpublished until after his death, a fact which points to the belief that he was not satisfied with it.

Leaving to philosophers by profession the task of appreciating and criticising Mr. Russell's philosophy of space, we may attempt to estimate the value of his book for mathematics. It has already been pointed out that in his criticism of Riemann and Helmholtz he has brought forward considerations which are mathematically important, and this is not the only place where he has had occasion to point to examples of the special philosophical vice of the mathematician, the tendency namely to mistake the sign for the thing signified (*cf.* Couturat "De l'Infini mathématique," p. 331). To mathematicians also his book should be interesting on account of its acute and novel treatment of familiar topics: thus—projective coordinates are numbers arbitrarily but systematically assigned to points of space "like the numbers of houses in a street" (p. 119). The ambiguity in the definition of distance, which is unavoidable on projective principles, does not show that distance is ambiguous, but that projective methods cannot adequately deal with distance (p. 35). The distinction between real and imaginary points is the distinction between quantities to which points correspond and quantities to which no points correspond (p. 44). The book is throughout well written, and is for the most part free from obscurity, and it may be recommended to all who wish to have clear ideas on matters of fundamental importance in mathematics.

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OUR BOOK SHELF.

A Bibliography of Gilbert White, the Natural Historian and Antiquarian of Selborne. Ey Edward A. Martin, F.G.S. Pp. xiii + 274. (Westminster: The Roxburghe Press, 1897.)

THERE are many places in England prettier than the little Hampshire village of Selborne, but none of them are so full of interest to the outdoor naturalist as the home of Gilbert White. Though more than a century has passed away since the simple student of nature's ways in the sleepy hollow of Selborne first gave the world the benefit of his observations and impressions, the book in which these notes are published is as fresh now as ever it was. The reason for this is, it seems to the writer, that Gilbert White was usually content to record facts as he found them, and he did not regard nature from the point of view of a pre-conceived theory. Accurate observations of natural objects and phenomena live for ever, but the explanation of such facts must alter from time to time as wider knowledge of the laws of nature is obtained.

The success of White's "Selborne" has had two unfortunate effects: it has made every country clergyman who can distinguish a martin from a swallow think that he is a Gilbert White, and it has caused the literary world to be deluged with so-called popular natural history works, which are often more remarkable for thoughts about nothing than for observations of something. We can, however, forgive the authors of such rhapsodies for inflicting their musings upon a busy world, because of the real naturalists which White's "Selborne" has created.

How large and widespread is the public to which the book appeals may be seen by the volume before us. Mr. Martin has found no less than seventy-three separate editions of our natural history classic; so the aggregate number of volumes published must be very great. The features of each of these editions are described in detail; hence Selbornites are now provided with interesting particulars of the various volumes which have refreshed the mind and administered to the intellectual enjoyment of thousands of nature-lovers the world over. Mr. Martin has not, however, confined his work to a mere list of editions of the "Natural History of Selborne"; he describes the naturalist himself and the main facts of his life, points out some of the chief observations and discoveries, gives a chapter on the village of Selborne, and devotes another to White's old house, "The Wakes." The work is thus more than a bibliography; it is a guide to the study of Gilbert White and his natural history, and as such will be prized by many of his disciples.

Reference is made on p. 71 to a suggestion of White's that entomology required some "neat plates" for its advancement, and it is stated that the idea has been carried out by the Science and Art Department. Surely there is a mistake here.