ON QUINTIC SURFACES HAVING A TACNODAL CONIC.

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1. In this communication I shall employ the following notation.

The symbols $(\alpha, \beta, \gamma, \delta)$ will be used to denote quadriplanar coordinates referred to a tetrahedron of reference $ABCD$; the symbols u_n, U_n will (unless otherwise stated) denote ternary quantics of (β, γ, δ); whilst such symbols as v_n, w_n, and the like will denote binary quantics of (γ, δ).

2. If S, S' denote two quadric surfaces, and U, V, W surfaces of degree $n-4$, the equation

(1) $$S^2 U + S S' V + S'^2 W = 0$$

represents a surface of the n^{th} degree having a nodal curve, which is the twisted quartic formed by the intersection of the quadrics S and S'. When the quadrics have a common generator, the nodal curve degrades into a straight line and a twisted cubic. I shall not however discuss these special nodal curves, but shall consider the case in which S' consists of a pair of coincident planes. Putting $S' = \alpha^2$, (1) becomes

(2) $$S^2 U + S V \alpha^2 + W \alpha^4 = 0$$

which represents a surface of the n^{th} degree having a tacnodal conic.

The plane α, which contains the tacnodal conic, intersects the surface in this conic twice repeated and in the residual curve $\alpha = 0$, $U = 0$; and in a communication shortly to be published in the Quarterly Journal, I have shown that the points of intersection of the tacnodal conic and the residual curve are singular points of the nature of cubic nodes, and that the nodal cone consists of a quadric cone and a plane touching the cone. In other words, the nodal cone is an improper cubic cone having a tacnodal generator. The number of these singular points is obviously $2n - 8$. A quintic surface has also four singular points, where the tacnode changes into a rhamphoid [1]) cusp (point de rebroussement de seconde espèce). The reader can easily prove the first of these theorems by making the vertex B of the tetrahedron of reference coincide with these singular points; and for a quintic surface this result will be proved in the course of the work.

[1]) ῥάμφος = un becco.

3. When U, V and W are planes, equation (2) represents a quintic surface having a tacnodal conic, and I shall proceed to investigate some of the properties of these surfaces.

Let p and q be the points of contact of a double tangent plane; then this plane will cut the tacnodal conic in two points P and Q, and the section of the surface by the plane will in general consist of a quintic curve having tacnodes at P and Q and ordinary nodes at p and q. But since the plane has one degree of freedom, the equation of the section will contain an arbitrary parameter θ, whose value depends upon the position of the plane. We may therefore determine θ as follows:

I. A conic can always be described touching the section at P and Q and passing through one of the points p or q; but if θ be determined so that the conic passes through p and q, the section will degrade into cubic curve and a conic, which touches the cubic at P and Q intersects it at p and q.

II. Let θ be determined so that P, p and q lie in the same straight line; then it would at first sight appear that the section consists of a quartic curve having a tacnode at Q and a straight line, which touches the quartic at p and intersects it at p and q. It will however be shown that the line in question passes through one of the singular points at which there is a cubic node, and that the section consists of the line Ppq, and a quartic curve having a tacnode at Q and an ordinary node at P.

III. Let θ be determined so that the plane is a triple tangent plane; then the section will consist of a cubic curve and a pair of straight lines, which touch the cubic at P and Q, intersect it at p and q and intersect one another at a point r.

We shall show that there are altogether nineteen lines lying in the surface.

4. *Through every point on the tacnodal conic two planes can be drawn, which intersect the surface in a conic and a cubic curve.*

It follows from (2) that the planes U and W intersect the surface in a conic and a cubic curve, but these are special planes. The plane W intersects the tacnodal conic in the two points which we shall choose as the vertices B and C of the tetrahedron of reference, whilst D will be any point on the tacnodal conic. Then

$$W = L\alpha + M\delta$$

also let

$$\Sigma = S - k\alpha^2,$$

where k is a constant. Substituting in (2), the equation of the surface may be written in the form

$$(3) \qquad \Sigma(\Sigma U + 2kU\alpha^2 + V\alpha^2) + \alpha^4(k^2 U + kV + L\alpha + M\delta) = 0$$

which shows that the plane

$$(4) \qquad k^2 U + kV + L\alpha + M\delta = 0$$

intersects the surface in a conic and a cubic curve, which touch one another at the two points where (4) intersects the tacnodal conic. If D be one of these points, the coefficient of δ in (4) must be zero, which furnishes a quadratic equation for determining k.

5. We shall now find the condition that the line AB shall lie in the surface.

We have shown that every line lying in the surface must pass through the tacnodal conic; we shall therefore choose the tetrahedron of reference so that A is the other point where AB cuts the quadric S, whilst C and D are any points on the tacnodal conic. Then the equation of the surface may be written in the form

$$(5) \qquad (\alpha u_1 + \Omega)^2 (L\alpha + U_1) + (\alpha u_1 + \Omega)(M\alpha + V_1)\alpha^2 + W_1 \alpha^4 = 0,$$

where

$$(6) \qquad u_1 = P\beta + Q\gamma + R\delta,$$

$$(7) \qquad \Omega = \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma,$$

and U_1, V_1, W_1 are linear fucntions of $(\beta, \gamma, \delta,)$. Equation (5), when written out at full length becomes

$$(8) \quad \begin{cases} \alpha^4(Mu_1 + W_1) + \alpha^3(M\Omega + V_1 u_1 + Lu_1^2) + \alpha^2(V_1\Omega + U_1 u_1^2 + 2L\Omega u_1) \\ \qquad + \alpha\Omega(L\Omega + 2u_1 U_1) + \Omega^2 U_1 = 0. \end{cases}$$

If AB is a line lying in the surface, (8) must vanish when $\gamma = \delta = 0$; hence if v_1, w_1, τ_1 be linear functions of (γ, δ), it follows that

$$(9) \quad \begin{cases} W_1 = -MP\beta + w_1, \\ V_1 = -LP\beta + v_1, \\ U_1 u_1 = 0, \quad \text{when} \quad \gamma = \delta = 0. \end{cases}$$

The last of (9) may be satisfied by making

$$(10) \qquad U_1 = \tau_1, \quad \text{or} \quad P = 0.$$

6. We shall now prove that the first of (10) furnishes the following theorem:

Through each of the points in which the residual line intersects the tacnodal conic, eight lines can be drawn which lie in the surface and are generators of a quadric cone, whose vertex is the point in question.

Equation (5) may be written in the form

$$(11) \quad \begin{cases} [\beta(P\alpha + \mu\delta + \nu\gamma) + \Sigma]^2(L\alpha + \tau_1) + [\beta(P\alpha + \mu\delta + \nu\gamma) + \Sigma](M\alpha - LP\beta + v_1)\alpha^2 \\ \qquad + (-MP\beta + w_1)\alpha^4 = 0, \end{cases}$$

where

$$\Sigma = \alpha(Q\gamma + R\delta) + \lambda\gamma\delta.$$

Arranging in powers of β (11) becomes

$$(12) \quad \begin{cases} \beta^2(P\alpha + \mu\delta + \nu\gamma)[(\mu\delta + \nu\gamma)(L\alpha + \tau_1) + P\alpha\tau_1] \\ + \beta\{(P\alpha + \mu\delta + \nu\gamma)[2\Sigma(L\alpha + \tau_1) + \alpha^2 v_1] + \alpha^2[M\alpha(\mu\delta + \nu\gamma) - LP\Sigma]\} \\ \qquad + \Sigma^2(L\alpha + \tau_1) + \Sigma(M\alpha + v_1)\alpha^2 + w_1\alpha^4 = 0. \end{cases}$$

The form of (11) shows that B is one of the points where the residual line $\alpha = 0$, $\tau_1 = 0$ intersects the tacnodal conic; whilst (12) shows that this point is a cubic node, whose nodal cone consists of a quadric cone and a plane which touches it. Also the line AB is a common generator of this quadric cone and the quartic cone which is the coefficient of β in (12); and since these cones have eight common generators, each of them must be a line lying in the surface. Similar considerations apply to the other point in which the residual line intersects the tacnodal conic; accordingly there are altogether sixteen lines lying in the surface, which together with the residual line make seventeen.

In (11) the point C may be any point whatever on the tacnodal conic; hence $\delta = 0$ is any arbitrary plane through AB. Putting $\delta = 0$ it will be found that the section of the surface consists of the line AB and a quartic curve having and ordinary node at B and a tacnode at C; and that the equation of the tacnodal tangent at C is $\nu\beta + Q\alpha = 0$.

The form of (5) shows that the plane $L\alpha + V_1 = 0$ touches the surface at every point of the residual line; and if E be the other point in which this line cuts the tacnodal conic, the section of the surface by this plane consists of the residual line twice repeated and a cubic curve which is touched by the planes ABD and AED at B and E.

7. Returning to equations (9), we must now consider the second condition, and we shall prove that:

Through each of the points where the plane W intersects the surface, a straight line can be drawn which lies in the surface. These two straight lines lie in the same plane, and the section of the surface by the plane consists of the straight lines and a cubic curve touching them at the points in question.

Equation (8) shows that when $P = 0$, AB will be a line lying in the surface provided W_1 is a linear function of (γ, δ) alone: but U_1 and V_1 may be arbitrary linear functions of (β, γ, δ). Let B and C be the points where the plane W_1 intersects the tacnodal conic; then $W_1 = \delta$; also let

$$(13)\qquad \begin{cases} U_1 = l\beta + m\gamma + n\delta, \\ V_1 = F\beta + G\gamma + H\delta, \end{cases}$$

then (5) becomes

$$(14)\qquad \begin{cases} [\beta(\mu\delta + \nu\gamma) + \Sigma]^2(L\alpha + l\beta + m\gamma + n\delta) \\ + [\beta(\mu\delta + \nu\gamma) + \Sigma](M\alpha + F\beta + G\gamma + H\delta) + \alpha^4\delta = 0. \end{cases}$$

Let $\delta = 0$, then (14) becomes

$$(15)\quad \gamma^2(\nu\beta + Q\alpha)^2(L\alpha + l\beta + m\gamma) + \gamma(\nu\beta + Q\alpha)(M\alpha + F\beta + G\gamma)\alpha^2 = 0$$

which shows that the section consists of the straight lines $\gamma = 0$, or AB; and $\nu\beta + Q\alpha = 0$ which passes through C; and the cubic curve

$$(16)\qquad \gamma(\nu\beta + Q\alpha)(L\alpha + l\beta + m\gamma) + (M\alpha + F\beta + G\gamma)\alpha^2 = 0.$$

The form of (16) shows that the lines

$$\gamma = 0, \qquad \nu\beta + Q\alpha = 0$$

are the tangents to the cubic at B and C.

8. *The four points where the quadric cone $V^2 = 4UW$ intersects the tacnodal conic are the points where the tacnode changes into a rhamphoid cusp.*

In equation (2) let

(17) $$U = Q^2\beta + U_1, \qquad V = Q'\beta + V_1, \qquad W = q^2\beta + W_1$$

where U_1, V_1, W_1 are planes passing through B. Then if B be one of the points in which the cone

(18) $$V^2 = 4UW$$

intersects the tacnodal conic, we must have $Q' = 2Qq$; and (2) becomes

(19) $$S^2U_1 + SV_1\alpha^2 + W_1\alpha^4 + \beta(QS + q\alpha^2)^2 = 0$$

Now the plane ABC may be any arbitrary plane through B; hence putting $\delta = 0$ in (19), and transforming birationally by means of the equations

$$\frac{\beta}{\alpha'\gamma'} = \frac{\alpha}{\alpha'\beta'} = \frac{\gamma}{\gamma'\beta'}$$

it will be found that the transformed curve will have a cusp at a certain point on AC. This shows that the original curve has a ramphoid cusp at B.

Equation (2) may also be written in the form

$$(2SV + V\alpha^2)^2 + \alpha^4(4UW - V^2) = 0$$

which shows that the cone (18) is a double tangent cone.

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