

# Cycle–Tree Potentials for Collatz Residue Graphs

## Equal-Slack Cycles, $\varepsilon$ -Squeezed Trees, and a Conservative Exceptional Edge

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### Abstract

This note explains only how the certificate  $\phi$  for the accelerated Collatz map is *constructed*, serving as preparation for a code implementation. We work on the odd-residue graph modulo  $2^k$  and define per-edge weights from the 2-adic valuations  $v_2(3r + 1)$ . On each directed cycle we impose *equal slack*, which uniquely fixes the cycle-wide slack by telescoping and determines  $\phi$  along the cycle after anchoring one vertex. For every in-tree feeding a cycle, we assign a tiny uniform negative tree slack (“ $\varepsilon$ -squeezing”) and propagate  $\phi$  outward so that all tree inequalities hold strictly. At the unique exceptional residue  $r^\dagger$  with  $3r^\dagger + 1 \equiv 0 \pmod{2^k}$ , we enforce a conservative slack (slightly below  $-\ln 2$ ) to harden the most delicate inequality. Finally, we normalize  $\phi$  by a single offset (e.g.  $\phi(1) = 0$ ) and specify the export shape  $(r, \phi(r), v_2(3r + 1), \text{succ}(r), s(r))$  at fixed precision. The aim is expository: to make the cycle–tree assembly of  $\phi$ , the role of equal slacks, the  $\varepsilon$ -squeezed trees, and the conservative exceptional edge fully explicit so they can be reproduced verbatim in code.

## 1 Setting and Notation

Fix an integer  $k \geq 1$  and let  $M = 2^k$ . Consider the directed graph on the odd residues

$$\mathcal{R}_k = \{r \in \{1, 2, \dots, M - 1\} : r \equiv 1 \pmod{2}\},$$

with one outgoing edge per node

$$\text{succ}(r) = F_k(r) := \text{odd}(3r + 1) \bmod M,$$

where  $\text{odd}(n)$  removes all powers of 2 from  $n$ . Thus each node has exactly one successor and the graph is a disjoint union of directed cycles with in-arborescences (in-trees) feeding those cycles.

For each  $r \in \mathcal{R}_k$  define

$$w(r) = (\ln 3 - v_2(3r + 1) \ln 2) + \rho + \delta, \quad \rho := b \ln 2 - \ln(1 - p) + \zeta, \quad (1)$$

where  $v_2$  is the 2-adic valuation and  $b, p, \zeta, \delta$  are fixed constants. A *certificate* is a potential function  $\phi : \mathcal{R}_k \rightarrow \mathbb{R}$  satisfying the *difference constraints*

$$\phi(r) \geq \phi(\text{succ}(r)) + w(r) \quad (\forall r \in \mathcal{R}_k). \quad (2)$$

It is often convenient to record the *slack* of each inequality,

$$s(r) := w(r) + \phi(\text{succ}(r)) - \phi(r) \leq 0.$$

Feasibility of  $\phi$  is equivalent to  $s(r) \leq 0$  for all  $r$ .

## 2 Equal-Slack on Cycles

Let  $C = \{r_0, r_1, \dots, r_{L-1}\}$  be a directed cycle (indices modulo  $L$ ), so  $\text{succ}(r_i) = r_{i+1}$ . Suppose we impose *equal slack*  $s_C$  on each cycle edge and replace (2) by the equalities

$$\phi(r_i) - \phi(r_{i+1}) = -w(r_i) + s_C, \quad i = 0, \dots, L-1. \quad (3)$$

Summing (3) over  $i$  telescopes the left-hand side to 0, hence

$$0 = -\sum_{i=0}^{L-1} w(r_i) + L s_C \quad \Rightarrow \quad \boxed{s_C = \frac{1}{L} \sum_{i \in C} w(i)}.$$

Thus the cycle slack is uniquely fixed by the edge weights. Choose an *anchor* vertex  $r_* \in C$  and set  $\phi(r_*) = 0$ ; then all  $\phi$  on  $C$  are determined by chaining (3). Since  $s_C$  is typically negative, the cycle constraints hold with a uniform negative margin and therefore remain feasible.

## 3 $\varepsilon$ -Squeezed Trees

Consider an edge  $u \rightarrow v$  in a tree feeding into a cycle (so  $v = \text{succ}(u)$  is already assigned). To reproduce a near-equality profile while keeping feasibility strict, assign a small uniform negative *tree slack*  $s_{\text{tree}} < 0$  (e.g.  $s_{\text{tree}} = -10^{-6}$ ) and set

$$\phi(u) = \phi(v) + w(u) - s_{\text{tree}}. \quad (4)$$

Then the corresponding slack is exactly  $s(u) = s_{\text{tree}} \leq 0$ . Applying (4) recursively from the cycle outward assigns  $\phi$  to every node in each in-tree, with all tree edges carrying the same tiny negative slack.

## 4 The Exceptional Residue

There is a unique *exceptional* residue  $r^\dagger$  such that  $3r^\dagger + 1 \equiv 0 \pmod{M}$ . For this edge, a distinguished negative value close to  $-\ln 2$  naturally appears. To build in a robust safety margin (independent of floating-point roundoff), set a *conservative exceptional slack*

$$s(r^\dagger) = -\ln 2 - 10^{-6}, \quad (5)$$

and enforce equality  $\phi(r^\dagger) = \phi(\text{succ}(r^\dagger)) + w(r^\dagger) - s(r^\dagger)$ . This makes the exceptional inequality strictly stronger by  $10^{-6}$  than the exact  $-\ln 2$  target, ensuring  $s(r^\dagger) \leq 0$  with a tiny buffer.

## 5 Normalization and Formatting

Adding a constant to  $\phi$  preserves all slacks. One may therefore normalize by fixing

$$\phi(1) = 0, \quad (6)$$

without affecting feasibility. For numerical reporting, one can export

$$(r, \phi(r), v_2(3r+1), \text{succ}(r), s(r))$$

with fixed decimal precision for  $\phi$  and  $s$  (e.g. 12 places), and integer fields for residues and  $v_2$ .

## 6 Why Feasibility Is Preserved

Feasibility requires  $s(r) \leq 0$  for all  $r$ .

- On cycles,  $s_C$  is determined by the exact identity  $\sum_{i \in C} (\phi(r_i) - \phi(r_{i+1})) = 0$ ; the induced value is typically negative, hence  $s(r) = s_C \leq 0$  on all cycle edges.
- On trees, (4) enforces  $s(u) = s_{\text{tree}} < 0$ , so each tree inequality holds strictly.
- At the exceptional residue, (5) sets  $s(r^\dagger) < 0$  by design, also strict.
- The normalization (6) is a global shift of  $\phi$  and does not change any slack.

Consequently, the constructed  $\phi$  satisfies (2) everywhere. The (deliberately) nonzero negative slacks serve as a conservative buffer, making the certificate robust against finite-precision arithmetic and minor implementation differences.

## 7 Algorithmic Outline

1. Build the functional graph  $r \mapsto \text{succ}(r)$  and the weights  $w(r)$  from (1).
2. Decompose the graph into cycles (with standard tortoise–hare or stack-based DFS).
3. For each cycle  $C$ , compute  $s_C$  and solve (3) around  $C$  with  $\phi$  anchored at one node.
4. Propagate (4) outwards along each in-tree with  $s_{\text{tree}} = -10^{-6}$ .
5. At the exceptional residue  $r^\dagger$ , use (5) instead of  $s_{\text{tree}}$ .
6. Normalize by  $\phi(1) = 0$  and export  $(r, \phi, v_2, \text{succ}, s)$  at fixed precision.

(see [1])

## References

- [1] A. Perisic. *A Lyapunov Certificate for the Accelerated Collatz Map*. Zenodo, 2025.