*On a Formula for the Multiplication of four Theta Functions.**

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ART. 1.—In a letter addressed to M. Hermite, and dated Berlin, Aug. 6, 1845,† Jacobi says:—

"Dans mes Leçons universitaires de Köuigsberg, moi aussi j'ai eu coutûme de partir des fonctions θ. Dans ces Leçons, en multipliant quatre séries $\sum_{i\,-\infty}^{+\infty} \epsilon^{-(ax+bi)^2}$ pour différentes valeurs de x, et en transformant les exposants par la formule

$$i^2+i'^2+i''^2+i'''^2 = \left(\frac{i+i'+i''+i'''}{2}\right)^2+\left(\frac{i+i'-i''-i'''}{2}\right)^2+\ldots$$

j'ai obtenu tout de suite une formule de laquelle découlent, comme cas particuliers et sans le moindre calcul, les expressions fractionnaires des fonctions elliptiques, les théorèmes sur l'addition des trois espèces, et plusieurs centaines de formules intéressantes auxquelles on ne saurait arriver que par un calcul algébrique fatiguant."

The formula to which this passage refers has not (it would seem) been given by any writer on elliptic functions. The object of the present paper is to enunciate and demonstrate it; and to justify what Jacobi says of it by showing that many of the fundamental formulæ of the theory of elliptic functions are either particular cases of it, or corollaries from it. For the sake of symmetry, however, it is convenient to employ the arithmetical equality

$$i^2+i'^2+i''^2+i'''^2 = \left(\frac{-i+i'+i''+i'''}{2}\right)^2+\left(\frac{i-i'+i''+i'''}{2}\right)^2 + \left(\frac{i+i'-i''+i'''}{2}\right)^2+\left(\frac{i+i'+i''-i'''}{2}\right)^2,$$

instead of that indicated by Jacobi.

* Read May 21, 1866. Professor De Morgan in the Chair.

† Jacobi, *Mathematische Werke*, Vol. I., p. 358.

ART. 2.—The Theta functions being defined by the equation

$$(1)\qquad \theta_{\mu,\mu'}(x) = \sum_{n=-\infty}^{n=+\infty} (-1)^{n\mu'} e^{i\pi\left[\frac{1}{4}(2n+\mu)^2\omega + (2n+\mu)\frac{x}{a}\right]},$$

(in which μ and μ' are given integral numbers, a any constant, ω an imaginary constant having for the coefficient of i* in its imaginary part a quantity different from zero and positive,) or, if $q = \epsilon^{i\pi\omega}$, $v = \epsilon^{\frac{i\pi}{a}}$, by the equation

$$(2)\qquad \theta_{\mu,\mu'}(x) = \sum_{n=-\infty}^{n=+\infty} (-1)^{n\mu'} q^{\frac{1}{4}(2n+\mu)^2} v^{(2n+\mu)x},$$

the formula for the multiplication of four Theta functions is

$$(3)\qquad 2\theta_{\mu_1,\mu'_1}(x_1)\times\theta_{\mu_2,\mu'_2}(x_2)\times\theta_{\mu_3,\mu'_3}(x_3)\times\theta_{\mu_4,\mu'_4}(x_4)$$

$$= \theta_{\sigma-\mu_1,\sigma'-\mu'_1}(s-x_1)\,\theta_{\sigma-\mu_2,\sigma'-\mu'_2}(s-x_2)\,\theta_{\sigma-\mu_3,\sigma'-\mu'_3}(s-x_3)\,\theta_{\sigma-\mu_4,\sigma'-\mu'_4}(s-x_4)$$

$$+\theta_{\sigma-\mu_1,\sigma'-\mu'_1+1}(s-x_1)\,\theta_{\sigma-\mu_2,\sigma'-\mu'_2+1}(s-x_2)\,\theta_{\sigma-\mu_3,\sigma'-\mu'_3+1}(s-x_3)\,\theta_{\sigma-\mu_4,\sigma'-\mu'_4+1}(s-x_4)$$

$$+(-1)^{\sigma'}\theta_{\sigma-\mu_1+1,\sigma'-\mu'_1}(s-x_1)\,\theta_{\sigma-\mu_2+1,\sigma'-\mu'_2}(s-x_2)\,\theta_{\sigma-\mu_3+1,\sigma'-\mu'_3}(s-x_3)$$
$$\times\,\theta_{\sigma-\mu_4+1,\sigma'-\mu'_4}(s-x_4)$$

$$+(-1)^{\sigma'+1}\theta_{\sigma-\mu_1+1,\sigma'-\mu'_1+1}(s-x_1)\,\theta_{\sigma-\mu_2+1,\sigma'-\mu'_2+1}(s-x_2)\,\theta_{\sigma-\mu_3+1,\sigma'-\mu'_3+1}(s-x_3)$$
$$\times\,\theta_{\sigma-\mu_4+1,\sigma'-\mu'_4+1}(s-x_4),$$

where $2s = x_1+x_2+x_3+x_4$, $2\sigma = \mu_1+\mu_2+\mu_3+\mu_4$, $2\sigma' = \mu'_1+\mu'_2+\mu'_3+\mu'_4$, the sums $\mu_1+\mu_2+\mu_3+\mu_4$ and $\mu'_1+\mu'_2+\mu'_3+\mu'_4$ being supposed even.

Its proof depends on the two identities

$$(4)\qquad \mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2 = (\sigma-\mu_1)^2+(\sigma-\mu_2)^2+(\sigma-\mu_3)^2+(\sigma-\mu_4)^2$$

$$(5)\qquad \mu_1\mu'_1+\mu_2\mu'_2+\mu_3\mu'_3+\mu_4\mu'_4 = (\sigma-\mu_1)(\sigma'-\mu'_1)+(\sigma-\mu_2)(\sigma'-\mu'_2)$$
$$+(\sigma-\mu_3)(\sigma'-\mu'_3)+(\sigma-\mu_4)(\sigma'-\mu'_4),$$

both of which admit of immediate verification.

Representing the product

$$\theta_{\mu_1,\mu'_1}(x_1)\,\theta_{\mu_2,\mu'_2}(x_2)\,\theta_{\mu_3,\mu'_3}(x_3)\,\theta_{\mu_4,\mu'_4}(x_4)$$

by Σ, we have, evidently,

$$(6)\qquad 2\Sigma = 2\Sigma(-1)^{\mu'_1n_1+\mu'_2n_2+\mu'_3n_3+\mu'_4n_4} \times q^{\frac{1}{4}[(2n_1+\mu_1)^2+(2n_2+\mu_2)^2+(2n_3+\mu_3)^2+(2n_4+\mu_4)^2]}$$
$$\times\, v^{(2n_1+\mu_1)x_1+(2n_2+\mu_2)x_2+(2n_3+\mu_3)x_3+(2n_4+\mu_4)x_4},$$

the sign of summation Σ extending to all integral values of n_1, n_2, n_3, n_4, from $-\infty$ to $+\infty$.

Let

$$(7)\qquad \begin{cases} N_1 = -n_1+n_2+n_3+n_4, \\ N_2 = n_1-n_2+n_3+n_4, \\ N_3 = n_1+n_2-n_3+n_4, \\ N_4 = n_1+n_2+n_3-n_4; \end{cases}$$

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* Here, and in the rest of this paper, i is used for $\sqrt{(-1)}$. The effect of the condition stated in the text is to render the analytical modulus of q inferior to unity, and thus ensure the convergence of the Theta series for all values of x real or imaginary.

so that, conversely,

$$(8)\qquad \begin{cases} 4n_1 = -N_1+N_2+N_3+N_4, \\ 4n_2 = N_1-N_2+N_3+N_4, \\ 4n_3 = N_1+N_2-N_3+N_4, \\ 4n_4 = N_1+N_2+N_3-N_4. \end{cases}$$

Transforming in (6) the index of q by the formula (4), and the indices of -1 and v by (5), we find

$$(9)\quad 2\Sigma = 2\Sigma\,(-1)^{\frac{1}{2}[N_1(\sigma'-\mu'_1)+N_2(\sigma'-\mu'_2)+N_3(\sigma'-\mu'_3)+N_4(\sigma'-\mu'_4)]}$$
$$\times\, q^{\frac{1}{4}[(N_1+\sigma-\mu_1)^2+(N_2+\sigma-\mu_2)^2+(N_3+\sigma-\mu_3)^2+(N_4+\sigma-\mu_4)^2]}$$
$$\times\, v^{(N_1+\sigma-\mu_1)(s-x_1)+(N_2+\sigma-\mu_2)(s-x_2)+(N_3+\sigma-\mu_3)(s-x_3)+(N_4+\sigma-\mu_4)(s-x_4)},$$

the sign of summation Σ extending to all values of N_1, N_2, N_3, N_4, defined by the equations (7); *i.e.*, to all integral values of N_1, N_2, N_3, N_4, from $-\infty$ to $+\infty$, which, substituted in the equations (8), give integral values to n_1, n_2, n_3, n_4. We have, therefore, to ascertain what values of N_1, N_2, N_3, N_4 satisfy this condition. In the first place, we see, from the equations (7), that the difference between any two of the four numbers N_1, N_2, N_3, N_4 is even; *i. e.*, that N_1, N_2, N_3, N_4 are all even or all uneven. (α) Let them be all even, and let $N_1=2\nu_1$, $N_2=2\nu_2$, $N_3=2\nu_3$, $N_4=2\nu_4$; substituting these values in the equations (8), we find that, in order to render n_1, n_2, n_3, n_4 integral, it is necessary and sufficient that $\nu_1+\nu_2+\nu_3+\nu_4$ should be even. (β) Let N_1, N_2, N_3, N_4 be all uneven, and let $N_1=2\nu_1+1$, $N_2=2\nu_2+1$, $N_3=2\nu_3+1$, $N_4=2\nu_4+1$; substituting as before, we find that, in order to render n_1, n_2, n_3, n_4 integral, it is necessary and sufficient that $\nu_1+\nu_2+\nu_3+\nu_4$ should be uneven. Separating the terms in (9), in which N_1, N_2, N_3, N_4 are all even, from those in which N_1, N_2, N_3, N_4 are all uneven, we obtain

$$(10)\quad 2\Sigma = 2\Sigma'\,(-1)^{\nu_1(\sigma'-\mu'_1)+\nu_2(\sigma'-\mu'_2)+\nu_3(\sigma'-\mu'_3)+\nu_4(\sigma'-\mu'_4)}$$
$$\times\, q^{\frac{1}{4}[(2\nu_1+\sigma-\mu_1)^2+(2\nu_2+\sigma-\mu_2)^2+(2\nu_3+\sigma-\mu_3)^2+(2\nu_4+\sigma-\mu_4)^2]}$$
$$\times\, v^{(2\nu_1+\sigma-\mu_1)(s-x_1)+(2\nu_2+\sigma-\mu_2)(s-x_2)+(2\nu_3+\sigma-\mu_3)(s-x_3)+(2\nu_4+\sigma-\mu_4)(s-x_4)}$$
$$+2\,(-1)^{\sigma'}\,\Sigma''\,(-1)^{\nu_1(\sigma'-\mu'_1)+\nu_2(\sigma'-\mu'_2)+\nu_3(\sigma'-\mu'_3)+\nu_4(\sigma'-\mu'_4)}$$
$$\times\, q^{\frac{1}{4}[(2\nu_1+\sigma-\mu_1+1)^2+(2\nu_2+\sigma-\mu_2+1)^2+(2\nu_3+\sigma-\mu_3+1)^2+(2\nu_4+\sigma-\mu_4+1)^2]}$$
$$\times\, v^{(2\nu_1+\sigma-\mu_1+1)(s-x_1)+(2\nu_2+\sigma-\mu_2+1)(s-x_2)+(2\nu_3+\sigma-\mu_3+1)(s-x_3)+(2\nu_4+\sigma-\mu_4+1)(s-x_4)},$$

the signs of summation Σ' and Σ'' referring to all values of ν_1, ν_2, ν_3, ν_4 from $-\infty$ to $+\infty$, for which the sum $\nu_1+\nu_2+\nu_3+\nu_4$ is even and uneven respectively. If, for a moment, we represent the general terms of the series Σ' and Σ'' by P and Q, we have, evidently,

$$2\Sigma'.\,P = \Sigma\,.\,P+\Sigma\,(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}P,\quad 2\Sigma''.\,Q = \Sigma\,.\,Q-\Sigma\,(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}Q,$$

the signs of summation Σ extending to all values of ν_1, ν_2, ν_3, ν_4 from $-\infty$ to $+\infty$ without any limitation. Substituting these values in

(10), we obtain, finally,

$$(11)\qquad 2\Sigma = \Sigma\,.\,\mathrm{P}+\Sigma\,(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}\,\mathrm{P}+(-1)^{\sigma'}\,\Sigma\,.\,\mathrm{Q} + (-1)^{\sigma'+1}\,\Sigma\,(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}\,\mathrm{Q},$$

an equation which, on replacing P and Q by the expressions which they represent, will be found to coincide with (3).

ART. 3.—We shall now develope some of the particular results included in the formula (3): in doing so, we shall have occasion to employ the equations

$$(12)\qquad \theta_{\mu+2,\mu'}(x) = (-1)^{\mu'}\theta_{\mu,\mu'}(x)\,;\quad \theta_{\mu,\mu'+2}(x) = \theta_{\mu,\mu'}(x),$$

$$(13)\qquad \theta_{\mu,\mu'}(-x) = (-1)^{\mu\mu'}\theta_{\mu,\mu'}(x),$$

which are immediate consequences of the equation of definition (1) or (2). From (12) we infer that there are only four distinct Theta functions, $\theta_{0,0}(x)$, $\theta_{0,1}(x)$, $\theta_{1,0}(x)$, $\theta_{1,1}(x)$; from (13) we infer that the first three of these are even functions, the last an uneven function; so that $\theta_{1,1}(0)$, and the three derived functions $\theta'_{0,0}(0)$, $\theta'_{0,1}(0)$, $\theta'_{1,0}(0)$ are all zero.

We successively attribute to the symbols

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ \mu_1, & \mu_2, & \mu_3, & \mu_4 \\ \mu'_1, & \mu'_2, & \mu'_3, & \mu'_4 \end{vmatrix}$$

the systems of values

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} x, & x, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} x, & x, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 1, & 1, & 0, & 0 \end{vmatrix},$$

$$\begin{vmatrix} x, & x, & 0, & 0 \\ 1, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} x, & x, & 0, & 0 \\ 1, & 1, & 0, & 0 \\ 1, & 1, & 0, & 0 \end{vmatrix};$$

and we obtain the equations

$$(14)\qquad \theta^4_{0,0}(0) = \theta^4_{0,1}(0)+\theta^4_{1,0}(0),$$

$$(15)\qquad \begin{cases} \theta^2_{0,0}(x)\,\theta^2_{0,0}(0) = \theta^2_{0,1}(x)\,\theta^2_{0,1}(0)+\theta^2_{1,0}(x)\,\theta^2_{1,0}(0), \\ \theta^2_{0,1}(x)\,\theta^2_{0,0}(0) = \theta^2_{0,0}(x)\,\theta^2_{0,1}(0)-\theta^2_{1,1}(x)\,\theta^2_{1,0}(0), \\ \theta^2_{1,0}(x)\,\theta^2_{0,0}(0) = \theta^2_{1,1}(x)\,\theta^2_{0,1}(0)+\theta^2_{0,0}(x)\,\theta^2_{1,0}(0), \\ \theta^2_{1,1}(x)\,\theta^2_{0,0}(0) = \theta^2_{1,0}(x)\,\theta^2_{0,1}(0)-\theta^2_{0,1}(x)\,\theta^2_{1,0}(0), \end{cases}$$

of which (14) arises from the first of (15) by putting 0 for x; and the four equations (15) are equivalent to two independent equations.

ART. 4.—Again, attributing to the elements x_1, x_2, x_3, x_4 the values $x-y$, $x+y$, 0, 0, we may obtain formulæ expressing the product of the

Theta functions of the sum and difference of two quantities in terms of the Theta functions of the two quantities themselves. Thus, to obtain the simplest possible expressions for the six products

$$\theta_{0,0}(x-y)\,\theta_{0,1}(x+y),\quad \theta_{0,0}(x-y)\,\theta_{1,0}(x+y),\quad \theta_{0,0}(x-y)\,\theta_{1,1}(x+y),$$
$$\theta_{0,1}(x-y)\,\theta_{1,0}(x+y),\quad \theta_{0,1}(x-y)\,\theta_{1,1}(x+y),\quad \theta_{1,0}(x-y)\,\theta_{1,1}(x+y)$$

(in which the two Theta functions are different), we attribute to the elements

$$\begin{vmatrix} \mu_1, & \mu_2, & \mu_3, & \mu_4 \\ \mu_1', & \mu_2', & \mu_3', & \mu_4' \end{vmatrix}$$

the systems of values

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 1 \end{vmatrix},\quad \begin{vmatrix} 0, & 1, & 0, & 1 \\ 0, & 0, & 0, & 0 \end{vmatrix},\quad \begin{vmatrix} 0, & 1, & 0, & 1 \\ 0, & 1, & 1, & 0 \end{vmatrix},$$
$$\begin{vmatrix} 0, & 1, & 0, & 1 \\ 1, & 0, & 1, & 0 \end{vmatrix},\quad \begin{vmatrix} 0, & 1, & 0, & 1 \\ 1, & 1, & 0, & 0 \end{vmatrix},\quad \begin{vmatrix} 1, & 1, & 0, & 0 \\ 0, & 1, & 0, & 1 \end{vmatrix};$$

and we obtain the results

(16) $$\theta_{0,0}(x-y)\,\theta_{0,1}(x+y)\,\theta_{0,0}(0)\,\theta_{0,1}(0) = \theta_{0,1}(y)\,\theta_{0,0}(y)\,\theta_{0,1}(x)\,\theta_{0,0}(x)-\theta_{1,1}(y)\,\theta_{1,0}(y)\,\theta_{1,1}(x)\,\theta_{1,0}(x),$$

(17) $$\theta_{0,0}(x-y)\,\theta_{1,0}(x+y)\,\theta_{0,0}(0)\,\theta_{1,0}(0) = \theta_{1,0}(y)\,\theta_{0,0}(y)\,\theta_{1,0}(x)\,\theta_{0,0}(x)+\theta_{1,1}(y)\,\theta_{0,1}(y)\,\theta_{1,1}(x)\,\theta_{0,1}(x),$$

(18) $$\theta_{0,0}(x-y)\,\theta_{1,1}(x+y)\,\theta_{0,1}(0)\,\theta_{1,0}(0) = \theta_{1,1}(y)\,\theta_{0,0}(y)\,\theta_{1,0}(x)\,\theta_{0,1}(x)+\theta_{1,0}(y)\,\theta_{0,1}(y)\,\theta_{1,1}(x)\,\theta_{0,0}(x),$$

(19) $$\theta_{0,1}(x-y)\,\theta_{1,0}(x+y)\,\theta_{0,1}(0)\,\theta_{1,0}(0) = \theta_{1,0}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,0}(x),$$

(20) $$\theta_{0,1}(x-y)\,\theta_{1,1}(x+y)\,\theta_{0,0}(0)\,\theta_{1,0}(0) = \theta_{1,0}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,0}(x),$$

(21) $$\theta_{1,0}(x-y)\,\theta_{1,1}(x+y)\,\theta_{0,0}(0)\,\theta_{0,1}(0) = \theta_{0,1}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{1,0}(x)+\theta_{1,1}(y)\,\theta_{1,0}(y)\,\theta_{0,1}(x)\,\theta_{0,0}(x),$$

of which the number may be doubled by changing $+y$ into $-y$.

It will be observed that the values attributed to $\begin{vmatrix} \mu_1, & \mu_2 \\ \mu_1', & \mu_2' \end{vmatrix}$ are determined in each of these equations by the indices of the Theta functions in the given product $\theta_{\mu_1,\mu_1'}(x-y)\,\theta_{\mu_2,\mu_2'}(x+y)$: the values of $\begin{vmatrix} \mu_3, & \mu_4 \\ \mu_3', & \mu_4' \end{vmatrix}$ are then determined by the conditions that 2σ and $2\sigma'$ must be even, and that, since $\theta_{1,1}(0)=0$, the combination $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ is inadmissible.

ART. 5.—If we differentiate the formulæ (16)—(21) with respect to y, and then put $y=0$, we arrive at the following equations, which serve to express the differential coefficient of a quotient of two Theta functions, in terms of the Theta functions themselves:—

$$(22)\quad \theta_{0,1}(x)\,\theta'_{0,0}(x)-\theta_{0,0}(x)\,\theta'_{0,1}(x)=\cdot\frac{\theta'_{1,1}(0)\,\theta_{1,0}(0)}{\theta_{0,0}(0)\,\theta_{0,1}(0)}\,\theta_{1,1}(x)\,\theta_{1,0}(x),$$

$$(23)\quad \theta_{1,0}(x)\,\theta'_{0,0}(x)-\theta_{0,0}(x)\,\theta'_{1,0}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,1}(0)}{\theta_{0,0}(0)\,\theta_{1,0}(0)}\,\theta_{1,1}(x)\theta_{0,1}(x),$$

$$(24)\quad \theta_{1,1}(x)\,\theta'_{0,0}(x)-\theta_{0,0}(x)\,\theta'_{1,1}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,0}(0)}{\theta_{0,1}(0)\,\theta_{1,0}(0)}\,\theta_{1,0}(x)\theta_{0,1}(x),$$

$$(25)\quad \theta_{1,0}(x)\,\theta'_{0,1}(x)-\theta_{0,1}(x)\,\theta'_{1,0}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,0}(0)}{\theta_{0,1}(0)\,\theta_{1,0}(0)}\,\theta_{1,1}(x)\,\theta_{0,0}(x),$$

$$(26)\quad \theta_{1,1}(x)\,\theta'_{0,1}(x)-\theta_{0,1}(x)\,\theta'_{1,1}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,1}(0)}{\theta_{0,0}(0)\,\theta_{1,0}(0)}\,\theta_{1,0}(x)\,\theta_{0,0}(x),$$

$$(27)\quad \theta_{1,1}(x)\,\theta'_{1,0}(x)-\theta_{1,0}(x)\,\theta'_{1,1}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{1,0}(0)}{\theta_{0,0}(0)\,\theta_{0,1}(0)}\,\theta_{0,1}(x)\,\theta_{0,0}(x).$$

ART. 6.—Each of the four products

$$\theta_{0,0}(x-y)\,\theta_{0,0}(x+y),\quad \theta_{0,1}(x-y)\,\theta_{0,1}(x+y),\quad \theta_{1,0}(x-y)\,\theta_{1,0}(x+y),$$
$$\theta_{1,1}(x-y)\,\theta_{1,1}(x+y)$$

(in which the two Theta functions are the same) can be expressed in six different ways in terms of the squares of the Theta functions of x and y: thus

$$(28)\quad \theta_{\mu,\mu'}(x-y)\,\theta_{\mu,\mu'}(x+y)\,\theta^2_{0,0}(0)$$
$$=\theta^2_{0,1}(y)\,\theta^2_{\mu,\mu'+1}(x)+(-1)^{\mu'}\theta^2_{1,0}(y)\,\theta^2_{\mu+1,\mu'}(x)$$
$$=\theta^2_{0,0}(y)\,\theta^2_{\mu,\mu'}(x)+(-1)^{\mu'}\theta^2_{1,1}(y)\,\theta^2_{\mu+1,\mu'+1}(x),$$

$$(29)\quad \theta_{\mu,\mu'}(x-y)\,\theta_{\mu,\mu'}(x+y)\,\theta^2_{0,1}(0)$$
$$=\theta^2_{0,0}(y)\,\theta^2_{\mu,\mu'+1}(x)+(-1)^{\mu'}\theta^2_{1,0}(y)\,\theta^2_{\mu+1,\mu'+1}(x)$$
$$=\theta^2_{0,1}(y)\,\theta^2_{\mu,\mu'}(x)+(-1)^{\mu'}\theta^2_{1,1}(y)\,\theta^2_{\mu+1,\mu'}(x),$$

$$(30)\quad \theta_{\mu,\mu'}(x-y)\,\theta_{\mu,\mu'}(x+y)\,\theta^2_{1,0}(0)$$
$$=(-1)^{\mu'}\theta^2_{0,0}(y)\,\theta^2_{\mu+1,\mu'}(x)-(-1)^{\mu'}\theta^2_{0,1}(y)\,\theta^2_{\mu+1,\mu'+1}(x)$$
$$=\theta^2_{1,0}(y)\,\theta^2_{\mu,\mu'}(x)-\theta^2_{1,1}(y)\,\theta^2_{\mu,\mu'+1}(x).$$

To form these equations, we represent by (A), (A′), (B), (B′), (C), (C′), the formulæ obtained by attributing to the indices in (3) the values

$$(\mathrm{A})\ \left|\begin{matrix}\mu, & \mu, & 0, & 0\\ \mu', & \mu', & 0, & 0\end{matrix}\right|,\qquad (\mathrm{A}')\ \left|\begin{matrix}\mu+1, & \mu+1, & 1, & 1\\ \mu'+1, & \mu'+1, & 1, & 1\end{matrix}\right|,$$

$$(B)\ \begin{vmatrix} \mu, & \mu, & 0, & 0 \\ \mu', & \mu', & 1, & 1 \end{vmatrix}, \qquad (B')\ \begin{vmatrix} \mu+1, & \mu+1, & 1, & 1 \\ \mu', & \mu', & 1, & 1 \end{vmatrix},$$

$$(C)\ \begin{vmatrix} \mu, & \mu, & 1, & 1 \\ \mu', & \mu', & 0, & 0 \end{vmatrix}, \qquad (C')\ \begin{vmatrix} \mu, & \mu, & 1, & 1 \\ \mu'+1, & \mu'+1, & 1, & 1 \end{vmatrix};$$

the indeterminates x_1, x_2, x_3, x_4 receiving the values $x-y$, $x+y$, 0, 0: the left-hand members of the three formulæ (A′), (B′), (C′) contain the factor $\theta_{1,1}^2(0)$, and are therefore zero; and the equations (A) ± (A′), (B) ± (B′), (C) ± (C′) will be found to coincide respectively with (28), (29), (30).*

One of the formulæ (29), by putting $\mu = 0$, $\mu' = 1$, becomes

$$(31)\quad \theta_{0,1}(x-y)\,\theta_{0,1}(x+y)\,\theta_{0,1}^2(0) = \theta_{0,1}^2(y)\,\theta_{0,1}^2(x) - \theta_{1,1}^2(y)\,\theta_{1,1}^2(x),$$

an equation which is particularly important in the theory of elliptic functions.

ART. 7.—Differentiating the equations (28)—(30) twice with respect to y, and putting $y = 0$, we obtain a system of equations, of which the following, derived from (31), is one:

$$(32)\qquad \frac{d}{dx}\left[\frac{\theta'_{0,1}(x)}{\theta_{0,1}(x)}\right] = \frac{\theta''_{0,1}(0)}{\theta_{0,0}(0)} - \frac{\theta'^2_{1,1}(0)\,\theta_{1,1}^2(x)}{\theta_{0,1}^2(0)\,\theta_{0,1}^2(x)}.$$

If in certain of the equations of this system† we put $x = 0$; or, more simply, if we differentiate the equations (15) twice with respect to x, and then put $x = 0$, we find

$$(33)\qquad \begin{cases} \theta_{0,0}^3\,\theta''_{0,0} = \theta_{0,1}^3\,\theta''_{0,1} + \theta_{1,0}^3\,\theta''_{1,0}, \\ \theta_{0,0}^2\,\theta'^2_{1,1} = \theta_{0,1}^2\,\theta_{1,0}\,\theta''_{1,0} - \theta_{1,0}^2\,\theta_{0,1}\,\theta''_{0,1}, \\ \theta_{0,1}^2\,\theta'^2_{1,1} = \theta_{0,0}^2\,\theta_{1,0}\,\theta''_{1,0} - \theta_{1,0}^2\,\theta_{0,0}\,\theta''_{0,0}, \\ \theta_{1,0}^2\,\theta'^2_{1,1} = \theta_{0,1}^2\,\theta_{0,0}\,\theta''_{0,0} - \theta_{0,0}^2\,\theta_{0,1}\,\theta''_{0,1}, \end{cases}$$

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* All the equations (16)—(21), and some of the equations (28)—(30), will be found in an excellent memoir of M. Betti, "La Teorica delle funzione ellittiche e sue applicazione" ("Tortolini," Vol. III.; New Series, p. 126). M. Betti derives these equations from the formula

$$\begin{aligned} &2\theta_{\mu,\nu}(x+y)\,\theta_{\mu',\nu'}(x-y)\,\theta_{\alpha,0}(0)\,\theta_{0,\beta}(0) \\ &= \theta_{\mu,\nu}(x)\,\theta_{\mu',\nu'}(x)\,\theta_{\alpha,0}(y)\,\theta_{0,\beta}(y) + (-1)^{\nu}\,\theta_{\mu+1,\nu}(x)\,\theta_{\mu'+1,\nu'}(x)\,\theta_{\alpha+1,0}(y)\,\theta_{1,\beta}(y) \\ &\quad + (-1)^{\nu}\,\theta_{\mu+1,\nu+1}(x)\,\theta_{\mu'+1,\nu'+1}(x)\,\theta_{\alpha+1,1}(y)\,\theta_{1,\beta+1}(y) \\ &\quad + \theta_{\mu,\nu+1}(x)\,\theta_{\mu',\nu'+1}(x)\,\theta_{\alpha,1}(y)\,\theta_{0,\beta+1}(y), \end{aligned}$$

where $\alpha = \mu - \mu'$, $\beta = \nu' - \nu$; which is a particular case of the formula (3), and which has also been given by M. Hermite ("Liouville," New Series, Vol. III., p. 27). The notation of M. Betti is different from that employed here, and by M. Hermite in the memoir just cited.

† Viz.: in (28) let $\mu = \mu' = 0$, and in (28), (29), (30) let $\mu = \mu' = 1$.

where $\theta_{0,0}$, $\theta''_{0,0}$, &c. are written for $\theta_{0,0}(0)$, $\theta''_{0,0}(0)$, &c. This system is equivalent to two independent relations between $\theta''_{0,0}$, $\theta''_{0,1}$, and $\theta''_{1,0}$.

ART. 8.—Lastly, in the equation (3), let $x_1+x_2+x_3=0$, $\mu_1=\mu_2=\mu_3=\mu_4=\mu$, $\mu'_1=\mu'_2=\mu'_3=\mu'_4=\nu$; differentiate the equation with respect to x_4, and afterwards put $x_4=0$: we find, on combining with one another (by addition or subtraction) the three equations, answering to the three combinations $\mu=0$, $\nu=0$; $\mu=0$, $\nu=1$; $\mu=1$, $\nu=0$,

$$(34)\quad \theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3)\,\theta'_{1,1}(0)$$
$$=(-1)^{\mu+\nu}\theta_{\mu,\nu}(0)\,[\theta_{\mu,\nu}(x_1)\,\theta_{\mu,\nu}(x_2)\,\theta'_{\mu,\nu}(x_3)$$
$$+\theta_{\mu,\nu}(x_1)\,\theta'_{\mu,\nu}(x_2)\,\theta_{\mu,\nu}(x_3)+\theta'_{\mu,\nu}(x_1)\,\theta_{\mu,\nu}(x_2)\,\theta_{\mu,\nu}(x_3)],$$

where, on the right-hand side, μ and ν may have any one of the three sets of values $\mu=0$, $\nu=0$; $\mu=0$, $\nu=1$; $\mu=1$, $\nu=1$.

ART. 9.—It remains to apply these formulæ to the demonstration of the fundamental properties of elliptic functions.

If $\sqrt{k}$, $\sqrt{k'}$ represent the fractions $\dfrac{\theta_{1,0}(0)}{\theta_{0,0}(0)}$, $\dfrac{\theta_{0,1}(0)}{\theta_{0,0}(0)}$, the three elliptic functions of the first species are defined by the equations

$$(35)\qquad \sin am\, x=\frac{1}{\sqrt{k}}\cdot\frac{1}{i}\cdot\frac{\theta_{1,1}(x)}{\theta_{0,1}(x)},$$

$$(36)\qquad \cos am\, x=\frac{\sqrt{k'}}{\sqrt{k}}\cdot\frac{\theta_{1,0}(x)}{\theta_{0,1}(x)},$$

$$(37)\qquad \Delta\, am\, x=\sqrt{k'}\cdot\frac{\theta_{0,0}(x)}{\theta_{0,1}(x)},$$

in which the constant a of equation (1) is to receive a certain value which will presently be assigned. Introducing these functions into the equations (14) and (15), we find

$$(38)\qquad k^2+k'^2=1,$$

$$(39)\qquad \cos^2 am\, x+\sin^2 am\, x=1,$$

$$(40)\qquad \Delta^2 am\, x+k^2\sin^2 am\, x=1.$$

Again, the differential equation (26) assumes the form

$$\frac{d\,.\sin am\, x}{dx}=\frac{1}{i}\,\frac{\theta_{0,0}(0)\,\theta'_{1,1}(0)}{\theta_{1,0}(0)\,\theta_{0,1}(0)}\cos am\, x\,\Delta\, am\, x.$$

To identify the function $\sin am\, x$ with the function u which satisfies the differential equation

$$\frac{du^2}{dx^2}=(1-u^2)(1-k^2u^2),$$

and the initial conditions $u=0$, $\frac{du}{dx}=+1$, when $x=0$, it is necessary and sufficient that the coefficient $\frac{1}{i}\frac{\theta_{0,0}(0)\,\theta'_{1,1}(0)}{\theta_{1,0}(0)\,\theta_{0,1}(0)}$ should be equal to unity. This condition is satisfied by assigning to the constant a, which has remained undetermined in the equations (1) and (2), the value

$$\pi\times\frac{\sum\limits_{-\infty}^{+\infty} q^{n^2}\times\sum\limits_{-\infty}^{+\infty}(-1)^n(2n+1)\,q^{\frac{1}{4}(2n+1)^2}}{\sum\limits_{-\infty}^{+\infty}(-1)^n q^{n^2}\times\sum\limits_{-\infty}^{+\infty} q^{\frac{1}{4}(2n+1)^2}}.$$

Adopting this value of the constant a, we obtain from (26), (25), and (22),

(41) $$\frac{d\,.\sin am\,x}{dx}=\cos am\,x\,\Delta\,am\,x,$$

(42) $$\frac{d\,.\cos am\,x}{dx}=-\sin am\,x\,\Delta\,am\,x,$$

(43) $$\frac{d\,.\Delta\,am\,x}{dx}=-k^2\sin am\,x\cos am\,x.$$

Next, dividing the equations (29), (19), and (16) by equation (31), and changing y into $-y$ in (16), we find

$$\frac{\theta_{0,0}(0)\,\theta_{1,0}(0)}{\theta^2_{0,1}(0)}\;\frac{\theta_{1,1}(x+y)}{\theta_{0,1}(x+y)}$$
$$=\frac{\theta_{1,0}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,0}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)-\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)},$$

$$\frac{\theta_{1,0}(0)}{\theta_{0,1}(0)}\;\frac{\theta_{1,0}(x+y)}{\theta_{0,1}(x+y)}$$
$$=\frac{\theta_{1,0}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,0}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)-\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)},$$

$$\frac{\theta_{0,0}(0)}{\theta_{0,1}(0)}\;\frac{\theta_{0,0}(x+y)}{\theta_{0,1}(x+y)}$$
$$=\frac{\theta_{0,1}(y)\,\theta_{0,0}(y)\,\theta_{0,1}(x)\,\theta_{0,0}(x)+\theta_{1,1}(y)\,\theta_{1,0}(y)\,\theta_{1,1}(x)\,\theta_{1,0}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)-\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)};$$

or, introducing the elliptic functions

(44) $$\sin am\,(x+y)=\frac{\cos am\,y\,\Delta\,am\,y\sin am\,x+\cos am\,x\,\Delta\,am\,x\sin am\,y}{1-k^2\sin^2 am\,x\sin^2 am\,y},$$

(45) $$\cos am\,(x+y)=\frac{\cos am\,x\cos am\,y-\Delta\,am\,y\sin am\,y\,\Delta\,am\,x\sin am\,x}{1-k^2\sin^2 am\,x\sin^2 am\,y},$$

(46) $$\Delta\,am\,(x+y)=\frac{\Delta\,am\,x\,\Delta\,am\,y-k^2\cos am\,y\cos am\,x\sin am\,y\sin am\,x}{1-k^2\sin^2 am\,x\sin^2 am\,y}.$$

These formulæ of addition may be obtained in various other well known forms, by using other combinations of the equations (16)—(21) with the equations (28)—(30).

ART. 10.—The elliptic function of the second species is defined by the equation

$$(47) \qquad Z(x) = \int_0^x k^2 \sin^2 am\, x\, dx.$$

Observing that $\qquad \sqrt{k} = \frac{\theta_{1,0}}{\theta_{0,0}} = \frac{1}{i}\,\frac{\theta'_{1,1}}{\theta_{0,1}},$

we may write the equation (32) in the form

$$k^2 \sin^2 am\, x = \frac{\theta''_{0,1}(0)}{\theta_{0,1}(0)} - \frac{d}{dx}\left[\frac{\theta'_{0,1}(x)}{\theta_{0,1}(x)}\right],$$

whence, integrating, we obtain

$$(48) \qquad Z(x) = x\,\frac{\theta''_{0,1}(0)}{\theta_{0,1}(0)} - \frac{\theta'_{0,1}(x)}{\theta_{0,1}(x)},$$

which is Jacobi's expression for the elliptic function of the second species.

If x_1, x_2, x_3 are three arguments of which the sum is zero, we have

$$Z(x_1) + Z(x_2) + Z(x_3) = -\frac{\theta'_{0,1}(x_1)}{\theta_{0,1}(x_1)} - \frac{\theta'_{0,1}(x_2)}{\theta_{0,1}(x_2)} - \frac{\theta'_{0,1}(x_3)}{\theta_{0,1}(x_3)}.$$

The right-hand member of this equation may be transformed, by means of the formula (34), so as to contain only elliptic functions of the first species. We thus obtain

$$(49) \quad Z(x_1) + Z(x_2) + Z(x_3) = \frac{\theta'_{1,1}(0)\,\theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3)}{\theta_{0,1}(0)\,\theta_{0,1}(x_1)\,\theta_{0,1}(x_2)\,\theta_{0,1}(x_3)}$$
$$= k^2 \sin am\, x_1 \sin am\, x_2 \sin am\, x_3,$$

which is the theorem of addition for elliptic functions of the second species.

ART. 11.—The elliptic function of the third species is defined by the equation

$$(50) \qquad \Pi(x, y) = \int_0^x \frac{k^2 \sin am\, y \cos am\, y\, \Delta\, am\, y \sin^2 am\, x\, dx}{1 - k^2 \sin^2 am\, y \sin^2 am\, x},$$

and is a function of two quantities, the argument x and the parameter y.

Dividing the equation (31) by $\theta^2_{0,1}(x)\,\theta^2_{0,1}(y)$, we find

$$1 - \frac{\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)} = 1 - k^2 \sin^2 am\, y \sin^2 am\, x$$
$$= \frac{\theta_{0,1}(x+y)\,\theta_{0,1}(x-y)\,\theta^2_{0,1}(0)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)}.$$

Take the logarithm of each side; differentiate with respect to y, and then integrate with respect to x from 0 to x: we obtain

$$(51)\qquad \Pi(x, y) = x\frac{\theta'_{0,1}(y)}{\theta_{0,1}(y)} + \tfrac{1}{2}\log\frac{\theta_{0,1}(x-y)}{\theta_{0,1}(x+y)},$$

the important equation by which Jacobi has expressed the function $\Pi(x, y)$ containing two indeterminates by means of functions of one indeterminate. If x_1, x_2, x_3 are three arguments of which the sum is zero, we have

$$(52)\quad \Pi(x_1, y)+\Pi(x_2, y)+\Pi(x_3, y) = \tfrac{1}{2}\log\frac{\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)\,\theta_{0,1}(x_3-y)}{\theta_{0,1}(x_1+y)\,\theta_{0,1}(x_2+y)\,\theta_{0,1}(x_3+y)}.$$

The quantity after the logarithmic sign can be transformed (and that in various ways) by means of the formula (3), so as to contain only elliptic functions of the first species. Thus, attributing to the indeterminates and indices in the formula (3) the values

$$\left|\begin{matrix} -y, & -x_1+y, & -x_2+y, & -x_3+y \\ 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \end{matrix}\right|,$$

we get

$$(53)\quad 2\theta_{0,1}(y)\,\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)\,\theta_{0,1}(x_3-y)$$
$$= \theta_{0,1}(2y)\,\theta_{0,1}(x_1)\,\theta_{0,1}(x_2)\,\theta_{0,1}(x_3)+\theta_{0,0}(2y)\,\theta_{0,0}(x_1)\,\theta_{0,0}(x_2)\,\theta_{0,0}(x_3)$$
$$+\theta_{1,1}(2y)\,\theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3)-\theta_{1,0}(2y)\,\theta_{1,0}(x_1)\,\theta_{1,0}(x_2)\,\theta_{1,0}(x_3).$$

In this equation change y into $-y$, and divide the first result by the second: the formula (52) becomes

$$(54)\quad \Pi(x_1, y)+\Pi(x_2, y)+\Pi(x_3, y) =$$
$$\tfrac{1}{2}\log\frac{1+\Delta\, am\, 2y\,[\Delta\, am\, x]-\cos am\, 2y\,[\cos am\, x]+\sin am\, 2y\,[\sin am\, x]}{1+\Delta\, am\, 2y\,[\Delta\, am\, x]-\cos am\, 2y\,[\cos am\, x]-\sin am\, 2y\,[\sin am\, x]},$$

where $[\sin am\, x]$, $[\Delta\, am\, x]$, $[\cos am\, x]$ are written as abbreviations for $k^2\sin am\, x_1 \sin am\, x_2 \sin am\, x_3$, $\frac{1}{k'^2}\Delta\, am\, x_1\,\Delta\, am\, x_2\,\Delta\, am\, x_3$, $\frac{k^2}{k'^2}\cos am\, x_1 \cos am\, x_2 \cos am\, x_3$. This is a very symmetrical form of the theorem of addition of functions of the third species: the equation of Legendre

$$(55)\quad \Pi(x_1, y)+\Pi(x_2, y)+\Pi(x_3, y) = \tfrac{1}{2}\log\frac{1+k^2\sin am\, y \sin am\, x_1 \sin am\, x_2 \sin am\,(x_3+y)}{1-k^2\sin am\, y \sin am\, x_1 \sin am\, x_2 \sin am\,(x_3-y)}$$

is, however, more convenient for computation. It may be obtained by attributing to the indeterminates and indices in the formula (3) the values $\begin{vmatrix} x_1-y, & x_2-y, & x_3, & 0 \\ 0\ , & 0\ , & 0, & 0 \\ 1\ , & 1\ , & 1, & 1 \end{vmatrix}$, $\begin{vmatrix} x_1-y, & x_2-y, & x_3, & 0 \\ 1\ , & 1\ , & 1, & 1 \\ 1\ , & 1\ , & 1, & 1 \end{vmatrix}$,

and adding the results: we thus find

(56) $2\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)\,\theta_{0,1}(x_3)\,\theta_{0,1}(0)$

$=\theta_{0,1}(x_1)\,\theta_{0,1}(x_2)\,\theta_{0,1}(x_3+y)\,\theta_{0,1}(y)+\theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3+y)\,\theta_{1,1}(y)$.

Changing y into $-y$, dividing the first result by the second, and substituting the resulting expression of $\dfrac{\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)}{\theta_{0,1}(x_1+y)\,\theta_{0,1}(x_2+y)}$ in the equation (52), we arrive at the formula (55).

Jacobi has given, in the *Fundamenta Nova*, two other expressions of this addition theorem, both of which are easily inferred from (52) by using appropriate particularisations of the formula (3).

On an Involution System of Circular Cubics, and description of the curve by points, when the double focus is on the curve.

By T. COTTERILL, M.A.

It follows from a theorem given by Chasles, that, if a system of circular cubics can pass through seven points, the tangents at each point sweep through equal angles; and consequently, if the system can break up in at least three ways into a circle and line, the circle described by the double focus or intersection of the tangents at the circular points and the conic of intersection of the tangents at any two of the other points are determined. The six intersections of four lines in a plane, and the focus of the inscribed parabola, afford an instance of such a system.

A particular case of this is given from the three points of a triangle (determined by a point on a circle and its intersections with a line), and the four centres of the circles touching its sides.

Any point on a circular cubic through seven such points determines another point on the curve, its confocal to the triangle, the line joining them being parallel to the real asymptote, so that a circular cubic is the locus of the real foci lying on parallel axes of conics inscribed to a triangle. The intersection of the tangents to the cubic at these