Note on some Relations between certain Elliptic and Hyperbolic Functions. By John Griffiths, M.A.

[*Read February 11th*, 1875.]

If the definite integrals

$$\int_{\theta_0}^{\theta} \sqrt{1-e^2\sin^2\theta}\,d\theta,\quad \int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}},\ \text{and}\ \int_{\theta_0}^{\theta}\sqrt{e'^2\operatorname{cosec}^2\theta-1}\,\operatorname{cosec}\theta\,d\theta$$

be denoted by $E_{\theta_0}^{\theta}$, $F_{\theta_0}^{\theta}$, and $H_{\theta_0}^{\theta}$ respectively, where $ee'=1$, I propose to show, in the first place, that they are connected by the equation

$$eH_{\theta_0}^{\theta}+E_{\theta_0}^{\theta}-(1-e^2)\,F_{\theta_0}^{\theta}+\left[\sqrt{1-e^2\sin^2\theta}\cot\theta\right]_{\theta_0}^{\theta}=0 \quad\ldots\ldots (1);$$

and also, if $\cos\theta\cos\phi-\sin\theta\sin\phi\sqrt{1-e^2\sin^2\mu}$

$$=\cos\mu=\cos\theta_0\cos\phi_0-\sin\theta_0\sin\phi_0\sqrt{1-e^2\sin^2\mu},$$

(where μ is a constant), to deduce from (1) the further relation

$$\frac{E_{\theta_0}^{\theta}+E_{\phi_0}^{\phi}}{H_{\theta_0}^{\theta}+H_{\phi_0}^{\phi}}=e^3\sin\theta\sin\phi\sin\theta_0\sin\phi_0 \quad\ldots\ldots\ldots\ldots\ldots (2).$$

[The integral $\int_{\theta_0}^{\theta}\sqrt{1-e^2\sin^2\theta}\,d\theta$ expresses, as we know, the length of the arc of the ellipse $x^2+\frac{y^2}{1-e^2}=1$ between the points P, P_0, whose abscissæ are $x=\sin\theta$, $x_0=\sin\theta_0$ respectively; and if $e'=\frac{1}{e}$, the integral $\int_{\theta_0}^{\theta}\sqrt{e'^2\operatorname{cosec}^2\theta-1}\,\operatorname{cosec}\theta\,d\theta$ denotes the length of the arc of the hyperbola $x^2-\frac{e^2y^2}{1-e^2}=1$ between the corresponding points P′, P_0', whose abscissæ are $x=\operatorname{cosec}\theta$, $x_0=\operatorname{cosec}\theta_0$.]

Taking, then, $H_{\theta_0}^{\theta}=\int_{\theta_0}^{\theta}\sqrt{e'^2\operatorname{cosec}^2\theta-1}\,\operatorname{cosec}\theta\,d\theta$, and $e'=\frac{1}{e}$, we have

$$-eH_{\theta_0}^{\theta}=-\int_{\theta_0}^{\theta}\frac{\sqrt{1-e^2\sin^2\theta}}{\sin^2\theta}\,d\theta$$

$$=\left[\sqrt{1-e^2\sin^2\theta}\cot\theta\right]_{\theta_0}^{\theta}+\int_{\theta_0}^{\theta}\frac{e^2\cos^2\theta}{\sqrt{1-e^2\sin^2\theta}}\,d\theta$$

$$=\left[\sqrt{1-e^2\sin^2\theta}\cot\theta\right]_{\theta_0}^{\theta}+\int_{\theta_0}^{\theta}\sqrt{1-e^2\sin^2\theta}\,d\theta-(1-e^2)\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}};$$

i. e., $$eH^{\theta}_{\theta_0}+E^{\theta}_{\theta_0}-(1-e^2)\,F^{\theta}_{\theta_0}+\left[\sqrt{1-e^2\sin^2\theta}\cot\theta\right]^{\theta}_{\theta_0}=0,$$
which is relation (1).

If θ, θ_0 be connected with two other amplitudes ϕ and ϕ_0, by the relations $\cos\theta\cos\phi-\sin\theta\sin\phi\sqrt{1-e^2\sin^2\mu}=\cos\mu$

$$=\cos\theta_0\cos\phi_0-\sin\theta_0\sin\phi_0\sqrt{1-e^2\sin^2\mu};$$

or, what is the same thing, by

$$\left.\begin{aligned}\cos\theta &= \cos\phi\cos\mu+\sin\phi\sin\mu\sqrt{1-e^2\sin^2\theta}\\ \cos\theta_0 &= \cos\phi_0\cos\mu+\sin\phi_0\sin\mu\sqrt{1-e^2\sin^2\theta_0}\end{aligned}\right\}\quad\ldots\ldots\ldots(3),$$

the equation (1) becomes

$$eH^{\theta}_{\theta_0}+E^{\theta}_{\theta_0}-(1-e^2)\,F^{\theta}_{\theta_0}+\left[\frac{\cos^2\theta-\cos\theta\cos\phi\cos\mu}{\sin\theta\sin\phi\sin\mu}\right]^{\theta}_{\theta_0}=0.$$

In the same way, we have

$$eH^{\phi}_{\phi_0}+E^{\phi}_{\phi_0}-(1-e^2)\,F^{\phi}_{\phi_0}+\left[\frac{\cos^2\phi-\cos\theta\cos\phi\cos\mu}{\sin\theta\sin\phi\sin\mu}\right]^{\phi}_{\phi_0}=0;$$

hence, by addition,

$$\begin{aligned}e\,(H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0})+(E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0})-(1-e^2)(F^{\theta}_{\theta_0}+F^{\phi}_{\phi_0})&\\ +\frac{\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi\cos\mu}{\sin\theta\sin\phi\sin\mu}&\\ -\frac{\cos^2\theta_0+\cos^2\phi_0-2\cos\theta_0\cos\phi_0\cos\mu}{\sin\theta_0\sin\phi_0\sin\mu}&=0;\end{aligned}$$

or, since (3) are equivalent to the following, viz.,

$$\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi\cos\mu=\sin^2\mu\,(1-e^2\sin^2\theta\sin^2\phi),$$
$$\cos^2\theta_0+\cos^2\phi_0-2\cos\theta_0\cos\phi_0\cos\mu=\sin^2\mu\,(1-e^2\sin^2\theta_0\sin^2\phi_0),$$

$$\begin{aligned}e\,(H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0})+(E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0})-(1-e^2)(F^{\theta}_{\theta_0}+F^{\phi}_{\phi_0})&\\ +\sin\mu\,(\operatorname{cosec}\theta\operatorname{cosec}\phi-e^2\sin\theta\sin\phi)&\\ -\sin\mu\,(\operatorname{cosec}\theta_0\operatorname{cosec}\phi_0-e^2\sin\theta_0\sin\phi_0)&=0;\end{aligned}$$

i. e., as we know that $E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0}=e^2\sin\mu\,(\sin\theta\sin\phi-\sin\theta_0\sin\phi_0)$,
and $F^{\theta}_{\theta_0}+F^{\phi}_{\phi_0}=0$, it follows that

$$e\,(H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0})+\sin\mu\,(\operatorname{cosec}\theta\operatorname{cosec}\phi-\operatorname{cosec}\theta_0\operatorname{cosec}\phi_0)=0;$$

and consequently $$\frac{E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0}}{H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0}}=e^3\sin\theta\sin\phi\sin\theta_0\sin\phi_0,$$
which is relation (2).

I may remark, that the locus of the point of intersection of the tangents to the ellipse $x^2 + \frac{y^2}{1-e^2} = 1$, at the points θ and ϕ, is a hyperbola confocal with the ellipse.

This may be proved as follows:—

The equations of the tangents in question are

$$X \sin\theta + \frac{Y}{1-e^2}\cos\theta = 1,$$

$$X \sin\phi + \frac{Y}{1-e^2}\cos\phi = 1,$$

or, say, $\quad X'\sin\theta + Y'\cos\theta = 1, \quad \text{and} \quad X'\sin\phi + Y'\cos\phi = 1,$

where $\quad X' = X, \quad \text{and} \quad Y' = \frac{Y}{\sqrt{1-e^2}};$

i. e., $\sin\theta$, $\sin\phi$ and $\cos\theta$, $\cos\phi$ are the roots of the two respective quadratics

$$(X'^2+Y'^2)\sin^2\theta - 2X'\sin\theta + 1 - Y'^2 = 0,$$
$$(X'^2+Y'^2)\cos^2\theta - 2Y'\cos\theta + 1 - X'^2 = 0.$$

Hence, since $\cos\theta\cos\phi - \sin\theta\sin\phi\sqrt{1-e^2\sin^2\mu} = \cos\mu$,

$$\frac{1-X'^2}{X'^2+Y'^2} - \frac{1-Y'^2}{X'^2+Y'^2}\sqrt{1-e^2\sin^2\mu} = \cos\mu,$$

which, on our replacing X', Y' by X, $\frac{Y}{\sqrt{1-e^2}}$, reduces to the following,

viz., $\frac{X^2}{1-\cos\mu} - \frac{Y^2}{\cos\mu + \sqrt{1-e^2\sin^2\mu}} = \frac{e^2}{1+\sqrt{1-e^2\sin^2\mu}}$; a hyperbola confocal with the ellipse $x^2 + \frac{y^2}{1-e^2} = 1$.

In the same way it may be shown that the locus of the intersection of tangents to the hyperbola $x^2 - \frac{e^2y^2}{1-e^2} = 1$, at the points P′, Q′, ($x = \operatorname{cosec}\theta$ and $x = \operatorname{cosec}\phi$) is an ellipse confocal with the hyperbola.

[Compare Mr. MacCullagh's theorem, viz.: If two tangents be drawn to an ellipse from any point of a confocal hyperbola, the difference of the arcs PK, QK is equal to the difference of the tangents TP, TQ. —See Dr. Salmon's "Conic Sections," p. 358, 5th ed.]

March 11*th*, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Messrs. A. B. Kempe, B.A., and S. A. Renshaw were elected Members; Mr. J. H. Röhrs, M.A., was proposed for election; and Mr. Harry Hart, M.A., was admitted into the Society.

Mr. Roberts gave an account of his paper, "On a Simplified Method of obtaining the Order of Algebraical Conditions."

Mr. Sylvester, F.R.S., spoke on the subject of "An Orthogonal Web," pointing out a curious paradox when the reticulation was not all in the same plane.

Mr. Tucker read a portion of Mr. Darwin's paper, "On some proposed forms of Slide-rule."

The following presents were received:—

"Bulletin de la Société Mathématique de France," Tome ii., Fev. No. 5.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome i., 2e Série, 1er Cahier. Paris, 1875.

"Bemerkungen zur Theorie der Ternären cubichen Formen von Axel Harnack." Erlangen, vom 8 Febr., 1875.

"Jahrbuch über die Fortschritte der Mathematik," viertes Band, Jahrgang 1872, Heft 3.

"Journal of the Institute of Actuaries," No. 97, Oct. 1874.

"Fifth Annual Report of the Association for the Improvement of Geometrical Teaching," January, 1875.

"Table des Fonctions Symétriques de Poids XI," dressée par le Chev. F. Faà de Bruno (extrait de la Théorie des Formes Binaires, du même auteur), Mars, 1875.

On a Simplified Method of obtaining the Order of Algebraical Conditions. By S. Roberts, M.A.

[*Read March* 11*th*, 1875.]

1. I propose to give some examples of a method of obtaining the order of the conditions for the co-existence of systems of equations. The method easily leads to the required expressions in the simpler cases, and shows the course of procedure where the actual expression is complicated and need not be evaluated.