A NOTE ON A THEOREM OF MR. HARDY'S

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1. In a paper entitled "An extension of a theorem on oscillating series", published in Vol. 12 of these *Proceedings*,* Mr. Hardy has given the following theorem:

If Σc_n *is summable* (λ, κ), *and*

$$c_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then Σc_n *is convergent.*

Assuming that κ is an integer (which is clearly permissible), Mr. Hardy's argument enables us to prove successively that the series is summable $(\lambda, \kappa-1)$, $(\lambda, \kappa-2)$, ..., $(\lambda, 1)$; but it fails to prove the last step of this induction, namely, that if c_n satisfies the condition of the theorem, and if Σc_n is summable $(\lambda, 1)$, then the series is convergent. It is the object of this note to supply this last step and thus to complete the proof of the theorem.

2. Using the notation of Mr. Hardy's paper, what we have to prove is this:

If
$$\int^{\infty} \frac{B(\omega)}{\omega^2}\, d\omega$$
is convergent, then

$$(1) \quad B(\omega) = o(\omega).\dagger$$

* Pp. 174–180.

† It is hardly necessary to point out that this is not a general convergence theorem true for all functions $B(\omega)$. It holds only in virtue of the structure of the special functions considered.

Suppose that (1) is false. Then there is a positive constant H, such that at least one of the inequalities

$$B(\omega) > H\omega,$$

$$B(\omega) < -H\omega$$

is satisfied for values of ω exceeding all limit. We shall shew that either of these hypotheses implies a contradiction. Let us take, for example, the first one.

Suppose that $\lambda_n \leqslant \omega < \lambda_{n+1}$. Let $\xi > \lambda_n$, and let $\lambda_{n+r} \leqslant \xi < \lambda_{n+r+1}$. Then

$$\begin{aligned} |B(\xi)-B(\omega)| &= |b_{n+1}+b_{n+2}+\ldots+b_{n+r}| \\ &< K(\lambda_{n+1}-\lambda_n+\lambda_{n+2}-\lambda_{n+1}+\ldots+\lambda_{n+r}-\lambda_{n+r-1}) \\ &= K(\lambda_{n+r}-\lambda_n) \\ &\leqslant K(\xi-\lambda_n). \end{aligned}$$

Further,

$$\begin{aligned} B(\xi) &\geqslant B(\omega)-|B(\xi)-B(\omega)| \\ &> H\omega-K(\xi-\lambda_n) \\ &= K\left(\lambda_n+\frac{H\omega}{K}-\xi\right). \end{aligned}$$

Hence, for

$$\lambda_n \leqslant \xi \leqslant \lambda_n+M\omega,$$

where

$$M = H/K,$$

we have

$$\frac{B(\xi)}{\xi^2} > \frac{K(\lambda_n+M\omega-\xi)}{\omega^2(1+M)^2};$$

and so

$$\int_{\lambda_n}^{\lambda_n+M\omega} \frac{B(\xi)}{\xi^2}\,d\xi > \frac{K^3}{\omega^2(H+K)^2}\int_{\lambda_n}^{\lambda_n+M\omega}(\lambda_n+M\omega-\xi)\,d\xi$$

$$= \frac{KH^2}{2(H+K)^2}.$$

That this should be true for values of ω exceeding all limit is incompatible with the convergence of

$$\int^{\infty} \frac{B(\xi)}{\xi^2}\, d\xi.$$

The theorem is therefore proved.*

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* There is also a small slip on p. 177 of Mr. Hardy's paper. The inequality

$$\frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < \frac{K}{\kappa+1}(\xi-\omega)\,\xi^{\kappa},$$

should be

$$\frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < K(\xi-\omega)\,\xi^{\kappa}.$$

The succeeding argument remains valid if the factor $\kappa+1$ is omitted in the three places where it occurs in a denominator.