Σ_0 becomes the Hessian of $f(u)$, 2Φ becomes $f(u)$, and Σ_1 vanishes; whence

$$\rho' = -\frac{H(u)}{f(u)}.$$

Again, the differential equation (A) becomes

$$2\frac{du}{\sqrt{f(u)}} = \frac{d\rho'}{\sqrt{4\rho'^3 - I\rho' - J}},$$

which is the general elliptic differential reduced to Weierstrass's canonical form by Hermite's substitution.

A very complete discussion of this subject will be found in chap. v. of Greenhill's *Elliptic Functions*, from the analytic side, and I offer this geometrical view of the matter only in order to show how it depends on the contravariants of the conics U and V.

On the Fundamental Theorem of Differential Equations. By W. H. YOUNG. Received and communicated January 9th, 1902.

The fundamental theorem of the modern theory of differential equations is Cauchy's existence theorem, dealing with the existence and uniqueness of a set of integrals satisfying given initial conditions, and the holomorphic character of the solution. This theorem has been stated in very precise language, and proved in various ways, by Picard and Painlevé, but some doubt has been expressed as to whether their proofs are rigorous. It has been suggested, in fact, that it has not been conclusively demonstrated that the holomorphic solution is unique even in the simplest case which can arise.

In the following note* it is proposed to give a brief account of the theorem in question, and to examine an example which has been put forward as typical of a large class of cases where the theorem fails.

* The note is substantially what I wrote in January, 1899, but did not publish, as I expected Painlevé or Picard to take the matter up. The former has now done so, but his treatment is too general to appeal to English readers. Indeed he does little more than repeat at length his previous definitions.

Taking the case of a single differential equation of the first order and first degree, Cauchy's theorem may be thus stated:—

Given a differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

and a pair of values a, b for which the function f is holomorphic, there exists one, and only one, integral of the equation which approaches the value b when x approaches the value a, and this integral is holomorphic.*

We add as gloss: When we say that "y approaches the value b when x approaches the value a," we mean that, if we consider small circles of radii ϵ and σ round the points b of the y plane and a of the x plane, and make x move up towards a along any path which, from and after a certain fixed point, enters and remains in the circle of radius σ, then y moves along a certain curve which from and after a certain fixed point (to be determined) enters and remains in the circle of radius ϵ; and this is to be true however small σ and ϵ are taken, provided only they are fixed. Such curves may be called "curves of approach" to the point in question. The student of precise theory of functions will recognise that this is, in fact, only a gloss and not a hypothesis, since in treating of the value of a function at a point, or the behaviour of a function in the neighbourhood of a point, we are working in the small (*im Kleinen*), and the geometry we can apply is only differential geometry. If we did not work in the small, we should find ourselves constantly hampered by quite unnecessary complications. For instance, starting from a point on a Riemann's surface, and considering such a commonplace function as an Abelian function with three or more periods having a given value at that point, the value at a neighbouring point is determinate, because we are working "in the small." But, if we allow the moving point to wander about at will on the Riemann's surface between the initial and final point, the Abelian function can, by a proper choice of path, be made to have a value at the final point as near as we please to any given value whatever. Thus the student who refused to work "in the small" would be tempted to say that such a function had an essential singularity or "point of indeterminateness" at every point of the Riemann's surface! The uselessness of such a mode of expression is self-evident.

* By "holomorphic at a point c" we mean that the function is developable in a series of positive integral powers of $(x-c)$ in the neighbourhood of that point.

Notice that there are virtually three independent statements in the theorem :—

(i.) That a solution exists, satisfying the given conditions, which is holomorphic at the point $x = a$.

(ii.) That this solution is the only one which is holomorphic at the point.

(iii.) That there is no non-holomorphic solution satisfying the conditions.

The first two statements are due to Cauchy ; proofs of the third have been given by Picard and Painlevé.

We confine our attention to (iii.), referring to any of the many treatises on analysis for an account of (i.) and (ii.).

Picard's elegant proof is given in his *Traité d'Analyse*, and is of this nature: He assumes the truth of (i.) and (ii.) for partial differential equations and deduces the truth of (iii.) for our case.

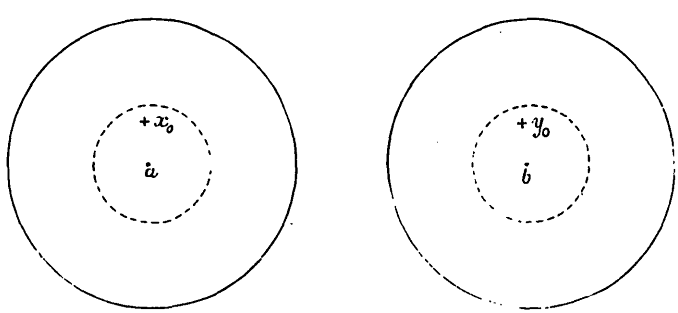
Painlevé has given more than one proof. His first proof is given on pp. 19, 20 of his Stockholm lectures. It is somewhat concisely stated, and its force may therefore be easily missed. It requires a knowledge of the domain in which the Cauchy integral is proved to be holomorphic.

Suppose $f(x, y)$ holomorphic for

$$|x-a| \leqslant r \quad \text{and} \quad |y-b| \leqslant \rho, \tag{2}$$

and let M be the modulus of its upper limit in this region ; then it is known that Cauchy's integral is holomorphic within a circle centre a and radius λ, where

$$\lambda = r\,(1-e^{-\rho/2Mr}). \tag{2'}$$



It at once follows that this is a Cauchy integral which has the value y_0 when $x = x_0$, and y_0 being *any* points within the circles (dotted in figure) centres a and b and radii $\frac{1}{2}r$ and $\frac{1}{2}\rho$, and is holomorphic within a circle of radius $\frac{1}{2}\lambda$ and centre x_0. Call these dotted circles C_1 and Γ_1.

We can now give the substance of Painlevé's proof.

Let L be a path of approach* to the point $x = a$.

Is it possible that for every such path, or for any one such path, a solution $y = \phi(x)$ exists satisfying the given couditions, and non-holomorphic at the point $x = a$?

In virtue of the hypothesis with regard to L on p. 235, we can evidently find a point H of L so near a that—

(1) Its distance from a is $< \frac{1}{2}\lambda$, and *a fortiori* $< \frac{1}{2}r$.

(2) The corresponding point y is within Γ_1.

(3) These statements are true for all points on L between a and this point H.

Assume:

(4) That among all these points one at least exists for which $\phi(x)$ is holomorphic.

(*I.e.*, assume that the portion of L between H and a is not singular for ϕ.)

Call this point x_0 and denote the corresponding y point by y_0. Then ϕ is the Cauchy integral corresponding to these values; for it is holomorphic at $x = x_0$.

Now, since $f(x, y)$ is holomorphic for

$$|x - x_0| < \tfrac{1}{2}r, \quad |y - y_0| < \tfrac{1}{2}\rho,$$

it follows that the Cauchy integral corresponding to the pair of values (x_0, y_0) is holomorphic at all points of the circle whose radius is $\frac{1}{2}\lambda$ and centre x_0. But, by (1), p. 237, this circle includes the point a. Hence ϕ is holomorphic at the point a. Hence it is the Cauchy integral at the point a, and the hypothesis that it could be non-holomorphic there will not hold.

The only doubt then that can exist is:—

Is there perhaps a function of x which satisfies the differential equation and which is non-holomorphic at all points of an arc of a curve of approach to a including its extremity (or asymptotic point) a, and which tends towards b as x moves on the curve towards a?

If this be the case, then a solution exists which has the value b for

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* The path L may have the point a as asymptotic point, and its length may increase indefinitely as x tends to a.

one, at any rate, of its values at a, and which is not the Cauchy integral.

Are there any functions of a single complex variable which are non-developable at all points of a certain curve lying in the region of existence of the function, and which none the less have a differential coefficient at each of these points?

We defer for a subsequent paper the discussion of this question. It will be seen shortly that it is not necessary to answer it for the purpose in hand if we make use of Painlevé's second method of treatment of the problem.

Let (x_0, y_0) be any pair of values in the domain in which $f(x, y)$ is holomorphic, and let the corresponding Cauchy integral be

$$y = \phi(x, y_0, x_0). \tag{A}$$

Let $(\bar{x}, \bar{y})$ be any pair of values of (x, y) satisfying equation (A), and therefore lying in the region of existence of ϕ, and in the domain in which f is holomorphic. Then, by the uniqueness of the Cauchy holomorphic solution, we get the same Cauchy integral (A) if we start with $(\bar{x}, \bar{y})$ instead of with (x_0, y_0), *i.e.*,

$$y = \phi(x, \bar{y}, \bar{x}),$$

is identical with (A). But (A) is satisfied by $x = x_0$, $y = y_0$; therefore

$$y_0 = \phi(x_0, \bar{y}, \bar{x}),$$

or, since $\bar{y}$, $\bar{x}$ were any pair, $y_0 = \phi(x_0, y, x)$ (A′)

is another way of writing (A).

Now give x_0 a definite numerical value a, and write

$$u = \phi(a, y, x), \tag{B}$$

where y of course no longer satisfies (A). Then u is a function of x and y which assumes the value b, when

$$x = a \quad \text{and} \quad y = b.$$

Change the dependent variable in our fundamental equation from y to u. We know that (B) is algebraically equivalent to

$$y = \phi(x, u, a). \tag{B′}$$

Hence our fundamental equation (1) becomes

$$\frac{\partial}{\partial x}\{\phi(x, u, a)\} + \frac{du}{dx}\frac{\partial}{\partial u}\{\phi(x, u, a)\} = f\{\phi(x, u, a), x\}.$$

But, since (A) satisfies (1), we have *identically*

$$\frac{\partial}{\partial x}\phi(x, y_0, x_0) = f\{\phi(x, y_0, x_0)\, x\}$$

for all values of x, y_0, and x_0; and therefore

$$\frac{\partial}{\partial x}\phi(x, u, a) \equiv f\{\phi(x, u, a), x\}.$$

Hence $$\frac{du}{dx}\frac{\partial}{\partial u}\phi(x, u, a) = 0;$$

and therefore, since ϕ certainly contains u, we have

$$\frac{du}{dx} = 0 \qquad (1')$$

as the new form of our equation (1).

Moreover, since when $x = a$, $y = b$ we have $u = b$, it follows that our initial conditions are

$$u = b, \quad \text{when} \quad x = a, \qquad (2')$$

or, more strictly, that, as x approaches a, u has to approach b in the usual way.

The obvious solution of equation (1′) subject to condition (2′) is

$$u = b;$$

whence, by (B′), $$y = \phi(x, b, a);$$

in other words, we are led by virtue of the uniqueness of the holomorphic solution back to equation (A).

Has the equation (1′) another solution?

In other words—

Can a function S be found and an arc Q (having x_0 for its limiting point) such that at all points of a certain domain in which Q lies the function S has a differential coefficient which is zero, but which along the arc Q is not developable?

This is, of course, impossible; for it is known that no function exists which has its differential coefficient zero at all points of a domain, except a constant.

For any point P of the domain can be joined to a fixed point A of the domain by means of a continuous line, consisting of a finite number of straight lines lying entirely within the domain. Since along each of these straight lines the differential coefficient is zero, we know, by the *Mengenlehre*, that the function is constant, and has therefore the same value at P as at A.

Thus the theorem (iii.) is completely proved.

We now proceed to discuss in some detail the typical example to which we referred in our opening remarks.

Consider the differential equation*

$$\frac{dy}{dz} = \frac{y^2}{y^2+z-y}, \tag{1}$$

and let us seek for an integral, other than the trivial one $y = 0$, which satisfies the condition of approaching the value 0 as z approaches the value c. According to Cauchy's theorem no such integral exists. It has been contended,† however, that a non-holomorphic integral exists which may be constructed as follows.

Take the complete integral of (1),

$$z = y + ae^{-1/y}, \tag{2}$$

where a is a constant of integration, and, writing

$$-\frac{1}{y} = -L + 2x\pi i, \tag{3}$$

where L is any particular chosen logarithm of $\frac{a}{c}$, let us give to x a series of integral values each numerically greater than the preceding; ultimately y and $z-c$ may in this way both be made as small as we please, and therefore it is asserted the non-holomorphic integral (2) satisfies the condition required (and this for all values of the arbitrary constant).

To examine the question completely, let all the quantities involved be considered as complex. We have three planes: an x-plane, a y-plane, and a z-plane. The x-plane is connected with each of the other planes by an analytical transformation. *The transformation of the x-plane into the y-plane is a simple inversion, given by the equation* (3), *the centre of inversion in the y-plane being the origin, and in*

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* See Forsyth, *Theory of Differential Equations*, Part 2, Vol. II., p. 81, where the equation in question is discussed. The notation adopted is only slightly different. We have written z for $\frac{ab-\alpha\beta}{b}z$, y for βy, a for $\frac{A}{b}$, c for $\frac{ab-\alpha\beta}{\beta}c$, and, for obvious reasons, x for k. We have omitted Forsyth's first differential equation, which is connected with the other by the substitution

$$w = -\frac{\alpha}{\beta}z + y.$$

A slight oversight with respect to the constant A is amended.

† Forsyth, *loc. cit.*

the x-plane the point $\frac{L}{2\pi i}$. Using polar coordinates referred to the centres of inversion r, θ in the x-plane, and ρ, ϕ in the y-plane, the equation (3) is equivalent to the two equations

$$\left.\begin{aligned} r\rho &= \frac{1}{2\pi} \\ \phi &= \frac{\pi}{2} - \theta \end{aligned}\right\}. \tag{3'}$$

Corresponding then to the *interior* of a small circle, centre the origin and radius η, in the y-plane, we have the *exterior* of a large circle, centre $\frac{L}{2\pi i}$ and radius $\frac{1}{2\pi\eta}$, in the x-plane.

Corresponding to that part of any curve of approach to the origin in the y-plane which lies inside the small circle in the y-plane, we have a portion of a curve exterior to the large circle in the x-plane and going off to infinity.

Again, from equations (2) and (3), we have

$$z = y + ce^{2x\pi i}, \tag{4}$$

where

$$y = \frac{1}{L - 2x\pi i}. \tag{3}$$

These equations define a transformation of the x-plane into the z-plane, and of the z-plane into the x-plane.

We have then to consider whether it is possible to find corresponding sets of values of z and y, the one set forming a curve of approach to the point c in the z-plane, and the other a curve of approach to the point O in the y-plane. If so, then, by definition, quantities ϵ and η can be found, such that, if we draw a circle centre c and radius ϵ in the z-plane, and a circle with centre O and radius η in the y-plane, both curves, on entering these circles, approach and never recede from the respective centres.

Assume the possibility of the point at issue, and inquire as to the region or regions of the x-plane to which the x-point is confined when the z-point has entered its circle of radius ϵ. By hypothesis the y-point must then have entered the corresponding circle radius η.

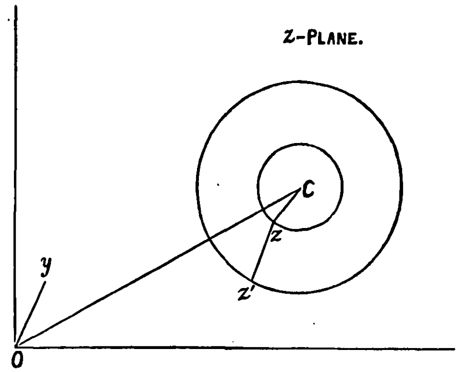
Let us introduce an auxiliary point z' given by

$$z' = ce^{2x\pi i}, \tag{5}$$

so that

$$z = y + z'. \tag{5'}$$

This shows us that (the y- and z-planes being taken for the moment as coinciding) the straight line $z'z$ is equal and parallel to the radius vector of y, and hence that z' must lie within a circle centre c and radius $\epsilon+\eta$.



Writing then $$x = x_1+ix_2,$$

and considering the length of the radius vector of z', we have

$$|c|-\epsilon-\eta \leqslant \text{mod of } z' \leqslant |c|+\epsilon+\eta,$$

or $$|c|-\epsilon-\eta \leqslant |c|\,e^{-2x_2\pi} \leqslant |c|+\epsilon+\eta.$$

This shows that

I. x_2 *must be a small quantity lying between the limits* $-\lambda_1$ *and* λ *where* λ_1 *and* λ_2 *are determinate small real positive quantities, viz.,*

$$-2\pi\lambda_1 = \textit{real logarithm of } 1-\frac{\epsilon+\eta}{|c|},$$

$$2\pi\lambda_2 = \textit{real logarithm of } 1+\frac{\epsilon+\eta}{|c|},$$

and evidently tend towards zero when either or both of the small quantiti ϵ *and* η *tend towards zero.*

Considering, on the other hand, the vectorial angle of z', we s that (since z' must lie between the tangents from the origin to th circle of radius $\epsilon+\eta$) :

II. $2x_1\pi$ *must differ from an integral multiple of* 2π *by a quanti less in absolute magnitude than the small angle—call it* $2\pi\sigma$*—wh sine is* $\dfrac{\epsilon+\eta}{|c|}$.

From I. and II. it follows that the point x must always, wh the z-point has entered the circle of radius ϵ, and the y-point the corresponding circle of radius η, lie in one or other of certain small parallelograms of the x-plane, the breadth of each parallelogram being 2σ, and the height $(\lambda_1+\lambda_2)$, and each parallelogram containing one (and of course only one) integral point of the real axis.

Further, as the x-point, and therefore also the y-point, moves along its curve of approach, both points enter circles of smaller and smaller radii. In other words, in the above investigation, we may diminish ϵ and η. It follows therefore, in accordance with I. and II., that the x-point remains in its parallelogram and moves towards the integral point belonging to it.

But, as remarked on p. 241 in connexion with equation (3′), the x-point must move in such a way as to remain outside the large circle in the x-plane whose centre is $\frac{L}{2\pi i}$, and whose radius is $\frac{1}{2\pi\eta}$, and therefore constantly increases as η decreases. The figure shows that this is inconsistent with the point remaining inside its parallelogram.

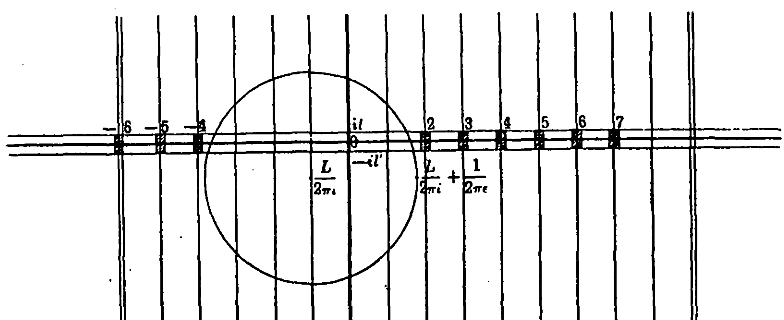


FIG. 1.

It has thus been conclusively shown by a *reductio ad absurdum* that, as the z-point moves along a curve of approach to c, the y-point cannot move along a corresponding curve of approach to its origin.

If we could make the z-point move *discontinuously*, we could make the x-point jump from parallelogram to parallelogram. The points of the parallelograms suitable for this purpose are of course the integral points of the real axis; for all the other points come to lie outside the parallelograms as ϵ and η diminish. If we make x jump along these integral points, z and y each jump along the points of an *abzählbare Menge* having the desired goals as points of condensation.

But our investigation shows that, if we attempt to draw any curve of approach through one of these *Mengen*, the corresponding curve will pass through the other *Menge*, but between each pair of points it will recede to a finite distance from the goal and come back again; so that it is not a "curve of approach."

It is of interest to take a particular curve of approach in the one plane, and see what happens to the curve corresponding to it in the other plane, and why it is not a curve of approach. The simplest case is obtained if we consider a curve of approach in the y-plane and inquire how the z-point must move.

Let us move along the real axis in the x-plane, and therefore in the y-plane move along the circle (Fig. 2)

$$y_1^2+y_2^2-\frac{y_1}{L_1}=0,$$

where $$y=y_1+iy_2,$$

and L_1 is the real part of L.

The point z will simultaneously describe the spiral

$$z=ce^{2x\pi i}+\frac{1}{L-2x\pi i},$$

where x is now a real parameter.

This spiral lies entirely outside or inside the circle

$$z=ce^{2x\pi i} \quad \text{or} \quad z_1^2+z_2^2=|c|^2,$$

according as $\frac{1}{L_1}$ is positive or negative.

Fig. 3 shows the spiral for x positive. For x negative we have to reflect the curve in the real axis. The dotted circle is the

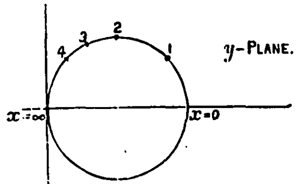


FIG. 2.

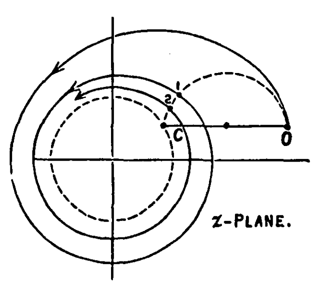


FIG. 3.

circle to which the whorls of the spiral approximate. The dotted semicircle cuts out the positive integral points. The negative integral points would be cut out by the lower semicircle of the same circle whose equation is

$$(z-c_1)^2+(z_2-c_2)^2=\frac{z_1-c_1}{L_1}.$$

Non-uniform Convergence, and the Integration of Series. By E. W. HOBSON, Sc.D., F.R.S. Read and received January 9th, 1902.

If the terms of an infinite series are functions of a real variable which are all continuous in a given interval taken as the field of the variable, and the series converges at every point of the interval, it is well known that the convergence of the series must be non-uniform in the neighbourhood of any point at which the sum of the series is discontinuous, but that non-uniformity of convergence in the neighbourhood of a point does not necessarily imply discontinuity of the sum at that point. In the case in which the sum of the series is continuous throughout the whole interval, the most general possible distribution of points of non-uniform convergence of the series has been obtained by Osgood.* He has shown that the points at which the measure of non-uniform convergence† exceeds an arbitrarily fixed positive number form a closed aggregate, non-dense in the given interval, and that the points at which the convergence is uniform form an everywhere dense aggregate.

In a very remarkable memoir,‡ Baire has proved that the sum of a series such as has been described is at most a point-wise discontinuous function, *i.e.*, in any sub-interval points can be found at which the function is continuous. The distribution of points of non-uniform convergence, which is of fundamental importance in the question of the integration of the series, was, however, not considered by Baire. In the present paper, it is shown by a method on the lines of that of Baire, that the most general dis-

* See his paper, "Non-uniform Convergence and the Integration of Series," *Amer. Jour. of Math.*, Vol. XIX., 1897.

† This term will be explained in the course of the paper.

‡ "Sur les fonctions de variables réelles," *Annali di Math.*, Vol. III., 1899.