ON THE INTEGRATION OF FOURIER SERIES

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1. In a companion paper * presented to the Society, I have discussed the circumstances under which one of the factors of the integrand of a definite integral may be replaced by a series, which is then multiplied term-by-term by the other factor of the integrand, and the new series so obtained integrated term-by-term. It was there pointed out that the series substituted for the factor in question need not have that factor for sum, and it need not, indeed, even converge. The more novel of the theorems obtained related, in fact, precisely to this case. Now, the series of Fourier can only be shown to converge, even in general, in a very restricted class of cases. On the other hand, what may conveniently be called *the integrated Fourier series*, obtained from the original series by replacing each term by its simplest indefinite integral, necessarily converges uniformly to the integral of the function associated with the original series. In accordance therefore with one of the theorems of the paper cited, we are at liberty to substitute for any function, which is summable in Lebesgue's manner, constituting one of the factors of the integrand of an integral, its corresponding Fourier series, whether this last converges or not, provided only that the remaining factor of the integrand is a function of bounded variation. This result is, however, the only one that follows immediately from the theorems of the paper cited. There are, none the less, a number of interesting and important cases, distinct from this one, in which the mode of procedure in question may be adopted. It is the main object of the present paper to set these forth in order. As far as I know, this has not been attempted in any existing text-book or memoir. In Hobson's Treatise the subject is not directly touched on, and earlier writers were necessarily ignorant of the considerations from which these results follow.

* W. H. Young, "On the Theory of the Application of Expansions to Definite Integrals" (1910), p. 463 *infra*.

Occasion is also taken to obtain a couple of simple results concerning what I have elsewhere called the semi-integrability above and below of a Fourier series, both *per se* and when multiplied by another function.

In general, it is of considerable importance in the theory to know whether an oscillating series is semi-integrable above and below. It enables us, for example, to assert with confidence that the upper and lower functions of the integrated series are integrals, a circumstance of which I have taken advantage in a previous paper * on "Trigonometrical Series" presented to this Society. In the case of Fourier series, however, we know that the integrated series, as already mentioned, converges uniformly, and, we may add, converges uniformly to an integral. Consequently the fact that a Fourier series is semi-integrable above and below is not in itself of the same importance. Similar remarks apply to the theorems concerning semi-integrability above and below when a Fourier series is multiplied by a function of the usual type, viz., a function of bounded variation. I have therefore not thought it necessary to linger on the present occasion over the properties in question. The other results stated in the paper are of use to the physicist and applied mathematician in so far as they may from time to time have occasion to make use of Fourier Series. They are as follows:—

The process of evaluating an integral referred to above is always allowable if—

(i) both factors are such that their squares are summable;

(ii) either of the factors is of bounded variation, while the other factor is summable;

(iii) the factor which is replaced by its Fourier series is summable, the other factor is bounded, and the series got by integrating the final series term-by-term between the desired limits converges;

(iv) the factor which is replaced by its Fourier series has a Harnack-Lebesgue integral, the other factor has bounded variation, and the series got by integrating the final series term-by-term converges.

Moreover, in the last two cases, if the series got by integrating the final series term-by-term does not converge, when summed in the ordinary way, it necessarily does so when summed in the Cesàro way, and the sum so obtained is the value of the integral.

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* W. H. Young, "On the Conditions that a Trigonometrical Series should have the Fourier Form" (1910), *supra*, p. 421.

The first of the above results is due to Fatou,* though stated by him in a slightly different manner. Of the results in (ii), one follows from the paper first cited; it is, however, much simpler to obtain both simultaneously by means of the theory of Fourier series, as is here done. These results are so important, and so easily obtained, that it is difficult to believe that they have not yet been stated; but I have not been able to find any such statement or any reference to such statement. With regard to (iii), it is practically certain that Lebesgue has definitely formulated it in his own mind, as only a single step is required to obtain it from his recently published work.† Of (iv), however, I have not seen any trace.

It should be added that I have extended these results to the case when one or both of the limits of integration are infinite, so far as this is possible without the introduction of additional irrelevant conditions.

One remark may be made in conclusion. In the early days of the history of Fourier series term-by-term integration was carried out without any *arrière pensée*, both in the case of Fourier series and when the series considered was that obtained by multiplying the Fourier series term-by-term by another function. A later school then arose and objected, with justice, to the uncritical nature of the investigations of their predecessors and the consequently inconclusive character of the results obtained. This earlier school of critics demanded that the series should be shewn to be uniformly convergent before being integrated term-by-term. Hence arose the almost interminable discussions with regard to the uniformity or the non-uniformity of the convergence of Fourier series. Finally, in the light of the most recent researches, it is seen that the considerations as to the uniformity of the convergence play a comparatively unimportant part in the theory of the integration of series, and that, in the case of Fourier series, the presence or absence of such uniformity has little or no bearing on the subject at all. Indeed, as is evident from the theorems stated above, even the question of the convergence of the series is frequently of secondary interest.

2. Fatou has shewn that if the a's and b's are the Fourier constants of $f(x)$, and A's and B's those of $g(x)$, then, provided only that the squares of f and g are summable,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \tfrac{1}{2}a_0 A_0 + \Sigma(a_n A_n + b_n B_n). \tag{1}$$

* P. Fatou, "Séries trigonométriques et Séries de Taylor" (1905), *Acta Math.*, Bd. 30, p. 335.

† *Loc. cit.*, *infra*, p. 456.

Hence we immediately have the following theorem:—

THEOREM 1.—*If $f(x)$ and $g(x)$ are such that their squares are summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of the series obtained by multiplying the Fourier series of either term-by-term by the other, that is*

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, g(x)\, dx = \int_c^z f(x)\, g(x)\, dx,$$

*where $s_n(x)$ is the sum of the first n terms of the Fourier series of $f(x)$, provided, in the case when the length of the interval of integration exceeds 2π, the function to be replaced by its Fourier series is periodic.**

First let (c, z) be any interval inside the interval $(-\pi, \pi)$, and let us replace $g(x)$ by the function which agrees with it inside this interval and is zero outside it. The new function is, of course, like the old, one whose square is summable, but its Fourier constants now take a new form, in which the limits of integration are c to z. Thus, the above equation (1) becomes

$$\int_c^z f(x)\, g(x)\, dx = \tfrac{1}{2}a_0 \int_c^z g(x)\, dx + \Sigma \left(a_n \int_c^z g(x) \cos nx\, dx + b_n \int_c^z g(x) \sin nx\, dx\right). \tag{2}$$

This proves the required result for any limits of integration between $-\pi$ and π.

Next suppose $f(x)$ periodic, and $g(x)$ not to be so. Then the equation (1) plainly holds for the limits $(\pi, 3\pi)$, provided the A's and B's be supposed obtained by integrating between π and 3π. A similar remark applies for the integral $(3\pi, 5\pi)$, and so on. Hence, adding all such equations, we see that equation (1) holds when the limits of integration on the left-hand side are any positive or negative odd multiples of π, provided only the limits of integration in the A's and B's are correspondingly adjusted. Hence equation (2) also holds as it stands, whatever the limits of integration c and z may be, so long as they are finite.

3. Before proving the next theorem, we must first show that *the Fourier series of a function of bounded variation has the property of what*

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* Throughout the present paper the word "periodic" is, for shortness, used to denote that the function to which the term is applied has the period 2π.

I have called bounded convergence *—in other words, that, where the convergence is not uniform, the non-uniformity has never infinite measure—that is, the peak and chasm functions are finite, and therefore bounded.

Let $f(x)$ be the function of bounded variation whose Fourier series is under consideration. Then the sum of the first $(2n+1)$ terms of its Fourier series is

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}[f(x+2z)+f(x-2z)]\frac{\sin mz}{\sin z}\,dz,$$

where m is the odd integer $(2n+1)$. It is clearly then sufficient to show that this expression, regarded as a function of the ensemble (x, m) has, when taken numerically, a finite upper bound. Further, we need only prove the statement for one of the two integrals into which the given integral naturally splits up—say, that in which $f(x+2z)$ occurs. Finally, we shall lose no generality if we suppose $f(x+2z)$ to be a positive monotone function of z, since the function of bounded variation $f(x+2z)$ may be expressed in the form $P(x+2z)-N(x+2z)$, where $P(u)$ and $N(u)$ are both positive monotone increasing functions, being the positive and negative variations of $f(u)$.

But if $f(x+2z)$ is a positive monotone increasing function of $2z$, the same is true of

$$zf(x+2z)/\sin z \quad \left(0 \leqslant z \leqslant \frac{\pi}{2}\right);$$

hence, by the Second Theorem of the Mean,

$$\int_0^{\frac{1}{2}\pi}\frac{zf(x+2z)}{\sin z}\,\frac{\sin mz}{z}\,dz = \frac{\pi}{2}f(x+\pi-0)\int_0^{k}\frac{\sin mz}{z}\,dz,$$

where k is some quantity between 0 and $\frac{1}{2}\pi$.

But the integral on the right-hand side lies numerically between 0 and π, and the factor multiplying it is a bounded function of x. Hence the required result follows.

4. We are now able to prove the following theorem :—

THEOREM 2.—*If one of the two functions $f(x)$ and $g(x)$ has bounded variation, while the other is summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of*

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* The limiting function is $\frac{1}{2}\{f(x+0)+f(x-0)\}$, and differs therefore from $f(x)$ only at a countable set of points. Hence, the limiting funct may, whenever it occurs under the sign of integration, be taken to be $f(x)$.

the series obtained by multiplying the Fourier series of either term-by-term by the other function, provided only that, if the length of the interval of integration exceeds 2π, the function whose Fourier series is employed is periodic.

To prove this we suppose the Fourier series of that of the two functions to be chosen which is a function of bounded variation. It then follows, by a known theorem * taken in conjunction with the result of § 3, that term-by-term integration between any finite limits of integration is allowable.

Denoting the function of bounded variation by $f(x)$, we then have, integrating between the limits $-\pi$ and π, the equation (1) of § 2. But this equation is clearly symmetrical with respect to $f(x)$, $g(x)$ and their respective Fourier constants, and holds, therefore, equally well whether it is $f(x)$ or $g(x)$, which is the function of bounded variation. Hence also the further reasoning of that article applies here equally whether it be $f(x)$ or $g(x)$, which is the function of bounded variation. Thus, both parts of our theorem are true.

5. In the last article we have not, however, used all the information at our disposal, and we can, accordingly, go somewhat farther than was possible in the first theorem. In the first place, the theorem quoted in my companion paper applies equally whether the limits of integration are finite or infinite, provided only, of course, that the function which is summable in that theorem is summable in the whole infinite interval, or, more accurately, that its absolute value has a Lebesgue integral in the whole of that interval.

With this understanding, the equation (2) of § 2 holds when either c or z is infinite, provided $g(x)$ is the function which is summable in the infinite interval in question, while $f(x)$ is the function of bounded variation, the boundedness of the variation referring to any and every finite interval.

In the second place, it is worth noting that the convergence of the integrated series is in accordance with the same theorem uniform. We

* For a proof of this theorem, which states that if $s_n(x)$ converges boundedly to its limit $f(x)$ as n increases, and $g(x)$ possesses a Lebesgue integral, $\int s_n(x)\,g(x)\,dx$ converges to $\int f(x)\,g(x)\,dx$, see my companion paper on "The Application of Expansions to Definite Integrals," § 4.

had no information enabling us to make this statement with regard to the corresponding series in Theorem 1.

Next suppose that $g(x)$ is a function of bounded variation in the whole infinite interval, and that throughout that interval it is summable; then $g(x)$ certainly has zero as unique limit when x increases indefinitely, for otherwise it could not possess an integral over the infinite interval. The function $f(x)$ is any summable periodic function, and we shall assume, in the first instance, that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

This assumption makes $\int_c^z f(x)\,dx$ periodic, and ensures that the Fourier series of $f(x)$ is deficient of its first term, so that the integrated series having only periodic terms, its n-th partial summation, say $S_n(x)$, is periodic. Hence, since when z lies between $-\pi$ and π, the integrated Fourier series in question converges boundedly to $\int_c^z f(x)\,dx$, the same will be true wherever z lies in the infinite interval of integration.

We are now able to apply Theorem 6 of the companion paper, the second of the sufficient conditions for integration over an infinite interval having been shewn to be satisfied. This proves that for such a function $f(x)$ the integral of $f(x)\,g(x)$ over the infinite interval may be evaluated by term-by-term integration in the manner desired.

If the condition imposed in the first instance on $f(x)$ is not satisfied, it is satisfied by $h(x)$, where

$$h(x) = f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = f(x) - \tfrac{1}{2}a_0.$$

Hence, since the n-th partial summation of the Fourier series of $h(x)$ is $s_n(x) - \frac{1}{2}a_0$, we have, by what has just been proved,

$$\operatorname*{Lt}_{n=\infty} \int_c^{\infty} \left[s_n(x) - \tfrac{1}{2}a_0\right] g(x)\,dx = \int_c^{\infty} \left[f(x) - \tfrac{1}{2}a_0\right] g(x)\,dx,$$

whence, since $g(x)$ has, by hypothesis, an integral over the infinite interval,

$$\operatorname*{Lt}_{n=\infty} \int_c^{\infty} s_n(x)\,g(x)\,dx = \int_c^{\infty} f(x)\,g(x)\,dx,$$

which proves that *Theorem 2 holds for an infinite interval of integration.*

6. In the next two theorems the function whose Fourier series is to be used is no longer a function of bounded variation. We therefore naturally

use Cesàro's mode of summation, instead of summing directly. The Cesàro partial summation is the mean of the first n partial summations, that is,

$$s'_n(x) = \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x)\left(1-\frac{1}{n}\right)+\dots$$
$$+[a_{n-1}\cos(n-1)x + b_{n-1}\sin(n-1)x]\left(1-\frac{n-1}{n}\right)$$
$$= \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(y)\left\{\frac{\sin\frac{1}{2}n(x-y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dy,$$

the latter expression being Fejér's integral, and it converges to $f(x)$ always except at a set of content zero, so that, for our purposes, we may work with $f(x)$ as the limiting function. Moreover, when $f(x)$ is bounded, it is easily seen that the convergence is bounded, for, if $f(x)$ is numerically less than K, Fejér's integral is less than K times what it becomes when we put $f(x) = 1$, in which case the Fourier series reduces to its first term, which is unity, so that the Fejér integral has also the value unity.

7. Corresponding exactly to Theorem 2, we have, therefore, using the Cesàro method of summation, the following theorem :—

THEOREM 3.—*If one of the two functions $f(x)$ and $g(x)$ is bounded, while the other is summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of the series obtained by multiplying the Fourier series of either term-by-term by the other function, provided this integrated product series converges, and assuming, if the interval of integration exceed 2π in length, that the function whose Fourier series is employed is periodic.*

To prove this * let us take the Fourier series of that one of the two functions which is bounded, say $f(x)$. Then, as mentioned in the preceding article, the Cesàro partial summation

$$s'_n = \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x)\left(1-\frac{1}{n}\right)+\dots$$
$$+[a_{n-1}\cos(n-1)x + b_{n-1}\sin(n-1)x]\left(1-\frac{n-1}{n}\right)$$

converges boundedly to a function, which, for our purposes, may be taken to be $f(x)$.

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* The argument is that used by Lebesgue, *Les Intégrales Singulières*, pp. 107 *seq.*, so far as it is here applicable.

Hence, by the theorem used in proving Theorem 2,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \underset{n=\infty}{\mathrm{Lt}}\, \frac{1}{\pi}\int_{-\pi}^{\pi} s_n'(x)\, g(x)\, dx$$

$$= \underset{n=\infty}{\mathrm{Lt}} \left\{ \tfrac{1}{2}a_0 A_0 + (a_1 A_1 + b_1 B_1)\left(1-\frac{1}{n}\right) + \dots \right.$$

$$\left. + (a_{n-1} A_{n-1} + b_{n+1} B_{n+1})\left(1-\frac{n-1}{n}\right) \right\},$$

the quantities A_i and B_i being the Fourier constants of $g(x)$, as the quantities a_i and b_i are the Fourier constants of $f(x)$. The expression on the extreme right of the equality (1) whose limit is to be taken, is the Cesàro partial summation of the series

$$\tfrac{1}{2}a_0 A_0 + \sum_{n=1} (a_n A_n + b_n B_n),$$

so that, if this latter series converges, its sum will be the limit in question, and will accordingly be equal to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

In other words, the equation (1) of § 2 holds in this case.

We have now only to repeat the argument of § 2, with the slight modification that we work in the first instance with the Cesàro partial summations instead of summing directly. Thus, the required theorem follows when the limits of integration lie in the first instance between $-\pi$ and π, and then, provided $f(x)$ is periodic for any finite interval of integration.

The symmetry of equation (1) of § 2 shews that, having been shewn to hold when $f(x)$ is bounded and $g(x)$ is summable, it holds equally when the rôles of these two functions are exchanged; the subsequent argument of § 2 then still holds, so that the theorem, having been shewn to be true in the former case, is equally true when $g(x)$ is bounded and $f(x)$ is summable. Thus, the theorem has been completely proved for every finite interval of integration.

COR.—*Even if the integrated product series does not converge directly, it will necessarily converge when summed in the Cesàro manner, and the sum so obtained is the value of the integral required.*

8. It is evident that we may extend the theorem of the preceding article to the case in which the interval of integration is infinite when $f(x)$

is the bounded function and $g(x)$ is the summable function. Then, as Theorem 2 of the companion paper holds equally whether the interval of integration is finite or infinite, the process carried out in the preceding article is still valid. Hence the extension is allowable.

9. We now come to the last of our theorems, which is concerned with the expression for the integral of the product of two functions. This is as follows :—

THEOREM 4.—*If one of the functions $f(x)$ and $g(x)$ is of bounded variation, while the other possesses a Harnack-Lebesgue integral, and if the series obtained by multiplying the Fourier series of either term-by-term by the other function converges when integrated between the chosen finite limits, then the integral of the product of the two functions between these limits is equal to the sum of the integrated series, provided only that, if the length of the interval of integration exceeds 2π, the function whose Fourier series is employed is periodic.*

It is clearly sufficient to prove the theorem when $f(x)$ is a function of bounded variation and $g(x)$ possesses a Harnack-Lebesgue integral. We then have, by the theory of Fejér's integral,

$$\tfrac{1}{2}[f(x+0)+f(x-0)] = \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{\sin \frac{1}{2}n(t-x)}{\sin \frac{1}{2}(t-x)}\right)^2 dt$$

$$= \operatorname*{Lt}_{n=\infty} \int_{x-c}^{x+c} f(t) \left(\frac{\sin \frac{1}{2}n(t-x)}{\sin \frac{1}{2}(t-x)}\right)^2 dt.$$

Put $t = x+u$, and we get

$$\tfrac{1}{2}[f(x+0)+f(x-0)] = \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{-c}^{c} f(u+x) \left(\frac{\sin \frac{1}{2}nu}{\sin \frac{1}{2}u}\right)^2 du$$

$$= \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{0}^{c} [f(u+x)+f(x-u)] \left(\frac{\sin \frac{1}{2}nu}{\sin \frac{1}{2}u}\right)^2 du$$

$$= \operatorname*{Lt}_{n=\infty} F(x, n) = \operatorname*{Lt}_{n=\infty} \{F_1(x, n)-F_2(x, n)\},$$

where $F_1(x, n)$ and $F_2(x, n)$ are monotone functions of x. Also the left-hand side is equal to $f(x)$, except at a set of content zero. Hence, by a theorem in the Harnack integration of series,* whatever function $g(x)$ may be,

* See § 3 of the companion paper, p. 466 *infra*.

provided only it possess a Harnack-Lebesgue integral, we have, since the sequence is certainly bounded,

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \int_{-\pi}^{\pi} g(x)\, F(x, n)\, dx.$$

Retracing our steps, with the modifications required by the presence of the factor $g(x)$, we accordingly have

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{-\pi}^{\pi} dx\, g(x) \int_{-\pi}^{\pi} f(t) \left(\frac{\sin \frac{1}{2}n(t-x)}{\sin \frac{1}{2}(t-x)}\right)^2 dt.$$

From this point the argument is precisely the same as in the proof of the preceding theorem.

We have, of course, the same corollary as in the preceding article.

COR.—*Even if the series obtained in the manner described does not converge in the ordinary manner, it will necessarily converge when summed in the Cesàro manner, and the sum so obtained is the value of the integral required.*

10. We have now to extend the theorem of the preceding article to the case of an infinite interval of integration.

When $f(x)$ is the function of bounded variation, the result of the preceding article remains true, since the argument used at the end of § 3 of the companion paper for an infinite interval of integration is again valid. Hence the theorem holds for an infinite interval of integration when $f(x)$ is a function of bounded variation.

Next let $g(x)$ be of bounded variation throughout the infinite interval, and let $f(x)$ possess a Harnack-Lebesgue integral in the same interval. When the limits c and z of integration lie in the closed interval $(-\pi, \pi)$, we know* that the integrated Fourier series of $f(x)$ is the Fourier series of $\int_c^z f(x)\, dx$, so that the Cesàro partial summation $s_n(x)$ of that series converges boundedly (in fact uniformly)† to $\int_c^z f(x)\, dx$. Hence, as in § 2 of the present paper, if

$$\int_{-\pi}^{\pi} f(x)\, dx = 0,$$

these facts remain true wherever the points c and z may lie in the infinite

* "On the Conditions that a Trigonometrical Series should have the Fourier Form," § 3, p. 425, *supra*.

† H. Lebesgue, *Sur les Intégrales singulières*, p. 89.

ERRATUM.

Vol. 9, p. 459.—The argument taken from p. 467 of the companion paper does not apply. When the intervals concerned are finite, but then only, the behaviour of a periodic function of bounded variation can be inferred from that of a monotone function.

integral of integration. In this case, *provided $g(x)$ has zero for unique limit when x moves off to infinity*, we may apply Theorem 6 of the companion paper, the second of the sets of sufficient conditions there given being satisfied. This proves the theorem for an infinite interval of integration in this case.

If, however, $\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ is different from zero, the above shews that the theorem is true when $f(x)$ is replaced by $f(x)-\frac{1}{2}a_0$. That is,

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^\infty [s_n(x)-\tfrac{1}{2}a_0]\,g(x)\,dx = \int_c^\infty [f(x)-\tfrac{1}{2}a_0]\,g(x)\,dx,$$

from which, *provided $g(x)$ possesses an integral* over the infinite interval*, the required result follows, viz.,

$$\underset{n=}{\mathrm{Lt}} \int_c^\infty s_n(x)\,g(x)\,dx = \int_c^\infty f(x)\,g(x)\,dx.$$

Here $s_n(x)$ is the partial summation, or the Cesàro partial summation, according as the Fourier series of $f(x)$, after term-by-term multiplication by $g(x)$ and integration, converges directly, or only when summed in the Cesàro manner.

11. In the previous part of the paper we have been concerned more with the function to which a given Fourier series belongs than with the Fourier series itself. If we fix our attention on the latter, in the case in which it does not converge in general or at all, it will have an upper function and a lower function, which may or may not be bounded, or even summable. If, however, we include for the moment among the summable functions those functions which give, when the Lebesgue process of dealing with unbounded integrands is applied, infinite values with definite signs, we can enunciate the following theorem :—

THEOREM 5.—*The upper and lower functions of every Fourier series are always summable, and the Fourier series itself is always semi-integrable, both above and below in the extended sense.*

In fact, the application of Poisson's method of dealing with Fourier series leads, with the use of Abel's Theorem, to the result that, wherever the function to which the Fourier series belongs is the differential coefficient of its integral, it lies between the upper and lower functions of the

* This will be an ordinary improper integral, since a function of bounded variation is, in every finite interval, integrable in the Riemann manner.

Fourier series. This is therefore true, except at a set of content zero. But the function to which the Fourier series belongs has necessarily a Lebesgue integral, for otherwise the Fourier series would not exist. Hence the upper and lower functions have Lebesgue integrals, in the sense explained above, and the integral of the function to which the series belongs is intermediate to these two integrals in value. Hence, bearing in mind that the integrated Fourier series converges to the integral of the function to which the series belongs, the result enunciated follows.

COR.—*If the upper and lower functions of a trigonometrical series be both bounded, the series is necessarily semi-integrable both above and below.*

This follows by the theorem* that such a series is necessarily a Fourier series; the result may also be extended to the case when the points at which the upper and lower functions of the series are unbounded are countable.

12. The result of the preceding article has been obtained by means of the known fact that a Fourier series, when integrated, converges to the integral of the function to which the Fourier series belongs. Similarly, by using Theorem 2 of the present paper, we can prove the following:—

THEOREM 6.—*The series obtained by multiplying the Fourier series of a function $f(x)$ term-by-term by a function $g(x)$ of bounded variation has its upper and lower functions always summable, and is always semi-integrable both above and below in the extended sense.*

First suppose $g(x)$ to be positive, and let $l(x)$ and $u(x)$ denote the lower and upper functions of the Fourier series. Then, by the argument of the preceding article, we have, except at most at a set of content zero,

$$l(x) \leqslant f(x) \leqslant u(x);$$

and therefore, multiplying by $g(x)$ and using Theorem 2, we have

$$\int l(x)\, g(x)\, dx \leqslant \operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx \leqslant \int u(x) g(x) dx,$$

which proves the theorem in this case, and proves also that the integrals $\int l(x) g(x) dx$ and $\int u(x) g(x) dx$ exist, if we include respectively $-\infty$ and $+\infty$ as possible values of these integrals, for the limit in the middle necessarily exists and is finite, being equal to the integral of the product of a summable function by a function of bounded variation.

* "On the Conditions that a Trigonometrical Series should have the Fourier Form," § 6, p. 427, *supra*.

Next suppose $g(x)$ not to be positive, and let $g_1(x)$ be equal to $g(x)$ when $g(x)$ is positive, and be zero elsewhere, while $g_2 = -g(x)$ when $g(x)$ is negative and is zero elsewhere. Then $g_1(x)$ and $g_2(x)$ are functions of bounded variation, so that the inequalities (1) hold both for $g_1(x)$ and for $g_2(x)$. These give at once,

$$\int [l(x)\, g_1(x) - u(x)\, g_2(x)]\, dx = \underset{n=\infty}{\mathrm{Lt}} \int s_n(x)\, g(x)\, dx$$
$$= \int [u(x)\, g_1(x) - l(x)\, g_2(x)]\, dx,$$

the first and last members of the inequality necessarily existing, at least to the extended sense above mentioned.

Now $l(x)\, g_1(x) - u(x)\, g_2(x) = l(x)\, g(x)$ when $g(x)$ is positive,

and $= u(x)\, g(x)$ when $g(x)$ is negative.

It is therefore the lower function of the succession $g(x)\, s_n(x)$. Similarly $u(x)\, g_1(x) - l(x)\, g_2(x)$ is the upper function. Hence the theorem follows, as enunciated.

COR.—*We have, of course, a corollary precisely similar to that of the preceding article.*