

Electrical Theorems in Connection with Parallel Cylindrical Conductors

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XI. *Electrical Theorems in Connection with Parallel Cylindrical Conductors.* By ALEXANDER RUSSELL, M.A., D.Sc.

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THE electrostatic problem of two conducting spheres having given electric charges and surrounded by a uniform dielectric has been completely solved. The capacity and potential coefficients of the spheres, the density of the surface charges, the potential at any point of the field and the mutual force between them can be computed in all cases without difficulty, to any required degree of accuracy.*

Unfortunately the apparently much simpler problem of two parallel cylindrical conductors has not been completely solved. It will be helpful, therefore, to give simplified proofs of the solutions already obtained and show how they can be extended. It is customary to make the assumptions that the cylinders are infinitely long, and that the other conductors of the system are at a great distance away from them. In this case it is shown that the three capacity coefficients are connected by two simple relations which determine the limits between which they must lie. In certain cases also their approximate values can be found. It is shown how the solution of the electrostatic problem enables us to solve the analogous problem of two parallel cylindrical conductors carrying high-frequency currents. In certain cases the exact values can be obtained of the current density on the surface, the inductance coefficients and the force between the conductors. In other cases useful approximations are given.

The Electric Force at any Point due to Two Parallel Cylindrical Conductors having Equal and Opposite Charges of Electricity.

Let us first consider the case of two thin parallel wires cutting the plane of the paper perpendicularly at A and B (Fig. 1). We shall suppose that they are infinitely long, that they have charges $+q$ and $-q$ per unit length respectively, and that the dielectric is air. To find the force at any point P in the plane of the paper join AP and BP and make the angle BPC equal to the angle BAP . Then PC will be the direction of the resultant force and its magnitude will equal $4qr/(r_1r_2)$ where $AB=2r$, $AP=r_1$, and $BP=r_2$. To prove

* Russell's "Alternating Currents," Vol. I., Ch. VIII., 2nd Ed.

this, we notice that the component force PQ due to the charge on the A wire $=2q/r_1$, the component force PS due to the charge on the B wire $=2q/r_2$, and the angle PSR = the angle APB . Hence the triangles APB and PSR are similar, and so the angle BPC equals the angle PAB . Also since $PR/PS = AB/r_1$, PR which is the resultant force, F at P , is given by

$$F = \frac{4qr}{r_1 r_2} \quad \dots \dots \dots (1)$$

The potential v at the point P is the sum of the potentials due to the charges on the two wires, and hence

$$v = 2q \log r_2 - 2q \log r_1 = 2q \log (r_2/r_1) \quad \dots \dots (2)$$

Since the angle BPC equals the angle BAP , the tangent PC to this circle at P gives the direction of the resultant force. Hence the tangent at every point of this circle and

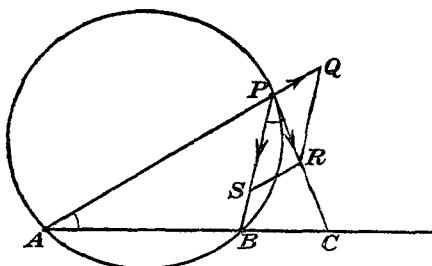


FIG. 1.

PR is the resultant force at any point P . The angle BPC equals the angle BAP . Hence every circle through A and B is a line of force.

therefore also at every point of any circle through A and B will give the direction of the resultant force. It follows that every circle which passes through A and B is a line of force. Again, if with centre at a point C on AB produced and with radius equal to $(CA \cdot CB)^{\frac{1}{2}}$ we describe a circle, any radius CP of this circle will be tangential to the circular line of force through ABP , and hence this circle will cut all the lines of force at right angles. It is, therefore, the cross-section of an equipotential surface. We see that all the equipotential surfaces round A and B are cylindrical in shape, that their axes are parallel and that the centres of their cross-sections lie on AB or BA produced. The equipotential surfaces surrounding A , we shall call the A cylinders, and those surrounding B the B cylinders. It is to be noticed that if we

take any A cylinder and any B cylinder, A and B are the inverse points of their circular cross-sections.

By Green's theorem we can suppose that any A cylinder and any B cylinder become conductors without affecting the distribution of the flux external to them. Similarly, if two of the A cylindrical surfaces become conducting the distribution of the flux between them will not be affected. The surface density σ also at any point on the surface of these conducting cylinders is given by $F/(4\pi)$, where F is the electric force at the point. Hence the surface density at the point P of the equipotential surface is given by

$$\sigma = \frac{qr}{\pi r_1 r_2} \dots \dots \dots (3)$$

Let the circles in Fig. 2 represent the sections of an A and a B cylinder respectively, and let their radii be a and b . Then, since A and B are the inverse points of the two circles,

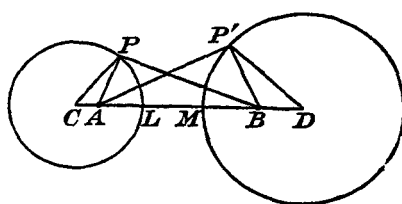


FIG. 2.

A and B are the inverse points of the two circles.

$CA \cdot CB = a^2$ and $DB \cdot DA = b^2$, where C and D are the centres of the two circles. The circle described on AB as diameter is the section of the smallest equipotential surface. It cuts the cylinders at right angles. If $2r$ be the diameter of this circle and if we denote the distance CD between the axes of the cylinders by c , we have

$$c^2 r^2 = 4s(s-a)(s-b)(c-s) \dots \dots \dots (4)$$

where

$$2s = a + b + c, \text{ (l.c., ante p. 164).}$$

Since $r_2/r_1 = BL/AL$, it is easy to show that

$$r_2/r_1 = BC/a = a/CA.$$

Hence, by (2), $v_1 = 2q \log (BC/a) = 2qa, \dots \dots \dots (5)$

where v_1 is the potential of the A cylinder and $a = \log (BC/a)$.

Hence, $\varepsilon^a = BC/a$ and $\varepsilon^{-a} = CA/a$,
 and thus $2r = CB - CA = a(\varepsilon^a - \varepsilon^{-a})$,
 and so $r = a \sinh \alpha$.

Hence,
$$\alpha = \sinh^{-1}(r/a) = \log_e \{r/a + (1 + r^2/a^2)^{\frac{1}{2}}\} \quad (6)$$

It will be seen that α can be readily computed by (4) and (6).

We see from (5) that whatever the radius of the B cylinder may be, we have

$$q/v_1 = 1/(2\alpha) \quad (7)$$

Similarly, if v_2 be the potential of the B cylinder whose radius is b , we have

$$q/v_2 = -1/(2\beta) \quad (8)$$

since all the B cylinders have a charge of $-q$ per unit length. The value of β is given by

$$\beta = \sinh^{-1}(r/b) = \log_e \{r/b + (1 + r^2/b^2)^{\frac{1}{2}}\} \quad (9)$$

We deduce from (7) and (8) that

$$\frac{q}{v_1 - v_2} = \frac{1}{2(\alpha + \beta)} \quad (10)$$

This is the formula for the capacity between the two cylinders—the capacity usually wanted in practice. Formulæ (7) and (8), however, are useful and instructive.

If the angle PCA in Fig. 2 be denoted by θ , we find by (3) that the surface density σ at P is given by

$$\sigma = \frac{qr}{\pi r_1 r_2} = \frac{q}{2\pi a} \cdot \frac{\sinh \alpha}{\cosh \alpha - \cos \theta} \quad (11)$$

Similarly, the surface density at any point P' on the B cylinder will be given by

$$\sigma = -\frac{q}{2\pi b} \cdot \frac{\sinh \beta}{\cosh \beta - \cos \varphi} \quad (12)$$

where φ is the angle $P'DA$ (Fig. 2).

If we write $\omega = \alpha + \beta$, we have

$$q/(v_1 - v_2) = 1/(2\omega), \quad (13)$$

and

$$\cosh \omega = (c^2 - a^2 - b^2)/(2ab). \quad (14)$$

Let us now consider the case of a cylinder inside a hollow conducting cylinder, the axes of the cylinders being parallel but not necessarily coincident. Let the radius of the inner cylinder be a , the inner radius of the outer cylinder be b ,

and let c be the distance between their axes. Then if q be the charge per unit length on the inner cylinder, $-q$ will be the induced charge per unit length on the inner side of the outer cylinder. The electric field between them, and therefore the potential difference between them, will be identically the same as that between two A cylinders whose radii are a and b respectively, the distance between their axes being c . The potentials v_1 and v_2 of these two cylinders when one of the B cylinders has a charge $-q$ per unit length, will be given by

$$v_1 = 2qa \quad \text{and} \quad v_2 = 2q\beta;$$

and hence

$$\frac{q}{v_1 - v_2} = \frac{1}{2(a - \beta)}. \quad \dots \quad (15)$$

This equation therefore gives us the capacity between the inner and the outer cylinder. If we denote $a - \beta$ by ω_1 we easily find that

$$\cosh \omega_1 = (a^2 + b^2 - c^2)/(2ab). \quad \dots \quad (16)$$

The surface densities at points on the surface of the inner cylinder, and on the inner surface of the outer cylinder, are given by (11) and (12) respectively.

Since q is the charge per unit length of the cylinder, the radius of which is a , it follows at once from (11) that

$$\int_0^\pi \frac{\partial \theta}{\cosh \alpha - \cos \theta} = \frac{\pi}{\sinh \alpha}. \quad \dots \quad (17)$$

This equation can easily be verified by the calculus.

The centroid line of the distribution of the electrical charge on the cylinder whose radius is a (Fig. 2) will from symmetry lie in the plane passing through the axes of the two cylinders. If \bar{x} be its distance from O , we have

$$\bar{x}q = 2 \int_0^\pi \sigma a^2 \cos \theta \partial \theta = qa \varepsilon^{-a}, \quad \text{by (17),}$$

$$\text{and thus} \quad \bar{x} = a \varepsilon^{-a} = OA. \quad \dots \quad (18)$$

Similarly, the centroid line of the charge on the B cylinder will pass through B .

It will be seen, therefore, that the inverse lines of the cylinders which pass through A and B respectively are the centroid lines of the electrical charges spread over the A and B cylindrical surfaces.

The Electrostatic Attraction Between any Pair of the A or B Cylinders is the Same.

Let us first suppose that the cylinders are external to one another. The attractive force on the *A* cylinder must remain constant if the electrostatic field surrounding it does not alter in magnitude or direction. The attraction on the *A* cylinder, therefore, is the same as if the *B* cylinder were a thin wire through *B* having a charge $-q$ per unit length. Since the attractions of the cylinders are equal and opposite, the required attraction will be equal to the attraction on this thin wire through *B*. But the attraction on this thin wire depends only on the number and direction of the lines of induction connected with it. It is independent, therefore, of the size of the *A* cylinder. Hence the attraction F per unit length between the two cylinders equals that between two thin wires having charges q and $-q$ per unit length respectively, and coincident with their centroid lines. It is therefore given by

$$F = \frac{2q^2}{\lambda \cdot AB} = \frac{q^2}{\lambda r}, \quad \dots \dots \dots (19)$$

where λ is the inductivity of the dielectric.

If K be the capacity per unit length between the two cylinders and W the energy stored up in the dielectric, we have

$$W = q^2 / (2K) = q^2 \omega / \lambda.$$

Hence, $F = \partial W / \partial c = (q^2 / \lambda) (\partial \omega / \partial c).$

Comparing this equation with (19), we see that

$$\frac{\partial \omega}{\partial c} = \frac{1}{r}. \quad \dots \dots \dots (20)$$

This equation can also be easily proved directly from the equations (4), (6) and (9), given above.

Similarly, when the sections of the cylinders are both *A* circles (Fig. 2), so that one of them is inside the other,

$$F = -\frac{q^2}{\lambda} \frac{\partial \omega_1}{\partial c} = \frac{q^2}{\lambda r}. \quad \dots \dots \dots (21)$$

The attraction, therefore, between any pair of cylinders which have the same inverse lines is the same.

The Stream Function.

Let u be the stream function corresponding to the potential function v . If u be zero at L (Fig. 3), we have

$$\begin{aligned} u &= \int_0^\theta \sigma a d\theta \\ &= \frac{q}{2\pi} \int_0^\theta \frac{\sinh \alpha}{\cosh \alpha - \cos \theta} d\theta, \\ &= \frac{q}{\pi} \tan^{-1} \frac{\tan (\theta/2)}{\tanh (\alpha/2)} \\ &= \frac{q}{2\pi} (\pi - \varphi), \quad \dots \dots \dots (22) \end{aligned}$$

where θ is the angle PCL and φ is the angle APB .

It follows, as we have shown above, that every line of force such as PQ (Fig. 3) is part of a circle which passes through

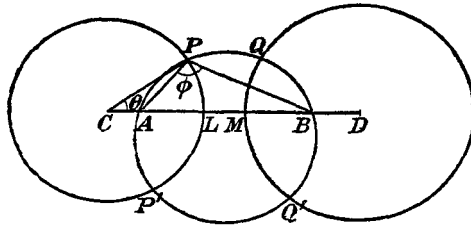


FIG. 3.

Every circle through A and B cuts the circles PLP' and QMQ' at right angles.

A and B . The capacity k_1 of the field per unit length between the portions PP' and QQ' of the cylinders intercepted by any one of these circles is given by

$$k_1 = \lambda \frac{\frac{q}{2\pi}(\pi - \varphi) + \frac{q}{2\pi}\varphi}{v_1 - v_2} = \lambda \frac{\frac{q}{2}}{2q\omega} = \frac{\lambda}{4\omega}. \quad \dots \dots (23)$$

The capacity k_1 is therefore equal to half of the capacity between the cylinders.

If R be the resistance per unit length between the two cylinders supposed of infinite conductivity, and if ρ be the resistivity of the medium between them, we have

$$KR = \rho\lambda/(4\pi),$$

where K is the corresponding capacity. Hence

$$R = \rho\omega/(2\pi), \quad \dots \dots \dots (24)$$

and the resistance R_1 of the medium between PP' and QQ' (Fig. 3) would be given by

$$R_1 = \rho\omega/(4\pi)^* \dots\dots\dots (25)$$

Similarly, if K' denote the thermal conductance between the two cylinders, and k' be the thermal conductivity, we have

$$K' = (4\pi/\lambda)k'K = 2\pi k'/\omega, \dots\dots\dots (26)$$

and the thermal conductance between PP' and QQ' (Fig. 3) will be $K'/2$.

Cylinders with Unequal Charges.

Since the sum of the electric charges in a self-contained system must always be zero, it follows that if the sum of the charges on the cylinders be not zero there must be other charged conductors in the system. To fix our ideas we shall suppose that the axis of the cylinder whose radius is a , is also the axis of a very large hollow cylinder, the inner radius of which is c . If v_3 be the potential of this outer cylinder which surrounds the other two, we have

$$v_1 = p_{11}q_1 + p_{12}q_2 + p_{13}q_3,$$

$$v_2 = p_{21}q_1 + p_{22}q_2 + p_{23}q_3,$$

and

$$v_3 = p_{31}q_1 + p_{32}q_2 + p_{33}q_3,$$

where $p_{11}, p_{12}, \dots\dots$ are the values of Maxwell's potential coefficients per unit length. If $v_1 = v_2 = v_3$ there can be no charges on the inner cylinders, and thus both q_1 and q_2 are zero. It follows, therefore, that $p_{13} = p_{23} = p_{33}$. Hence

$$v_3 = p_{33}(q_1 + q_2 + q_3).$$

We see that $1/p_{33}$ is the outside capacity per unit length of the hollow cylinder, and by taking this cylinder large enough it can be made as large as ever we please. Hence if we assume that the cylinders are surrounded by a co-axial hollow cylinder at a great distance from them, we can write $p_{13} = p_{23} = p_{33} = 0$.

Our equations simplify to

$$v_1 = p_{11}q_1 + p_{12}q_2, \dots\dots\dots (27)$$

and

$$v_2 = p_{22}q_2 + p_{12}q_1, \dots\dots\dots (28)$$

In the particular case when $q_1 = -q_2 = q$, we have by (7) and (8)

$$v_1 = 2q\alpha, \text{ and } v_2 = -2q\beta.$$

* Cf. G. Carey Foster and O. J. Lodge, "Proc. Phys. Soc.," Vol. I., 115, 1875.

Hence, substituting in the simplified equations, we get

$$p_{11}=p_{12}+2\alpha, \text{ and } p_{22}=p_{12}+2\beta \quad . \quad (29)$$

Solving the equations for q_1 and q_2 in terms of v_1 and v_2 we get

$$q_1=k_{11}v_1+k_{12}v_2 \text{ and } q_2=k_{22}v_2+k_{12}v_1,$$

where $k_{11}=p_{22}/\Delta$, $k_{22}=p_{11}/\Delta$, $k_{12}=-p_{12}/\Delta$, and $\Delta=p_{11}p_{22}-p_{12}^2$. The coefficients of v_1 and v_2 are called the capacity coefficients, and by considering the case when $q_1=-q_2$, we see that

$$k_{11}=\frac{1}{2\alpha}+\frac{\beta}{\alpha}k_{12}, \text{ and } k_{22}=\frac{1}{2\beta}+\frac{\alpha}{\beta}k_{12}. \quad . \quad (30)$$

We also have

$$k_{11}=\frac{1}{2\alpha+2\beta p_{12}/(2\beta+p_{12})}, \text{ and } k_{12}=\frac{-1}{2(\alpha+\beta)+4\alpha\beta/p_{12}}.$$

Since p_{12} is positive, we see that

k_{11} must lie in value between $1/(2\alpha)$ and $1/(2\alpha+2\beta)$,

k_{22} must lie in value between $1/(2\beta)$ and $1/(2\alpha+2\beta)$,

and that $1/(2\alpha+\beta)$ is a superior limit to $-k_{12}$. For instance, if a be very great compared with b , α will be very small compared with β , and thus k_{22} will equal $1/(2\beta)$ very approximately.

From the equations (27), (28) and (29) we deduce that

$$\begin{aligned} v_1-v_2 &= 2(q_1\alpha-q_2\beta) \\ &= (q_1+q_2)(\alpha-\beta)+(q_1-q_2)(\alpha+\beta). \quad . \quad (31) \end{aligned}$$

Whatever may be the values of the charges on the conductors this relation always holds. When $q_1+q_2=0$, it gives the capacity between the conductors, and when $q_1=q_2=q$ we have

$$\frac{q}{v_1-v_2}=\frac{1}{2(\alpha-\beta)}. \quad . \quad . \quad . \quad (32)$$

Comparing this with equation (15) we see that when the charges on the cylinders are equal, the ratio of the charge to the difference of potentials is the same as for a cylinder of radius a and an enveloping cylinder of inner radius b , provided that the circular cross-sections have the same inverse points in the two cases. The energy stored in the field, however, in the latter case is $(v_1-v_2)^2/4(\alpha-\beta)$, whilst in the former case it is $(v_1^2-v_2^2)/4(\alpha-\beta)$.

When the charge on the B cylinder is zero

$$v_1-v_2=2q_1\alpha, \quad . \quad . \quad . \quad . \quad (33)$$

In this case $q_1/(v_1 - v_2)$ is a constant ($1/2\alpha$) which can be easily found. It is interesting to notice that the value of this constant is the same whichever of the B cylinders is chosen.

A Cylinder Inside a Hollow Cylinder, their Axes being Parallel.

If q_1 , q_2 , and v_1 , v_2 be the charges and potentials of the cylinder (radius a) and sheath (inside radius b) respectively, it is easy to show that

$$q_1 = \frac{1}{2(\alpha - \beta)} v_1 - \frac{1}{2(\alpha - \beta)} v_2, \quad \dots \quad (34)$$

and

$$q_2 = \left\{ C + \frac{1}{2(\alpha - \beta)} \right\} v_2 - \frac{1}{2(\alpha - \beta)} v_1, \quad \dots \quad (35)$$

where C is the capacity per unit length of the outer cylinder with respect to external bodies.

Hence also,

$$v_1 = \left\{ \frac{1}{C} + 2(\alpha - \beta) \right\} q_1 + \frac{1}{C} q_2, \quad \dots \quad (36)$$

and

$$v_2 = \frac{1}{C} q_2 + \frac{1}{C} q_1. \quad \dots \quad (37)$$

Hence when C can be found, we know the complete solution. We see that

$$k_{11} = -k_{12} = \frac{1}{2(\alpha - \beta)} = C_0 \text{ (say)}, \quad \dots \quad (38)$$

and

$$k_{22} = C + C_0. \quad \dots \quad (39)$$

Also

$$p_{11} = \frac{1}{C} + \frac{1}{C_0}, \quad p_{22} = p_{12} = \frac{1}{C}. \quad \dots \quad (40)$$

If W denote the electrostatic energy,

$$\begin{aligned} W &= \frac{1}{2} p_{11} q_1^2 + p_{12} q_1 q_2 + \frac{1}{2} p_{22} q_2^2 \\ &= \frac{(q_1 + q_2)^2}{2C} + \frac{q_1^2}{2C_0}. \quad \dots \quad (41) \end{aligned}$$

Hence we deduce the following three theorems :—

(a) If $q_1 + q_2$ is constant, W is a minimum when q_1 is zero, and, therefore, by (34) when $v_1 = v_2$.

(b) If $q_1 = a$ constant, W is a minimum when $q_2 = -q_1$, and in this case by (37), $v_2 = 0$.

(c) If $q_2 = a$ constant, W is a minimum when

$$q_1 = - \{C_0/(C+C_0)\} q_2, \text{ and then by (36), } v_1 = 0.$$

$$\begin{aligned} \text{We also have } W &= \frac{1}{2}k_{11}v_1^2 + k_{12}v_1v_2 + \frac{1}{2}k_{22}v_2^2 \\ &= \frac{1}{2}C_0(v_1 - v_2)^2 + \frac{1}{2}Cv_2^2 \quad . \quad . \quad . \quad (42) \end{aligned}$$

Hence—

(a) If $v_1 - v_2$ is constant, W is a minimum when $v_2 = 0$, and therefore when $q_2 = -q_1$.

(b) If v_2 is constant, W is a minimum when $v_1 = v_2$. In this case $q_1 = 0$.

(c) If v_1 is constant, W is a minimum when $v_2 = \frac{C_0}{C+C_0}v_1$.

In this case $q_2 = 0$. A study of these theorems is instructive.

The force F per unit length between the cylinders when λ is the inductivity of the dielectric is given by

$$F = -\frac{q_1^2}{\lambda r} = -\frac{\lambda(v_1 - v_2)^2}{4(\alpha - \beta)^2 r} \quad . \quad . \quad . \quad (43)$$

The equilibrium is unstable when the cylinders are co-axial, both when the charge on the inner cylinder or when the potential difference between them is maintained constant. In the former case they move so that the potential energy stored in the field is diminished and in the latter case so that it is increased.

Approximate Values of the Electrostatic Coefficients for Parallel Cylinders.

As a preliminary to finding approximate values for the potential coefficients in equations (27) and (28), let us consider the case of a concentric main, the radius of the inner cylinder being a , and the inner radius of the outer being d . The potential v at a point P between the cylinders distant r from the common axis is given by

$$v = 2q_1 \log \frac{d}{r}, \quad . \quad . \quad . \quad . \quad (44)$$

where q_1 is the charge per unit length on the inner cylinder. We see that for a given value of q_1 the greater the value of d , the greater will be the value of the coefficients of q_1 in this equation. If d is infinite, v is also infinite. This follows

because the work done in taking unit charge from the infinite cylinder to infinity is infinite.

Let us now consider the case of a very long charged prolate spheroid, the other conductor being a confocal spheroid (practically a sphere) at infinity. If l be the length of the axis of the spheroid and v be the potential at a point P on the equatorial plane at a distance r from the axis, we have

$$v=2q_1 \log \frac{l}{r} (45)$$

very approximately, where q_1 is the charge on the surface intercepted between any two planes perpendicular to the axis and at unit distance apart.

Formulæ (44) and (45) prove that the actual value of the potential at a point near a charged cylinder even when it is at a great distance away from the ends of the cylinder depends both on the length of the cylinder and on the location of the necessary complementary charge. We notice, however, that the electric force at the point P is to a high degree of approximation independent both of the length and of the position of the complementary charge. Similarly if we have two parallel cylinders at a great distance away from the conductors carrying their complementary charge, we infer that the values of the potential coefficients will vary both with the length of the cylinders and with the position of the other conductors. We are led to infer also that the force per unit length between the cylinders is practically independent of their length and of the location of the complementary charge.

Considering now the case of the concentric main, let us suppose that d is very large and that we have a thin uncharged wire parallel to the axis of the main at a distance c from it, where c is great compared with a but very small compared with d . Since the field is practically undisturbed by the presence of this thin wire, its potential v_2 will be given by

$$v_2=2q_1 \log \frac{d}{c}.$$

Hence we see that

$$p_{12}=2 \log \frac{d}{c}, (46)$$

and therefore by (29)

$$p_{11}=2\alpha+2 \log \frac{d}{c}; \text{ and } p_{22}=2\beta+2 \log \frac{d}{c} . . (47)$$

If we make d infinite, the potential coefficients become infinite, but the capacity coefficients are given by

$$k_{11}=k_{22}=-k_{12}=\frac{1}{2(a+\beta)}=\frac{1}{2\log(c^2/ab)} \quad . \quad . \quad (48)$$

approximately.

The Electrostatic Forces Between the Cylinders.

From (27), (28), (46) and (47) we get

$$v_1=2\left(a+\log\frac{d}{c}\right)q_1+2\log\frac{d}{c}\cdot q_2 \quad . \quad . \quad (49)$$

and

$$v_2=2\log\frac{d}{c}\cdot q_1+2\left(\beta+\log\frac{d}{c}\right)\cdot q_2 \quad . \quad . \quad (50)$$

Hence if λ be the inductivity, the electromagnetic energy W stored in the field is given by

$$W\lambda=\frac{1}{2}\left(2a+2\log\frac{d}{c}\right)q_1^2+2\log\frac{d}{c}\cdot q_1q_2+\frac{1}{2}\left(2\beta+2\log\frac{d}{c}\right)q_2^2 \quad (51)$$

$$=aq_1^2+\beta q_2^2+\log\frac{d}{c}\cdot (q_1+q_2)^2. \quad . \quad . \quad . \quad (52)$$

Hence

$$\begin{aligned} F &= \frac{\partial W}{\partial c} \\ &= \frac{1}{\lambda} \left\{ q_1^2 \frac{\sinh a \cosh \beta}{r \sinh \omega} + q_2^2 \frac{\cosh a \sinh \beta}{r \sinh \omega} - \frac{(q_1+q_2)^2}{c} \right\} \quad (53) \\ &= \frac{1}{2\lambda c^2 r} \{ q_1^2 (c^2 + b^2 - a^2 - 2cr) + q_2^2 (c^2 + a^2 - b^2 - 2cr) - 4q_1 q_2 cr \} \quad (54) \end{aligned}$$

By Kelvin's theorem (l.c. *ante*, p. 150) the conductor must move so as to diminish the electrostatic energy W . Hence when F is positive, that is when W increases with c the force is attractive, and when F is negative it is repulsive. We may write equation (54) in the form

$$F = \frac{1}{2\lambda c^2 r} (Aq_1 - Bq_2)(Aq_1 - Cq_2), \quad . \quad . \quad (55)$$

where

$$A^2 = c^2 + b^2 - a^2 - 2cr,$$

$$B = (2cr + N)/A, \quad C = (2cr - N)/A,$$

and

$$N^2 = (b^2 - a^2)^2 + c^2(4r - c).$$

We see that when q_1 and q_2 are of opposite signs, F is always positive and, therefore, the force is always attractive. When, however, q_1 and q_2 have the same sign F is attractive, zero or repulsive according as q_1/q_2 is greater than B/A or less than C/A , equals either of these quantities or lies in value between them.

Particular cases are of interest. If $q_1 = -q_2 = q$ (54) reduces to (19), which is exactly true. If, in addition $a = b$, we have

$$F = \frac{2q^2}{\lambda(c^2 - 4a^2)^{\frac{1}{2}}} \dots \dots \dots (56)$$

When $q_1 = q_2 = q$, the formula reduces to

$$F = -\frac{(4r - c)q^2}{\lambda cr} \dots \dots \dots (57)$$

In the particular case when the B cylinder is so thin that we may write $b = 0$, and therefore $2cr = c^2 - a^2$, we get

$$F = \frac{2q_2 \{a^2 q_2 - (c^2 - a^2) q_1\}}{c(c^2 - a^2)} \dots \dots \dots (58)$$

If q_2 is zero, F vanishes. When q_1 is zero the attractive force is given by

$$F = -\frac{2a^2 q_2^2}{c(c^2 - a^2)} \dots \dots \dots (59)$$

We see from (58) that when q_1 and q_2 have the same sign the force is attractive, when q_2/q_1 is greater than $(c^2 - a^2)/a^2$;

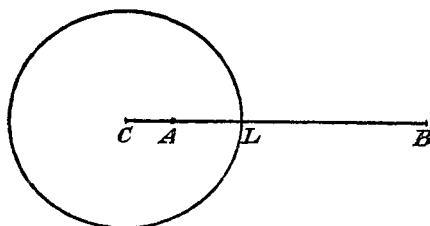


FIG. 4.

$$CA.CB = CL^2.$$

The wire which is the image of the wire through B passes through A .

it vanishes when q_2/q_1 has this value and it is repulsive, when q_2/q_1 is less than $(c^2 - a^2)/a^2$.

A direct geometrical proof of (58) can easily be given by the method of images as follows:—

Let C and B (Fig. 4) be the centres of the cross-sections of the cylinder and wire respectively. If the charge per unit

length on the wire be q_2 , its image in the cylinder will be a wire through A having a charge $-q_2$ per unit length, where $CA \cdot CB = a^2$. Hence if the total charge on the cylinder be q_1 per unit length we can replace the cylinder by two parallel wires through C and A respectively, which have charges $q_1 + q_2$ and $-q_2$ per unit length respectively. Hence we see at once that the force F is given by

$$F = \frac{2q_2(q_1 + q_2)}{c} - \frac{2q_2^2}{c - a^2/c}$$

$$= \frac{2q_2 \{a^2 q_2 - (c^2 - a^2) q_1\}}{c(c^2 - a^2)},$$

which is equation (58).

High Frequency Currents.

We know that at very high frequencies the currents distribute themselves over the surface of the cylinders in such a way that there are no magnetic lines of force produced in the metal. We thus see that when the currents are equal to $+I$ and $-I$ the current density i per unit of the circumference of the A cylinder is given by

$$i = \frac{I}{2\pi a} \cdot \frac{\sinh \alpha}{\cosh \alpha - \cos \theta} \quad \dots \quad (60)$$

See Fig. 2 and compare with equation (11).

Similarly for the B cylinder,

$$i = -\frac{I}{2\pi b} \cdot \frac{\sinh \beta}{\cosh \beta - \cos \theta} \quad \dots \quad (61)$$

Since $l/r = 1 - e \cos \theta$ is the equation to an ellipse referred to the focus as origin and the major axis as initial line, l being the semi latus rectum and e the eccentricity, we see that the length Cp of the radius vector of the ellipse MpM' in Fig. 5 gives the current density i at the point P on the cylinder A when carrying high frequency currents, the eccentricity of the ellipse being $\text{sech } \alpha$ and its major axis $(I/\pi a) \coth \alpha$.

The ratio of the greatest to the least current density on the A cylinder equals CM/CM' , which equals $\coth^2 (\alpha/2)$.

Let us now consider the case when a potential difference e_1 per unit length is applied to the A cylinder and a P.D. e_2 per unit length to the B cylinder. If I_1 and I_2 are the currents

produced on the cylinders, then, since with high frequencies the resistance terms can be neglected, our equations are

$$e_1 = L_{11} \frac{\partial I_1}{\partial t} + L_{12} \frac{\partial I_2}{\partial t} \quad . \quad . \quad . \quad (62)$$

and

$$e_2 = L_{22} \frac{\partial I_2}{\partial t} + L_{12} \frac{\partial I_1}{\partial t}, \quad . \quad . \quad . \quad (63)$$

where L_{11} , L_{22} and L_{12} are the inductance coefficients with high frequency currents.

Integrating these equations we get

$$\Phi_1 = L_{11} I_1 + L_{12} I_2 \quad . \quad . \quad . \quad (64)$$

and

$$\Phi_2 = L_{22} I_2 + L_{12} I_1, \quad . \quad . \quad . \quad (65)$$

where Φ_1 and Φ_2 are the linkages of the lines of induction with the currents in the A and B cylinders respectively.

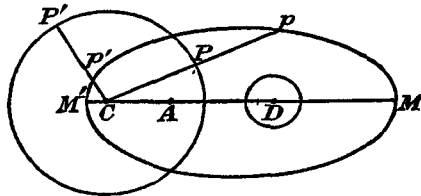


FIG. 5.

The current density at the point P on the cylinder equals Cp , where C is the focus of the ellipse whose major axis is in the same line as $C D$, and for which $C M = (I/2\pi a) \coth(\alpha/2)$; $C M' = (I/2\pi a) \tanh(\alpha/2)$ and the eccentricity $= \text{sech } \alpha = \frac{2ca}{c^2 + a^2 - b^2}$.

Let us first suppose that the cylinder A is inside the cylinder B . Since the currents are so distributed that the resultant magnetic force inside the metal of either conductor is zero we see by comparing (36) and (37) with (64) and (65) that

$$L_{11} = 1/C + 2(\alpha - \beta), \quad \text{and} \quad L_{12} = 1/C = L_{22}. \quad (66)$$

If $I_1 = -I_2$ the self inductance L , per unit length, of the circuit formed by the two cylinders is given by

$$L = L_{11} + L_{22} - 2L_{12} = 2(\alpha - \beta). \quad . \quad . \quad . \quad (67)$$

When the circuits are in parallel $e_1 = e_2$, and hence

$$I_1 = \frac{L_{22} - L_{12}}{L_{11} + L_{22} - 2L_{12}} (I_1 + I_2) = 0 \quad . \quad . \quad . \quad (68)$$

since $L_{22} - L_{12} = 0$. Thus all the current flows on the outside of the outer conductor.

If the outer cylinder is insulated so that $e_2=0$, we have $L_{22}I_2+L_{12}I_1=0$, and so $I_1=-I_2$. The induced current on the outer cylinder is thus equal and opposite to the current on the inner cylinder. Comparing also with the analogous electrostatic problem, we see that it is on the inside of the outer cylinder.

If the inner cylinder is insulated so that $e_1=0$, we have

$$I_1 = -\frac{L_{12}}{L_{11}} I_2 = -\frac{C_0}{C+C_0} I_2.$$

On the inner surface of the outer cylinder, therefore, there must flow a current $C_0 I_2 / (C+C_0)$, and on the outer surface a current $C I_2 / (C+C_0)$. When this occurs the electromagnetic energy

$$\frac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_{22} I_2^2 \quad . \quad . \quad . \quad (69)$$

has a minimum value, since I_2 is constant.

Since L_{12} and L_{22} are independent of c , the distance between the axes of the cylinders, the force F per unit length between them is given by

$$F = \frac{1}{2} \frac{\partial L_{11}}{\partial C} I_1^2 = \frac{1}{2} \frac{\partial}{\partial C} \left\{ \frac{1}{C} + 2(\alpha - \beta) \right\} I_1^2 = -\frac{I_1^2}{r} \quad . \quad . \quad (70)$$

the equilibrium being stable when the cylinders are co-axial.

When the cylinders are solid and parallel to one another we have

$$e_1 = \left(2\alpha + 2 \log \frac{d}{c} \right) \frac{\partial I_1}{\partial t} + 2 \log \frac{d}{c} \cdot \frac{\partial I_2}{\partial t} \quad . \quad . \quad . \quad (71)$$

$$\text{and} \quad e_2 = \left(2\beta + 2 \log \frac{d}{c} \right) \frac{\partial I_2}{\partial t} + 2 \log \frac{d}{c} \cdot \frac{\partial I_1}{\partial t} \quad . \quad . \quad . \quad (72)$$

approximately. The greater the values of d/c , c/a and c/b the more accurate will be the equations. Hence

$$L_{11} = 2\alpha + 2 \log \frac{d}{c}; \quad L_{22} = 2\beta + 2 \log \frac{d}{c}; \quad L_{12} = 2 \log \frac{d}{c}.$$

Thus the self inductance L per unit length of the circuit formed by the two cylinders in series will be given by

$$L = L_{11} + L_{22} - 2L_{12} = 2(\alpha + \beta) \quad . \quad . \quad . \quad (73),$$

an equation which is exactly true at all distances apart.

Comparing (10) and (73) we see that $LK=1$.

If W be the electromagnetic energy stored in the field at any instant, we have

$$W = a I_1^2 + \beta I_2^2 + \log \frac{d}{c} \cdot (I_1 + I_2)^2 \quad . \quad . \quad . \quad (74)$$

When the cylinders are in parallel we get

$$I_1 = \frac{\beta}{\alpha + \beta} (I_1 + I_2), \quad (75)$$

or

$$I_2 = \frac{\alpha}{\alpha + \beta} (I_1 + I_2).$$

At every instant, therefore, the ratio I_1/I_2 equals β/α . Since $\beta/\alpha = \sinh^{-1}(r/b)/\sinh^{-1}(r/a)$, we see that if b is greater than a , I_1/I_2 is less than unity. Hence the smaller conductor carries the smaller current. The density of the current on the smaller conductor is also less than on the larger conductor. It is to be noticed that (75) is the condition that (74) has a minimum value when $I_1 + I_2$ is a constant.

If F be the instantaneous value of the E.M.F. acting on the cylinders per unit length, we have

$$F = \frac{\partial W}{\partial c} = I_1^2 \frac{\sinh \alpha \cosh \beta}{r \sinh \omega} + I_2^2 \frac{\cosh \alpha \sinh \beta}{r \sinh \omega} - \frac{(I_1 + I_2)^2}{c} \quad (76)$$

$$= \frac{1}{2c^2r} \{I_1^2(c^2 + b^2 - a^2 - 2cr) + I_2^2(c^2 + a^2 - b^2 - 2cr) - 4crI_1I_2\} \quad (77)$$

$$= \frac{1}{2c^2r} (AI_1 - BI_2)(AI_1 - CI_2) \quad (78)$$

where A , B and C have the same values as in (55).

Now by Kelvin's theorem* if I_1 and I_2 are maintained constant the conductors move so as to increase the potential energy. Since W increases with c when F is positive, it follows that when F is positive the force is repulsive, and when negative it is attractive. We deduce from (78) that when I_1 and I_2 are of opposite signs the force is always repulsive. When, however, the currents I_1 and I_2 are flowing in the same direction the force is attractive when I_1/I_2 lies in value between B/A and C/A . When the ratio of these currents equals either of these limiting values the force vanishes, and when it lies outside those limits the force is repulsive.

In practice we are concerned with the average value of the force F taken over a whole period. If ϕ be the phase difference between the currents I_1 and I_2 , if F' denote the average value of F , and I_1' and I_2' be the effective values of the currents, we have by (78)

$$F' = \frac{1}{2c^2r} \{A^2I_1'^2 + B^2I_2'^2 - A(B+C)I_1'I_2'\cos \phi\} \quad . . (79)$$

* Russell's "Alternating Currents," Vol. I., p. 41.

When $\cos \varphi = 1$ or -1 , the equation is practically identical with (78). In any case, however, it can easily be put into factors and discussed in a similar way.

In the particular case when $I_1 = I_2 = I$, we have $\cos \varphi = 1$, and

$$F' = -\frac{I^2}{cr} (4r - c), \quad (80)$$

the force being always attractive.

When $I_1 = -I_2 = I$, $\cos \varphi = -1$, and hence

$$F' = \frac{I^2}{r}, \quad (81)$$

the force being repulsive. It is to be noticed that (80) is only an approximate formula, while (81) is an exact formula.

When the cylinder B is a very fine wire, formulæ (76) to (78) are extremely accurate. Putting $b=0$ and $2cr=c^2-a^2$, in 77 we get

$$F = \frac{2I_2 \{a^2 I_2 - (c^2 - a^2) I_1\}}{C(c^2 - a^2)}. \quad (82)$$

It can be readily shown by the method of images that this formula is exact. We have also

$$F' = \frac{2I_2' \{a^2 I_2' - (c^2 - a^2) I_1' \cos \varphi\}}{C(c^2 - a^2)}. \quad . . . (83)$$

We see that when I_2'/I_1' is greater than $(c^2 - a^2) \cos \varphi / a^2$ the force is repulsive. When this ratio equals $(c^2 - a^2) \cos \varphi / a^2$ the force vanishes, and when it is less than this value it is attractive.

It follows also from (83) that when the distance between the wire and the axis of the cylinder is less than

$$a \{(I_1' \cos \varphi + I_2') / I_1' \cos \varphi\}^{\frac{1}{2}}$$

the force is repulsive, when it is equal to it the force vanishes, and when it is greater than this value it is attractive.

Hence when

$$c = a \{(I_1' \cos \varphi + I_2') / I_1' \cos \varphi\}^{\frac{1}{2}} \quad . . . (84)$$

the wire is in a position of stable equilibrium.

In conclusion, physicists should bear in mind that the potential and capacity coefficients of conductors have perfectly determinate values. Even in simple cases values are difficult to find by calculation, but in every case they can be

found accurately by experiment. If we compare (27) and (28) with (64) and (65) we see at once that $L_{11}=p_{11}$, $L_{12}=p_{12}$ and $L_{22}=p_{22}$.* It follows that the inductance coefficients of conductors for high frequency alternating currents can be found very simply by determining experimentally the potential coefficients of the conductors for electrostatic charges.

ABSTRACT.

Many problems in connection with parallel cylindrical conductors occur in practical electrical work. The formulæ for the capacity between the conductors and for the effective inductance are well known, but the values of the capacity and potential coefficients and of the inductance coefficients have not yet been determined. It is shown that for the case of a cylinder inside a cylindrical tube their values can in all cases be easily computed. When the cylinders are external to one another, it is proved that the three capacity coefficients are connected by two very simple relations. Limiting values between which these coefficients must lie are found, and methods of obtaining closely approximate values in special cases are given. Whatever the charges on the cylinders may be, provided that the other conductors of the system are remote, the mutual force between them can be calculated to high accuracy when their distance apart is great or when the radius of one is small compared with that of the other.

Practically identical formulæ enable us to find the current-density and the inductance coefficients with high-frequency currents, both for a cylinder inside a cylindrical tube and for two parallel cylinders. In the latter case it is shown that when the phase difference between the currents is less than 90 deg the mechanical force between the cylinders is repulsive when they are close together and attractive when they are far apart. At a definite distance apart, therefore, the cylinders when carrying high-frequency currents are in stable equilibrium. Since the potential coefficients can always be determined experimentally, it follows that the inductance coefficients for high-frequency currents which are equal to them are also found by the same experiments.

DISCUSSION.

Dr. D. OWEN said that in the first paragraph of the section on High Frequency Currents, the author said "We thus see—&c." He did not quite know the justification for this.

Mr. F. J. W. WHIPPLE said it surprised him that the solutions to these problems had not all been already worked out. It appeared clear to him that one could write down the solutions in θ functions without much trouble. By so doing many difficulties might be got over since these functions were tabulated. Dr. Russell handled infinities rather familiarly, and he was not certain of the reliability of some of the solutions. Then he obtained a result for a thin wire and assumed it to hold for a thick one. One did not know how far the assumption was justified.

Dr. RUSSELL, in reply, said the object of the Paper was to obtain approximate solutions to clear the ground for a complete discussion of the problem.

Dr. Owen's point was explained in one of Kelvin's works, to which he would send him the reference.

* Cf. *l.c. ante*, Vol. I., p. 201.