Hence they are both intersected by the line of action of the single resultant to which the system of forces reduces in certain positions of the body.

This is Minding's Theorem.

On certain Extensions of Frullani's Theorem.

By C. LEUDESDORF, M.A.

[*Read April* 11*th*, 1878.]

The following investigation is intended to be supplementary to two papers by Mr. Elliott, on "Certain Multiple Definite Integrals," published in No. 106 and No. 113 of the *Proceedings.* Starting with an amended form of the theorem generally called Frullani's, the author deduces certain elegant extensions of this theorem to the cases of integrals of the 2nd, 3rd, ... 6th order; and in the second paper (§ 5) he writes down (though without proof) the theorem for the general case of the n-tuple integral to which he is led by his results in the cases investigated. These results are shewn to be true when certain conditions hold good among the involved constants; but these conditions are unsymmetrical in form, and their number in any given case seems subject to no law. For the integrals of order 1, 2, ... 6, the number of constants involved is 2, 4, ... 12, while the number of conditions found is 0, 1, 2, 4, 5, 7. There is thus no means of determining, for the general case, how many conditions must hold among the $2n$ constants, nor yet what these conditions are; so that there is no guarantee that it shall be possible to satisfy the conditions with the disposable constants. It is however difficult, on reading Mr. Elliott's papers, to avoid concluding that he is correct in asserting that the general theorem *does* hold, and the following proof of it may perhaps be of interest, as the subject is considered from rather a different point of view from that taken in the two papers mentioned; while the conditions found are symmetrical in form and less than the number of disposable constants, being $n-1$ in number when there are $2n$ constants.

As in the second paper, let $S(p, q, r \ldots)$ denote any symmetric function of $p, q, r \ldots$ which does not become infinite for any positive values of $p, q, r \ldots$ from zero to infinity, both inclusive. Denote

$$\int_0^\infty \frac{\phi(ax)}{x}\,dx \text{ by } [a], \quad \int_0^\infty\int_0^\infty S(ax, by)\,\frac{dx\,dy}{xy} \text{ by } [ab],$$

$$\int_0^\infty\int_0^\infty\int_0^\infty S(ax, by, cz)\,\frac{dx\,dy\,dz}{xyz} \text{ by } [abc], \text{ \&c.}$$

Let also $\log a$, $\log b$, $\log c$ ... be denoted by α, β, γ .... Then Frullani's Theorem in Mr. Elliott's form may be written

$$[a]-[b] = (\alpha-\beta)\,\{\phi(\infty)-\phi(0)\} \qquad (1).$$

By two applications of this,

$$\int_0^\infty\int_0^\infty \{S(ax, cy)-S(bx, cy)-S(ax, dy)+S(bx, dy)\}\,\frac{dx\,dy}{xy}$$

becomes first

$$(\alpha-\beta)\int_0^\infty \{S(\infty, cy)-S(0, cy)-S(\infty, dy)+S(0, dy)\}\,\frac{dy}{y},$$

and finally

$$(\alpha-\beta)(\gamma-\delta)\{S(\infty, \infty)-2S(\infty, 0)+S(0, 0)\},$$

which result may be written

$$\begin{aligned}(\alpha-\beta)(\gamma-\delta)\,S(\infty-0)^2 &= [ac]-[bc]-[ad]+[bd]\\ &= [(a-b)(c-d)] \qquad (2),\end{aligned}$$

by an extension of the notation already used. Similarly

$$[(a-b)(c-d)(e-f)]$$

or

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty \{S(ax, cy, ez)-S(bx, cy, ez)+\ldots\}\,\frac{dx\,dy\,dz}{xyz}\\ &= [ace]-[bce]-[ade]+[bde]\\ &\quad -[acf]+[bcf]+[adf]-[bdf]\end{aligned}$$

$$\begin{aligned}&= (\alpha-\beta)\\ &\times\int_0^\infty\int_0^\infty \left\{\begin{matrix} S(\infty, cy, ez)-S(0, cy, ez)-S(\infty, dy, ez)+S(0, dy, ez)\\ -S(\infty, cy, fz)+S(0, cy, fz)+S(\infty, dy, fz)-S(0, dy, fz)\end{matrix}\right\}\frac{dy\,dz}{yz}\\ &= (\alpha-\beta)(\gamma-\delta)\\ &\quad\times\int_0^\infty \left\{\begin{matrix} S(\infty, \infty, ez)-S(\infty, 0, ez)-S(0, \infty, ez)+S(0, 0, ez)\\ -S(\infty, \infty, fz)+S(\infty, 0, fz)+S(0, \infty, fz)-S(0, 0, fz)\end{matrix}\right\}\frac{dz}{z}\\ &= (\alpha-\beta)(\gamma-\delta)(\epsilon-\zeta)\\ &\qquad\times\{S(\infty, \infty, \infty)-3S(\infty, \infty, 0)+3S(\infty, 0, 0)-S(0, 0, 0)\}\\ &= (\alpha-\beta)(\gamma-\delta)(\epsilon-\zeta)\,S(\infty-0)^3 \qquad (3).\end{aligned}$$

And, in the same way, it may be shewn that

$$\begin{aligned}&[(a_1-b_1)(a_2-b_2)\,\ldots\,(a_n-b_n)]\\ &\qquad = (\alpha_1-\beta_1)(\alpha_2-\beta_2)\,\ldots\,(\alpha_n-\beta_n)\,S(\infty-0)^n \qquad (4).\end{aligned}$$

And if, by another extension of the notation used, $[a\bar{b}]$ be employed to denote $\int_0^\infty S(ax, b)\,\frac{dx}{x}$, $[a\bar{b}\bar{c}]$ to denote $\int_0^\infty S(ax, b, c)\,\frac{dx}{x}$, $[ab\bar{c}]$ to denote $\int_0^\infty \int_0^\infty S(ax, by, c)\,\frac{dx\,dy}{xy}$, &c., it is seen that

$$[a(b-c)] = (\beta-\gamma)\,[a\overline{\infty-0}],$$
$$[ab(c-d)] = (\gamma-\delta)\,[ab\overline{\infty-0}],$$
&c. &c.,

$$[a(b-c)(d-e)]$$
$$= (\beta-\gamma)(\delta-\epsilon)\int_0^\infty \{S(ax,\infty,\infty)-2S(ax,\infty,0)+S(ax,0,0)\}\,\frac{dx}{x}$$
$$= (\beta-\gamma)(\delta-\epsilon)\,[a(\overline{\infty-0}^2]$$
$$[a(b-c)(d-e)(f-g)]$$
$$= (\beta-\gamma)(\delta-\epsilon)(\zeta-\eta)\,[a\overline{\infty-0}^3],\ \text{\&c. \&c.} \quad\ldots\ldots\ (5).$$

Taking now the case of the integral of the second order,

$$[ab]-[a'b'] = [\{r-(r-a)\}\{r-(r-b)\}]-[\{r-(r-a')\}\{r-(r-b')\}]$$

(introducing a new quantity r whose logarithm is ρ)

$$= [rr-r(r-a+r-b)+(r-a)(r-b)]$$
$$-[rr-r(r-a'+r-b')+(r-a')(r-b')]$$
$$= [(r-a)(r-b)]-[(r-a')(r-b')]$$
$$+[r(a-a'+b-b')],$$
$$= \{(\rho-\alpha)(\rho-\beta)-(\rho-\alpha')(\rho-\beta')\}\,S(\infty-0)^2$$
$$+(\alpha+\beta-\alpha'-\beta')\,[r\overline{\infty-0}],\ \text{by (2) and (5)},$$
$$= (\alpha\beta-\alpha'\beta')\,S(\infty-0)^2,$$

if only $\alpha+\beta = \alpha'+\beta'$.

Taking next the triple integral,

$$-[abc]+[a'b'c']$$
$$= [(r-a)(r-b)(r-c)]-[(r-a')(r-b')(r-c')]$$
$$-[r\{(r-a)(r-b)+(r-b)(r-c)+(r-c)(r-a)\}$$
$$-r\{(r-a')(r-b')+(r-b')(r-c')+(r-c')(r-a')\}]$$
$$+[rr(a+b+c-a'-b'-c')]$$
$$= \{(\rho-\alpha)(\rho-\beta)(\rho-\gamma)-(\rho-\alpha')(\rho-\beta')(\rho-\gamma')\}\,S(\infty-0)^3$$
$$-\{(\rho-\alpha)(\rho-\beta)+(\rho-\beta)(\rho-\gamma)+(\rho-\gamma)(\rho-\alpha)\}\,[r\overline{\infty-0}^2]$$
$$-(\rho-\alpha')(\rho-\beta')-(\rho-\beta')(\rho-\gamma')-(\rho-\gamma')(\rho-\alpha')$$
$$+(\alpha+\beta+\gamma-\alpha'-\beta'-\gamma')\,[rr\overline{\infty-0}],\ \text{by (3) and (5)}.$$

But

$$(\rho-\alpha)(\rho-\beta)(\rho-\gamma)-(\rho-\alpha')(\rho-\beta')(\rho-\gamma')$$
$$= -\rho^2(\Sigma\alpha-\Sigma\alpha')+\rho(\Sigma\alpha\beta-\Sigma\alpha'\beta')-(\alpha\beta\gamma-\alpha'\beta'\gamma'),$$
$$\left.\begin{array}{l}(\rho-\alpha)(\rho-\beta)+(\rho-\beta)(\rho-\gamma)+(\rho-\gamma)(\rho-\alpha)\\ -(\rho-\alpha')(\rho-\beta')-(\rho-\beta')(\rho-\gamma')-(\rho-\gamma')(\rho-\alpha')\end{array}\right\}$$
$$= 2\rho(\Sigma\alpha-\Sigma\alpha')-(\Sigma\alpha\beta-\Sigma\alpha'\beta'),$$
$$\alpha+\beta+\gamma-\alpha'-\beta'-\gamma' = \Sigma\alpha-\Sigma\alpha';$$

so that

$$[abc]-[a'b'c']$$
$$=\{\rho^2(\Sigma a-\Sigma a')-\rho(\Sigma a\beta-\Sigma a'\beta')+(a\beta\gamma-a'\beta'\gamma')\}\,S(\infty-0)^3$$
$$+\{2\rho^2(\Sigma a-\Sigma a')-\rho(\Sigma a\beta-\Sigma a'\beta')\}\,[r\,\overline{\infty-0}^2]$$
$$-\{\rho^2(\Sigma a-\Sigma a')\}\,[r^2\,\overline{\infty-0}];$$

and when $\Sigma a=\Sigma a'$ and $\Sigma a\beta=\Sigma a'\beta'$,

$$[abc]-[a'b'c']=(a\beta\gamma-a'\beta'\gamma')\,S(\infty-0)^3.$$

The cases of the double and the triple integral only have been taken for simplicity; but it is clear that precisely similar investigations will hold good for the cases of the integrals of order 4, 5, ... n. The general theorem may be thus enunciated:—

Denote the sums of the homogeneous products of orders 1, 2, ... n of the n quantities $\log\frac{r}{a_1}$, $\log\frac{r}{a_2}$, ... $\log\frac{r}{a_n}$ by Σ_1, Σ_2, ... Σ_n; and let Σ'_1, Σ'_2, ... Σ'_n denote the same with regard to the n quantities $\log\frac{r}{a'_1}$, $\log\frac{r}{a'_2}$, ... $\log\frac{r}{a'_n}$; then

$$\begin{aligned}(-1)^n\{[a_1a_2\dots a_n]-[a'_1a'_2\dots a'_n]\}=\;&(\Sigma_n-\Sigma'_n)\,S(\infty-0)^n\\&-(\Sigma_{n-1}-\Sigma'_{n-1})\,[r\,\overline{\infty-0}^{n-1}]\\&+(\Sigma_{n-2}-\Sigma'_{n-2})\,[r^2\,\overline{\infty-0}^{n-2}]\\&-\&c.\\&+(-1)^n(\Sigma_1-\Sigma'_1)\,[r^{n-1}\,\overline{\infty-0}];\end{aligned}$$

and, just as in the cases of $n=2$, $n=3$, it is seen that $\Sigma_n-\Sigma'_n$, $\Sigma_{n-1}-\Sigma'_{n-1}$, ... can each be expressed in terms of $\sigma_n-\sigma'_n$, $\sigma_{n-1}-\sigma'_{n-1}$, ..., where σ_r stands for the sum of the homogeneous products of order r of the n quantities $\log a_1$, $\log a_2$, ... $\log a_n$, and σ'_r similarly with regard to $\log a'_1$, $\log a'_2$, ... $\log a'_n$. So that, when the $n-1$ conditions $\sigma_1=\sigma'_1$, $\sigma_2=\sigma'_2$, ... $\sigma_{n-1}=\sigma'_{n-1}$ are satisfied,

$$\begin{aligned}&[a_1a_2\dots a_n]-[a'_1a'_2\dots a'_n]\\&\quad=(-1)^n(\Sigma_n-\Sigma'_n)\,S(\infty-0)^n\\&\quad=(-1)^n S(\infty-0)^n\left\{\begin{array}{l}(\rho-a_1)(\rho-a_2)\dots(\rho-a_n)\\-(\rho-a'_1)(\rho-a'_2)\dots(\rho-a'_n)\end{array}\right\}\\&\quad=(-1)^n S(\infty-0)^n\left\{\begin{array}{l}-\rho^{n-1}(\sigma_1-\sigma'_1)\\+\rho^{n-2}(\sigma_2-\sigma'_2)\\-\&c.\\+(-1)^n(\sigma_n-\sigma'_n)\end{array}\right\}\\&\quad=(\sigma_n-\sigma'_n)\,S(\infty-0)^n;\end{aligned}$$

or

$$\int_0^\infty\!\!\int_0^\infty\dots\{S(a_1x_1,a_2x_2,\dots a_nx_n)-S(a'_1x_1,a'_2x_2,\dots a'_nx_n)\}\frac{dx_1dx_2\dots dx_n}{x_1x_2\dots x_n}$$
$$=(\log a_1\log a_2\dots\log a_n-\log a'_1\log a'_2\dots\log a'_n)\,S(\infty-0)^n,$$

in which form Mr. Elliott enunciates the result in a Note added to his second paper.

Referring to the first paper, it is easily seen that a precisely similar theorem holds good with regard to the integrals there considered; in fact, that

$$\int_0^\infty\int_0^\infty \dots \{\phi(a_1x_1+a_2x_2+\dots+a_nx_n)-\phi(a_1'x_1, a_2'x_2, \dots a_n'x_n)\}\frac{dx_1dx_2\dots dx_n}{x_1x_2\dots x_n}$$
$$=(-1)^{n-1}(\log a_1 \log a_2 \dots \log a_n - \log a_1' \log a_2' \dots \log a_n')\{\phi(\infty)-\phi(0)\}.$$

This result is of course really included in the one proved above; but it may be established independently of this by an exactly similar method; the fundamental theorem, corresponding to (4), being in this case

$$[(a_1-b_1)(a_2-b_2)\dots(a_n-b_n)]$$
$$=(-1)^{n-1}(a_1-\beta_1)(a_2-\beta_2)\dots(a_n-\beta_n)\{\phi(\infty)-\phi(0)\}.$$

$[a]$ standing for $\int_0^\infty \phi(ax)\frac{dx}{x}$, $[ab]$ for $\int_0^\infty\int_0^\infty \phi(ax+by)\frac{dx\,dy}{xy}$, &c.

May 9th, 1878.

Lord RAYLEIGH, F.R.S., President, in the Chair.

Messrs. W. M. Hicks and T. R. Terry were elected Members, and Prof. G. M. Minchin was admitted into the Society.

Messrs. Brioschi, Darboux, Gordan, Sophus Lie, and Mannheim were elected Honorary Foreign Members.

Prof. Henrici, F.R.S., communicated a paper by Dr. Klein, of Munich, "Uber die Transformation der elliptischen Functionen." Prof. Cayley, F.R.S., gave an account of a paper "On the Theory of Groups." Prof. Kennedy read his "Notes on the Solution of Statical Problems connected with Linkwork, and other Plane Mechanisms." Mr. J. W. L. Glaisher, F.R.S., briefly gave an account of his "Generalized Form of certain Series." Mr. A. B. Kempe spoke on "Conjugate Four-piece Linkages."

The following presents were made to the Society:—

"Aperçu historique sur l'origine et le développement des méthodes en géométrie," par M. Chasles, 2nd edition, 1875. From the Author.

"Sur les écarts de la ligne géodésique et des sections planes normales entre deux points rapprochés d'une surface courbe," par F. J. van de

Berg. (Harlem, extrait des Archives Néerlandaises, t. xii., pp. 353—398.)

"Monatsbericht," Feb. 1878.

"Educational Times," May, 1878.

"Atti della R. Accademia dei Lincei," Anno cclxxv., Serie terza.

"Transunti," Vol. ii., Fasc. 5°, Aprile 1878.

"Memorie della Classe di Scienze Fisiche, Matematiche e Naturali," Vol. i., dispensa 1ma, 2a; Roma, 1877.

"Memorie della Classe di Scienze Morali, Storiche e Filologiche," Vol. i.; Roma, 1877.

"Memorie della R. Accademia di Scienze, Lettere ed Arti in Modena," Tom. xvii.; Modena, 1877.

"Annali di Matematica," Tom. ix., Fasc. i.; Marzo 1878.

"Verallgemeinerung des Gaussischen Criterium für den quadratischen Rest-character einer Zahl in Bezug auf eine andere." Hr. Ernst Schering. (Extract from "Monatsbericht," 22 Juni, 1876.)

Dr. Kronecker, "Ueber Abelsche Gleichungen" (read April 16), Dec. 1877; "Notiz über Potenzreihen," Jan. 21, 1878; "Ueber Sturmsche Functionen," Feb. 14, 1878. (Extracts from "Monatsbericht.")

"Die fundamentalen Anzahlen und Ausartungen der cubischen Plancurven nullten Geschlechts," von H. Schubert in Hamburg. ("Math. Annalen." Bd. xiii., zweite Abhandlung der "Beiträge zur Abzählenden Geometrie.")

"Die Bestimmung und Ausgleichung der aus Beobachtungen abgeleiteten Wahrscheinlichkeiten," von Wilhelm Lazarus in Hamburg.

On the Transformation of Elliptic Functions.

By Dr. F. KLEIN.*

[*Read May 9th*, 1878.]

My investigations into the solution of equations of the fifth degree have led me to some results concerning the Transformation of Elliptic Functions which I wish to lay before the Mathematical Society. A full discussion of these will soon appear in the "Mathematische Annalen," t. xiv.

Let $\int \frac{dx}{\sqrt{a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4}}$ be an elliptical integral. We have then the invariants

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2,$$

* The Secretaries are indebted to Prof. Henrici for the following translation.